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COMMON FIXED POINT RESULT SATISFYING RATIONAL CONTRACTION IN COMPLEX VALUED G_b-METRIC SPACE

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Abstract. In recent years, researchers have become increasingly interested in studying fixed point results within complex-valued metric spaces. This is because these spaces are useful for solving complex equations and have unique topological properties. This paper adds to the development of fixed point theory by presenting a common fixed point theorem for selfmappings that meet a rational contraction condition in complex-valued metric spaces. The goal of this work is to generalize and extend existing fixed point results, making them more applicable to advanced systems. To achieve this, we use a detailed analytical approach within complex-valued metric spaces, which provide a more flexible framework than traditional metric spaces. Our method focuses on proving the existence and uniqueness of common fixed points for such mappings, building on earlier theorems in this field. An example is included to show how the results can be applied, proving their validity in mathematical analysis. The findings not only expand but also unify several known results, opening new directions for further study. Mathematically, this research offers stronger tools for working with complex-valued functions and contractions. While this work is mainly theoretical, it lays a foundation for practical applications in areas like quantum mechanics and computational mathematics in the future.

Keywords: common fixed point; rational contraction; self mappings.

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1. INTRODUCTION

Metric space theory has broad applications not only in mathematics but also in various quantitative sciences, serving as a fundamental framework for analytical studies. The notion of complex spaces is of great relevance in science; see, e.g., [16].

In 1976, Jungck [11] generalized the Banach contraction principle, proving a fixed-point theorem for pairs of mappings (with the assumption that at least one of them is continuous), see also [12, 13]. Sessa [18] introduced weak commutativity conditions for mappings in fixed point considerations, while Pant [17] further developed the theory of common fixed points of noncommuting mappings.

In 2011, Azam et al. [3] introduced the notion of complex-valued metric spaces and allowed the metric function to take the values in the set of complex numbers instead of real numbers. They proved some common fixed point theorems in complex-valued metric spaces. Morales and Rojas [14, 15] have established the existence and uniqueness of fixed points and common fixed points for a broad category of contraction mappings with rational expressions. The contractive inequality of these mappings is regulated by functions that remain stable at zero.

In 2014, Aghajani et al. [1] introduced the concept of G_b-metric spaces, which has since been instrumental in establishing fixed-point theorems in partially ordered spaces. This development builds upon earlier advancements in metric space theory, particularly in the study of contraction mappings and continuity, which have inspired extensive research, also see [8, 9].

More recently, Ege [5] extended these ideas by introducing complex-valued G_b-metric spaces, exploring their fundamental properties and establishing several fixed-point results [6, 7]. This work contributes to the ongoing generalization of metric concepts, enhancing their applicability in both theoretical and applied contexts, also see [10].

Further developments in this field include the work of Beniwal et al. [4], who established common fixed-point theorems for compatible self-mappings in extended parametric S_b-metric spaces, supported by corollaries, examples, and graphical representations. Their results were also applied to guarantee the existence of solutions to integral equations. Anjana et al. [2] studied coincidence points and common fixed points for pairs of compatible self-mappings in partially ordered b-metric spaces, presenting their findings with discrete examples and graphical illustrations. Additionally, Shukla et al. [19] investigated complex-valued fuzzy metric spaces, introducing α-admissible mappings and proving fixed-point theorems under symmetric contractive conditions.

In this paper, we establish a common fixed-point theorem for mappings satisfying rational contraction conditions in complex-valued G_b-metric spaces, contributing to the growing body of research in this area

2. PRELIMINARIES

First, we recollect some elementary descriptions of complex-valued metric spaces. Let $\mathbb{C}$ be the set of complex numbers, and let $\alpha_1, \alpha_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$\alpha_1 \precsim \alpha_2 \text{ if and only if } \operatorname{Re}(\alpha_1) \leq \operatorname{Re}(\alpha_2) \text{ and } \operatorname{Im}(\alpha_1) \leq \operatorname{Im}(\alpha_2).$$

It follows that $\alpha_1 \precsim \alpha_2$ if one of the following conditions is satisfied:

1. $\operatorname{Re}(\alpha_1) = \operatorname{Re}(\alpha_2)$ and $\operatorname{Im}(\alpha_1) = \operatorname{Im}(\alpha_2)$;
2. $\operatorname{Re}(\alpha_1) < \operatorname{Re}(\alpha_2)$ and $\operatorname{Im}(\alpha_1) = \operatorname{Im}(\alpha_2)$;
3. $\operatorname{Re}(\alpha_1) = \operatorname{Re}(\alpha_2)$ and $\operatorname{Im}(\alpha_1) < \operatorname{Im}(\alpha_2)$;
4. $\operatorname{Re}(\alpha_1) < \operatorname{Re}(\alpha_2)$ and $\operatorname{Im}(\alpha_1) < \operatorname{Im}(\alpha_2)$.

Definition 2.1. [1] In a non-void set Π, a mapping $d : \Pi \times \Pi \to \mathbb{C}$ fulfills the following properties for every $z, w, a \in \Pi$:

- $0 \precsim d(z,w)$ and $d(z,w) = 0 \Leftrightarrow z = w$;
- $d(z,w) = d(w,z)$;
- $d(z,w) \precsim d(z,a) + d(a,w)$.

The pair (Π, d) is known as a complex-valued metric space.

Definition 2.2. [5] In a non-void set Π, a mapping $G_b : \Pi \times \Pi \times \Pi \to \mathbb{C}$ with $s \geq 1$ fulfills the following properties for every $z, w, t, a \in \Pi$:

- (CG_b1) $G_b(z,w,t) = 0$ if $z = w = t$;
- (CG_b2) $0 \prec G_b(z,z,w)$ with $z \neq w$;
- (CG_b3) $G_b(z,z,t) \precsim G_b(z,w,t)$ with $w \neq t$;

- (CG_b4) $G_b(z,w,t) = G_b(\pi\{z,w,t\})$, where π is a permutation of z,w,t;
- (CG_b5) $G_b(z,w,t) \preccurlyeq s\{G_b(z,a,a) + G_b(a,w,t)\}$.

The pair (Π, G_b) is called a complex-valued G_b-metric space.

Proposition 2.3. [5] *Let* (Π, G_b) *be a complex-valued* G_b*-metric space. Then, for each* $z,w,t \in \Pi$*:*

(1) $G_b(z,w,t) \preccurlyeq s\{G_b(z,z,w) + G_b(z,z,t)\}$*;*

(2) $G_b(z,w,w) \preccurlyeq 2sG_b(w,z,z)$.

Definition 2.4. [5] Let (Π, G_b) be a complex-valued G_b-metric space. A sequence $\{z_n\}$ in Π is said to converge to a point $u \in \Pi$ if for each $a \in \mathbb{C}, a \succ 0$, there exists a positive integer n_0 such that for all $m,n \geq n_0$, $G_b(z_n, z_m, u) \prec a$.

Definition 2.5. [5] Let (Π, G_b) be a complex-valued G_b-metric space. A sequence $\{z_n\}$ in Π is said to be Cauchy if for each $a \in \mathbb{C}, a \succ 0$, there exists a positive integer n_0 such that for all $m,n,l \geq n_0$, $G_b(z_n, z_m, z_l) \prec a$.

Definition 2.6. [5] A complex-valued G_b-metric space is called complete if every Cauchy sequence in Π converges in Π.

The concept of compatible maps in metric spaces was introduced by Jungck [12] in 1986.

Definition 2.7. [12] Two self-mappings A and B are called compatible if there exists a sequence $\{z_n\}$ such that

$$\lim_{n\to\infty} G_b(ABz_n, BAz_n, BAz_n) = 0 \text{ or } \lim_{n\to\infty} G_b(BAz_n, ABz_n, ABz_n) = 0,$$

whenever $\{z_n\}$ is a sequence in Π such that $\lim_{n\to\infty} Az_n = u = \lim_{n\to\infty} Bz_n$ for some $u \in \Pi$.

Definition 2.8. [12] Let A and B be self-mappings on a non-void set Π. Then A and B are said to be weakly compatible if they commute at their coincidence points, i.e., $Az = Bz$ for some $z \in \Pi$ implies that $ABz = BAz$.

Definition 2.9. [12] Let A and B be self-mappings on a non-empty set Π. If $w = Az = Bz$ for some $z \in \Pi$, then z is called a coincidence point of A and B, and w is called a point of coincidence A and B.

The concept of compatible maps in G_b metric space is given by [7]

Definition 2.10 ([7])**.** Let (Π, G_b) be a complex valued G_b-metric space and H, K be mappings from (Π, G_b) into itself. The mappings H, K are called compatible if there exists a sequence $\{z_n\}$ such that

$\lim_{n\to\infty} G_b(HKz_n, KHz_n, KHz_n) = 0$ or $\lim_{n\to\infty} G_b(KHz_n, HKz_n, HKz_n) = 0$, whenever $\{z_n\}$ is a sequence in Π such that $\lim_{n\to\infty} Hz_n = \varphi = \lim_{n\to\infty} Kz_n$, for some $\varphi \in \Pi$.

Example 2.11 ([7])**.** Let $\Pi = [-1,1]$ and (Π, G_b) a complex valued G_b-metric space such that $G_b(h,j,k) = |h-j|^2 + |j-k|^2 + |k-h|^2$, for all $h,j,k \in \Pi$, where $s = 2$. Define two self-mappings $H, K : \Pi \to \Pi$ by $H(h) = h$ and $K(h) = \frac{h}{3}$. Consider a sequence $h_n = \frac{1}{2n}$, we get

$$\lim_{n\to\infty} G_b(KHh_n, HKh_n, HKh_n) = \lim_{n\to\infty} G_b\left(\frac{1}{6n}, \frac{1}{6n}, \frac{1}{6n}\right) = 0,$$

and also,

$$\lim_{n\to\infty} Hh_n = H\left(\frac{1}{2n}\right) = 0 \text{ and } \lim_{n\to\infty} Kh_n = K\left(\frac{1}{2n}\right) = \left(\frac{1}{6n}\right) = 0.$$

Therefore, mappings H, K are compatible.

3. MAIN RESULTS

Theorem 3.1. *Let (S, G) be a complete complex-valued G_b-metric space with $s \geq 1$, and let $A, B, C, D : \Pi \to \Pi$ be self-mappings of S satisfying the following conditions:*

(1) (CG_b1) $A(\Pi) \subseteq D(\Pi)$ and $B(\Pi) \subseteq C(\Pi)$;

(2) (CG_b2) $G_b(Az, Bw, Bt) \precsim \frac{\lambda}{s^2} R(z,w,t)$ for $\lambda \in (0,1)$ and for all $z, w, t \in \Pi$, where

$$R(z,w,t) = \max \begin{cases} G_b(Cz, Dw, Dt), \\ G_b(Cz, Az, Az), \\ G_b(Dw, Bw, Bt), \\ \frac{1}{2}\left[G_b(Dw, Az, Az) + G_b(Cz, Bw, Bt)\right], \\ \frac{G_b(Cz,Az,Az)G_b(Dw,Bw,Bt)}{1+G_b(Cz,Dw,Dt)} \end{cases};$$

(3) (CG_b3) The pair (A, C) is compatible, and the pair (B, D) is weakly compatible;

(4) (CG_b4) *Either A or C is continuous.*

Then, A,B,C,D *have a unique common fixed point in* Π*.*

Proof. Let $p_0 \in \Pi$ be an arbitrary point. From condition (CG_b1), there exist z_1, z_2 such that $w_0 = Dz_1 = Az_0$ and $w_1 = Cz_2 = Bz_1$. We can construct sequences $\{w_n\}$ and $\{z_n\}$ in Π such that

$$w_{2n} = Dz_{2n+1} = Az_{2n} \quad \text{and} \quad w_{2n+1} = Cz_{2n+2} = Bz_{2n+1}. \tag{3.1}$$

Using condition (CG_b2), we have

$$G_b(w_{2n}, w_{2n+1}, w_{2n+1}) = G_b(Az_{2n}, Bz_{2n+1}, Bz_{2n+1}) \preccurlyeq \frac{\lambda}{s^2} R(z_{2n}, z_{2n+1}, z_{2n+1}),$$

where

$$R(z_{2n}, z_{2n+1}, z_{2n+1}) = \max \begin{cases} G_b(Cz_{2n}, Dz_{2n+1}, Dz_{2n+1}), \\ G_b(Cz_{2n}, Az_{2n}, Az_{2n}), \\ G_b(Dz_{2n+1}, Bz_{2n+1}, Bz_{2n+1}), \\ \frac{1}{2}\left[G_b(Dz_{2n+1}, Az_{2n}, Az_{2n}) + G_b(Cz_{2n}, Bz_{2n+1}, Bz_{2n+1})\right], \\ \frac{G_b(Cz_{2n}, Az_{2n}, Az_{2n}) G_b(Dz_{2n+1}, Bz_{2n+1}, Bz_{2n+1})}{1 + G_b(Cz_{2n}, Dz_{2n+1}, Dz_{2n+1})} \end{cases}.$$

$$= \max \left\{ \begin{array}{l} G_b(w_{2n-1}, w_{2n}, w_{2n}), \\ G_b(w_{2n-1}, w_{2n}, w_{2n}), \\ G_b(w_{2n}, w_{2n+1}, w_{2n+1}), \\ \frac{1}{2}\left[G_b(w_{2n}, w_{2n}, w_{2n}) + G_b(w_{2n-1}, w_{2n+1}, w_{2n+1})\right], \\ \frac{G_b(w_{2n-1}, w_{2n}, w_{2n}) G_b(w_{2n}, w_{2n+1}, w_{2n+1})}{1 + G_b(w_{2n-1}, w_{2n}, w_{2n})} \end{array} \right\}$$

which is either $G_b(w_{2n-1}, w_{2n}, w_{2n})$ or $G_b(w_{2n}, w_{2n+1}, w_{2n+1})$.

If $R(z_{2n}, z_{2n+1}, z_{2n+1}) = G_b(w_{2n}, w_{2n+1}, w_{2n+1})$, then

$$\left(1 - \frac{\lambda}{s^2}\right) G_b(w_{2n}, w_{2n+1}, w_{2n+1}) \preccurlyeq 0,$$

which is a contradiction since $s \geq 1$ and $\lambda \in (0,1)$. Therefore,

$$G_b(w_{2n}, w_{2n+1}, w_{2n+1}) \preccurlyeq \frac{\lambda}{s^2} G_b(w_{2n-1}, w_{2n}, w_{2n}).$$

Similarly,

$$G_b(w_{2n+1},w_{2n+2},w_{2n+2}) \precsim \frac{\lambda}{s^2}G_b(w_{2n},w_{2n+1},w_{2n+1}).$$

It follows that

$$G_b\left(w_n,w_{n+1},w_{n+1}\right) \precsim \frac{\lambda}{s^2}G_b\left(w_{n-1},w_n,w_n\right)$$
$$\precsim \left(\frac{\lambda}{s^2}\right)^2 G_b\left(w_{n-2},w_{n-1},w_{n-1}\right)+\dots+\precsim \left(\frac{\lambda}{s^2}\right)^n G_b\left(w_0,w_1,w_1\right),$$

implies that

$$\begin{aligned}\left|G_b\left(w_n,w_{n+1},w_{n+1}\right)\right| &\le \frac{\lambda}{s^2}\left|G_b\left(w_{n-1},w_n,w_n\right)\right|\\ &\le \left(\frac{\lambda}{s^2}\right)^2\left|G_b\left(w_{n-2},w_{n-1},w_{n-1}\right)\right|+\dots\\ &\le \left(\frac{\lambda}{s^2}\right)^n\left|G_b\left(w_0,w_1,w_1\right)\right|.\end{aligned}$$

For every $n,m\in\mathbb{N}$, with $n<m$, we have:

$$\begin{aligned}&\left|G_b\left(w_n,w_m,w_m\right)\right| \le \left|s\left\{G_b\left(w_n,w_{n+1},w_{n+1}\right)+G_b\left(w_{n+1},w_m,w_m\right)\right\}\right|\\ &\le \left|sG_b\left(w_n,w_{n+1},w_{n+1}\right)+s^2\left\{G_b\left(w_{n+1},w_{n+2},w_{n+2}\right)+G_b\left(w_{n+2},w_m,w_m\right)\right\}\right|\\ &\vdots\\ &\le \left|sG_b\left(w_n,w_{n+1},w_{n+1}\right)+s^2G_b\left(w_{n+1},w_{n+2},w_{n+2}\right)+\dots+s^{m-n+1}G_b\left(w_{m-1},w_m,w_m\right)\right|\\ &\le \left|s\left(\frac{\lambda}{s^2}\right)^n G_b\left(w_0,w_1,w_1\right)+s^2\left(\frac{\lambda}{s^2}\right)^{n+1} G_b\left(w_0,w_1,w_1\right)+\dots+s^{m-n+1}\left(\frac{\lambda}{s^2}\right)^{m-1} G_b\left(w_0,w_1,w_1\right)\right|\\ &= \left[s\left(\frac{\lambda}{s^2}\right)^n\right]\left[1+s\left(\frac{\lambda}{s^2}\right)+\dots+s^{m-n}\left(\frac{\lambda}{s^2}\right)^{m-n-1}\right]\left|G_b\left(w_0,w_1,w_1\right)\right|\\ &= \left(\frac{\lambda^n}{s^{2n-1}}\right)\left[\frac{1}{1-\frac{\lambda}{s}}\right]\left|G_b\left(w_0,w_1,w_1\right)\right|\end{aligned}$$

as $n\to\infty$, we have:

$$\left|G_b\left(w_n,w_m,w_m\right)\right| \le \left(\frac{\lambda^n}{s^{2n-1}}\right)\left[\frac{1}{1-\frac{\lambda}{s}}\right]\left|G_b\left(w_0,w_1,w_1\right)\right| \to 0. \tag{3.2}$$

Thus, $\{w_n\}$ is a Cauchy sequence in Π. Since Π is complete, there exists $u\in\Pi$ such that $w_n\to u$ i.e., $G_b\left(w_n,w_m,w_m\right)<a$ as $n\to\infty$, for some $a\in\mathbb{C}$.

And for its sub-sequences, we also have

$$Dz_{2n+1}\to u,\quad Az_{2n}\to u,\quad Cz_{2n+2}\to u,\quad \text{and}\quad Bz_{2n+1}\to u.$$

From (CG_b4), if C is continuous, then

$$CCz_{2n} \to Cu \quad \text{and} \quad CAz_{2n} \to Cu \quad \text{as} \quad n \to \infty.$$

Further, (A,C) is compatible, which implies that

$$CAz_{2n} = ACz_{2n} \to Cu.$$

Indeed,

$$G_b\left(ACz_n, Cu, Cu\right) \preccurlyeq s\left\{G_b\left(ACz_n, Cu, CAz_n\right) + G_b\left(CAz_n, CAz_n, Cu\right)\right\}$$

$$\left|G_b\left(ACz_n, Cu, Cu\right)\right| \le s\left|G_b\left(ACz_n, Cu, CAz_n\right)\right| + s\left|G_b\left(CAz_n, CAz_n, Cu\right)\right|$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

We want to show that $Cu = u$. On the contrary, we suppose that $Cu \neq u$. Then,

$$G_b(Cu,u,u) \preccurlyeq sG_b\left(Cu, ACz_{2n}, ACz_{2n}\right) + s^2 G_b\left(ACz_{2n}, Bz_{2n+1}, Bz_{2n+1}\right) + s^3 G_b\left(Bz_{2n+1}, u, u\right).$$

Using (CG_b2), by taking $z = Cz_{2n}$ and $w = t = Bz_{2n+1}$, we proceed with the proof

$$G_b\left(ACz_{2n}, Bz_{2n+1}, Bz_{2n+1}\right) \preccurlyeq \frac{\lambda}{s^2} R\left(Cz_{2n}, z_{2n+1}, z_{2n+1}\right), \tag{3.3}$$

where

$$R\left(Cz_{2n}, z_{2n+1}, z_{2n+1}\right) = \max\left\{\begin{array}{l} G_b\left(CCz_{2n}, Dz_{2n+1}, Dz_{2n+1}\right), \\ G_b\left(CCz_{2n}, ACz_{2n}, ACz_{2n}\right), \\ G_b\left(Dz_{2n+1}, Bz_{2n+1}, Bz_{2n+1}\right), \\ \frac{1}{2}\left[G_b\left(Dz_{2n+1}, ACz_{2n}, ACz_{2n}\right) + G_b\left(CCz_{2n}, Bz_{2n+1}, Bz_{2n+1}\right)\right], \\ \frac{G_b(CCz_{2n}, ACz_{2n}, ACz_{2n}) G_b(Dz_{2n+1}, Bz_{2n+1}, Bz_{2n+1})}{1+G_b(CCz_{2n}, Dz_{2n+1}, Dz_{2n+1})} \end{array}\right\}.$$

Taking the limit as $n \to \infty$, we get:

$$R(Cu,u,u) = \max\left\{\begin{array}{l} G_b(Cu,u,u), \\ G_b(Cu,u,u), \\ G_b(u,Cu,Cu), \\ \frac{1}{2}\left[G_b(u,Cu,Cu) + G_b(Cu,Cu,Cu)\right], \\ \frac{G_b(Cu,Cu,Cu) G_b(u,Cu,Cu)}{1+G_b(Cu,u,u)} \end{array}\right\}.$$

This is either $G_b(u,Cu,Cu)$ or $G_b(Cu,u,u)$.

If $R(Cu,u,u) = G_b(u,Cu,Cu)$, then from 3.3 and , we have:

$$|G_b(u,Cu,Cu)| \leq \frac{\lambda}{s^2}\,|G_b(u,Cu,Cu)|\,.$$

We have:

$$\left(1-\frac{\lambda}{s^2}\right)|G_b(u,Cu,Cu)| \leq 0,$$

which is not possible.

Also, if $R(Cu,u,u) = G_b(Cu,u,u) \preccurlyeq 2sG_b(u,Cu,Cu)$, then

$$|G_b(u,Cu,Cu)| \leq \frac{2\lambda s}{s^2}\,|G_b(u,Cu,Cu)| = \frac{2\lambda}{s}\,|G_b(u,Cu,Cu)|\,,$$

which is not always true.

Hence, in both cases, we have:

$$|G_b(u,Cu,Cu)| = 0 \quad \text{and} \quad Cu = u.$$

Now, our aim is to show $Au = u$. On the contrary, suppose that $Au \neq u$. Then:

$$G_b(Au,u,u) \preccurlyeq sG_b\left(Au,Bz_{2n+1},Bz_{2n+1}\right) + sG_b\left(Bz_{2n+1},u,u\right).$$

Using (CG_b2), by taking $z = u$, $w = z_{2n+1}$, and $t = z_{2n+1}$, we have:

$$G_b\left(Au,Bz_{2n+1},Bz_{2n+1}\right) \preccurlyeq \frac{\lambda}{s^2}R\left(u,z_{2n+1},z_{2n+1}\right), \tag{3.4}$$

where

$$R\left(u,z_{2n+1},z_{2n+1}\right) = \max\left\{\begin{array}{l} G_b\left(Cu,Dz_{2n+1},Dz_{2n+1}\right), \\ G_b(Cu,Au,Au), \\ G_b\left(Dz_{2n+1},Bz_{2n+1},Bz_{2n+1}\right), \\ \frac{1}{2}\left[G_b\left(Dz_{2n+1},Au,Au\right) + G_b\left(Cu,Bz_{2n+1},Bz_{2n+1}\right)\right], \\ \frac{G_b(Cu,Au,Au)G_b(Dz_{2n+1},Bz_{2n+1},Bz_{2n+1})}{1+G_b(Cu,Dz_{2n+1},Dz_{2n+1})} \end{array}\right\},$$

Taking the limit as $n \to \infty$, we get:

$$R\left(u,z_{2n+1},z_{2n+1}\right) = G_b(u,Au,Au).$$

Therefore, (3.4) becomes:

$$G_b(Au,u,u) \preccurlyeq \frac{\lambda}{s^2}G_b(u,Au,Au) \preccurlyeq \frac{2s\lambda}{s^2}G_b(Au,u,u) = \frac{2\lambda}{s}G_b(Au,u,u).$$

This implies:

$$\left(1-\frac{2\lambda}{s}\right)|G_b(Au,u,u)| \leq 0.$$

This is a contradiction, and hence $Au = u$.

Next, we prove $Du = Bu$. Since $A(\Pi) \subseteq D(\Pi)$, there exists $v \in \Pi$ such that $u = Au = Dv$. First, we show that $Dv = Bv$.

Consider:

$$G_b(Dv,Bv,Bv) = G_b(Au,Bv,Bv) \preccurlyeq \frac{\lambda}{s^2}R(u,v,v), \tag{3.5}$$

where

$$R(u,v,v) = \max\left\{\begin{array}{l} G_b(Cu,Dv,Dv), \\ G_b(Cu,Au,Au), \\ G_b(Dv,Bv,Bv), \\ \frac{1}{2}\left[G_b(Dv,Au,Au)+G_b(Cu,Bv,Bv)\right], \\ \frac{G_b(Cu,Au,Au)G_b(Dv,Bv,Bv)}{1+G_b(Cu,Dv,Dv)} \end{array}\right\},$$

$$= G_b(u,Bv,Bv).$$

Then, (3.5) becomes:

$$G_b(u,Bv,Bv) \preccurlyeq \frac{\lambda}{s^2}G_b(u,Bv,Bv).$$

Further,

$$|G_b(u,Bv,Bv)| \leq \frac{\lambda}{s^2}|G_b(u,Bv,Bv)|,$$

which is a contradiction. Hence,

$$|G_b(Dv,Bv,Bv)| = 0 \quad \text{and} \quad Dv = Bv = u.$$

Since (B,D) is weakly compatible, we have:

$$BDv = DBv = v \quad \text{and} \quad Du = Bu.$$

Now, we will prove that $Bu = u$. On the contrary, suppose that $Bu \neq u$. Then:

$$G_b(u,Du,Du) = G_b(Au,Du,Du) \preccurlyeq \frac{\lambda}{s^2}R(u,u,u), \tag{3.6}$$

where

$$R(u,u,u) = \max\left\{\begin{array}{l} G_b(Cu,Du,Du), \\ G_b(Cu,Au,Au), \\ G_b(Du,Bu,Bu), \\ \frac{1}{2}\left[G_b(Du,Au,Au)+G_b(Cu,Bu,Bu)\right], \\ \frac{G_b(Cu,Au,Au)G_b(Du,Bu,Bu)}{1+G_b(Cu,Du,Du)} \end{array}\right\}.$$

$$= G_b(u,Bu,Bu)$$

and 3.6 becomes,

$$G_b(u,Bu,Bu) \preccurlyeq \frac{\lambda}{s^2}G_b(u,Bu,Bu), |G_b(u,Bu,Bu)| \leq \frac{\lambda}{s^2}|G_b(u,Bu,Bu)|,$$

again not possible. Hence $Bu = u$

Now, next we will prove $Du = u$. On contrary, we suppose that $Du \neq u$.

$$G_b(u,Du,Du) = G_b(Au,DBu,DBu) = G_b(Au,BDu,BDu) \preccurlyeq \frac{\lambda}{s^2}R(u,Du,Du), \tag{3.7}$$

where

$$R(u,Du,Du) = \max\left\{\begin{array}{l} G_b(Cu,DDu,DDu), \\ G_b(Cu,Au,Au), \\ G_b(DDu,BDu,BDu), \\ \frac{1}{2}\left[G_b(DDu,Au,Au)+G_b(Cu,BDu,BDu)\right], \\ \frac{G_b(Cu,Au,Au)G_b(DDu,BDu,BDu)}{1+G_b(Cv,DDu,DDu)} \end{array}\right\}.$$

$$= G_b(u,Du,Du).$$

From 3.7, it follows that:

$$G_b(u,Du,Du) \preccurlyeq \frac{\lambda}{s^2}G_b(u,Du,Du),$$

which implies:

$$\left(1-\frac{\lambda}{s^2}\right)|G_b(u,Du,Du)| \leq 0,$$

which is not possible. Hence, $Du = u$.

When C is continuous, we have:

$$Au = Bu = Cu = Du = u,$$

i.e., the mappings A, B, C, D have u as their common fixed point. Similar results hold when A is continuous.

Now, we prove the uniqueness of the fixed point. Let u^* be a common fixed point of A, B, C, D such that $u \neq u^*$. Using (CG_b2), we get:

$$G_b\left(Au, Bu^*, Bu^*\right) \precsim \frac{\lambda}{s^2} R\left(u, u^*, u^*\right),$$

where

$$\begin{aligned} R\left(u, u^*, u^*\right) &= \max \left\{ \begin{array}{l} G_b\left(Cu, Du^*, Du^*\right), \\ G_b(Cu, Au, Au), \\ G_b\left(Du^*, Bu^*, Bu^*\right), \\ \frac{1}{2}\left[G_b\left(Du^*, Au, Au\right) + G_b\left(Cu, Bu^*, Bu^*\right)\right], \\ \frac{G_b(Cu, Au, Au) G_b(Du^*, Bu^*, Bu^*)}{1 + G_b(Cu, Du^*, Du^*)} \end{array} \right\}, \\ &= \text{Either } \frac{G_b\left(u^*, u, u\right)}{2} \text{ or } G_b\left(u, u^*, u^*\right). \end{aligned}$$

If $R\left(u, u^*, u^*\right) = \frac{1}{2} G_b\left(u^*, u, u\right) \precsim \frac{2s}{2} G_b\left(u, u^*, u^*\right)$, then:

$$\left|G_b\left(u, u^*, u^*\right)\right| \leq \frac{\lambda}{s}\left|G_b\left(u, u^*, u^*\right)\right|,$$

which is a contradiction. Hence, $u = u^*$.

Similarly, if $R\left(u, u^*, u^*\right) = G_b\left(u, u^*, u^*\right)$, we again arrive at a contradiction. This shows that $u = u^*$. Thus, the mappings A, B, C, D have u as their common fixed point.

□

Example 3.2. Let $S = [-1, 1]$ and define

$$G_b(z, w, t) = \max(|z - w|, |w - t|, |t - z|)^2 + i \max(|z - w|, |w - t|, |t - z|)^2$$

for all $z, w, t \in S$. This is a complex-valued G_b-metric space with $s = 2$. Consider the mappings $A, B, C, D : S \to S$ defined by

$$Az = \frac{z}{24}, \quad Bz = \frac{z}{36}, \quad Cz = \frac{z}{2}, \quad Dz = \frac{z}{3}.$$

Consider the expression:

$$\begin{aligned} G_b(Az,Bw,Bt) &= \max\left(\left|\frac{z}{24}-\frac{w}{36}\right|,\left|\frac{w}{36}-\frac{t}{36}\right|,\left|\frac{t}{36}-\frac{z}{24}\right|\right)^2 \\ &\quad + i\max\left(\left|\frac{z}{24}-\frac{w}{36}\right|,\left|\frac{w}{36}-\frac{t}{36}\right|,\left|\frac{t}{36}-\frac{z}{24}\right|\right)^2 \\ &= \frac{1}{144}\left[\left(\left|\frac{z}{2}-\frac{w}{3}\right|\right)^2 + i\max\left(\left|\frac{z}{2}-\frac{w}{3}\right|\right)^2\right]. \end{aligned}$$

Also, we have:

$$\begin{aligned} G_b(Cz,Dw,Dt) &= \max\left(\left|\frac{z}{2}-\frac{w}{3}\right|,\left|\frac{w}{3}-\frac{t}{3}\right|,\left|\frac{t}{3}-\frac{z}{2}\right|\right)^2 \\ &\quad + i\max\left(\left|\frac{z}{2}-\frac{w}{3}\right|,\left|\frac{w}{3}-\frac{t}{3}\right|,\left|\frac{t}{3}-\frac{z}{2}\right|\right)^2. \end{aligned}$$

From the above, it follows that:

$$G_b(Az,Bw,Bt) = \frac{1}{144}G_b(Cz,Dw,Dt).$$

All conditions of the theorem are satisfied with $\lambda = \frac{1}{36}$, and 0 is the unique common fixed point of A,B,C,D.

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All the authors contributed equally to prepare this paper.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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