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COMMON FIXED POINT THEOREMS FOR T-REICH CONTRACTION MAPPING IN CONE METRIC SPACES

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Abstract. In the present paper we prove common fixed point theorem for T-Reich contraction mapping in the setting of cone metric space.

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1. Introduction

Recently, Huang and Zhang [1] generalize the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtain some fixed point theorems for mappings satisfying different contraction conditions in cone metric spaces. Subsequently many authors have generalized the results of Huang and Zhang and have studied fixed point theorems for different types of cones, see for instance [9],[10],[11],[12],[13],[14] etc. In sequel, J.R. Morales and E. Rojas [3],[4],[5] obtained sufficient conditions for the existence of a unique fixed point of T-Kannan contractive, T-zamfirescu, T-contractive mappings etc, in complete cone metric spaces. Afterwards; R.Sumitra et al [9] have proved common

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fixed point theorem for a Banach pair of mappings satisfying T-Hardy Rogers type contraction condition in cone metric spaces. In the sequel, we need a definition which was introduced and called Banach operator of type k by subrahmanyam [7]. Recently Chen and Li [8] extended the concept of Banach operator of type k of Banach operator pair and proved various best approximation results using common fixed point theorems for f-nonexpansive mappings.

The purpose of this paper is to prove common fixed point theorem for T-Reich contraction mappings in cone metric spaces. Our results generalize and extend the result [9].

2. Preliminary

In this section we recall the definition of cone metric spaces and some of their properties (see [1]).

The following notions will be used in order to prove the main results.

Definition 2.1[1]: Let E be a real Banach space and P a subset of $E.P$ is called a cone if and only if:

i): P is closed, non-empty, and $P \neq \{0\}$;

ii): $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,

iii): $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{o\}$.

Given a cone $P \subset E$, a partial ordering is defined as $\leqslant$ on E with respect to P by $x \leqslant y$ if and only if $y - x \in P$.It is denoted as $x \ll y$ will stand for $y - x \in intP, where\, intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that, for all $x, y \in E, 0 \leqslant x \leqslant y$ implies $\| x \| \leqslant K \| y \|$. (2.1.1.)

The least positive number K satisfying (2.1.1) is called normal constant of P.

Definition 2.2[1]: Let X be a non-empty set. Suppose E is real Banach space, P is a cone with $int\mathrm{P} \neq \phi$ and $\leqslant$ is a partial ordering with respect to P. If the following $d : X \times X \to E$ satisfies,

(i) $0 \leqslant d\,(x, y)$**:** for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$**:** for all $x, y \in X$;

(iii) $d(x, y) \leqslant d(x, z) + d(y, z)$**:** for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example2.3[1] Let $E = \mathbb{R}^2, P = \{(x, y) \in E : x, y \geqslant 0\} \subset R^2, X = \mathbb{R}\, and\ d : X \times X \to E$ such that $d\,(x, y) = (|\ x - y\ |, \propto |\ x - y\ |)$, where $\propto \geqslant 0$ is a constant. Then (X, d) is cone metric space.

Lemma 2.4[1]: Let (X, d) be a cone metric space and P be a normal cone wih normal constant K.

(i): A sequence $\{x_n\}$ in X converges to x, if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(ii): A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 2.5[4]: Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence in X. Then,

(i)$\{x_n\}$**converges:** to $x \in X$ if for every $c \in E$ with $o \ll c$, there is $n_o \in N$, the set of all natural numbers such that for all $n \geqslant n_o$,

$d(x_n, x) \ll c$**.:**

It: is denoted by $\lim_{n\to\infty} x_n = x\ or\ x_n \to x, (n \to \infty)$.

(ii): If for every $c \in E$, there is a number $n_o \in N$ such that for all $m, n \geqslant n_o$.

$d(x_n, x_m) \ll c$**.:**

then: $\{x_n\}$is called a Cauchy sequence in X.

(iii): $(X, d)$is a complete cone metric space if every Cauchy sequence in X is convergent.

(iv): A self mapping $T : X \to X$ is said to be continuous at a point $x \in X$, if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} Tx_n = Tx$ for every $\{x_n\}$ in X.

Definition 2.6 - A self mapping T of a metric space (X, d) is said to be a contraction mapping, if there exists a real number $0 \leqslant k < 1$ such that for all $x, y \in X$.

$d(Tx, Ty) \leqslant kd(x, y)$ (2.6.1)

Definition 2.7 [2]: Let T and f be two self-mappings of a metric space (X, d). The self mappin f of X is said to be T-contraction, if there exists a real number $0 \leqslant k < 1$ such that

$d(Tfx, Tfy) \leqslant kd(Tx, Ty)$ (2.7.1)

for all $x, y \in X$.

If $T = I$, the identity mapping, then the Definition 2.7 reduces to Banach contraction mapping.

The following example shows that a T-contraction mapping need not be a contraction mapping.

Example 2.8: Let $X = [0, \infty)$ be with the usual metric. Let define two mappings $T, f : X \to X$ as

$fx = \beta x, \beta > 1$

$Tx = \frac{\propto}{x^2}, \propto \in R.$

It is clear that, f is not contraction but f is T-contraction, since

$$d(Tfx, Tfy) = \mid \frac{\propto}{\beta^2 x^2} - \frac{\propto}{\beta^2 y^2} \mid = \frac{1}{\beta^2} \mid Tx - Ty \mid .$$

Definition 2.9 **[2]:** Let T be a self mapping of a metric space (X, d). Then

(i) A mapping T is said to be sequentially convergent, if the sequence $\{y_n\}$ in X is convergent whenever $\{Ty_n\}$ is convergent.

(ii) A mapping T is said to be subsequentially convergent, if $\{y_n\}$has a convergent subsequence whenever $\{Ty_n\}$is convergent.

Definition 2.10 **[7]:** Let T be a self mapping of a normed space X. Then T is called a Banach operator of type k if

$$\| T^2x - Tx \| \leqslant k \| Tx - x \|$$

for some $k \geqslant 0$ and for all $x \in X$.

This concept was introduced by subrahmanyam [7], then Chen and Li [8] extended this as following:

Definition 2.11 **[8]:** Let T and f be two self mappings of a non-empty subset M of a normed linear space X . Then (T, f) is a Banach operator pair, if any one of the following conditions is satisfied:

(i) $T[F(f)] \subseteq F(f)$ ie $F(f)$ is T-invariant.

(ii) $fTx = Tx$ for each $x \in F(f)$.

(iii) $fTx = Tfx$ for each $x \in F(f)$.

(iv) $\| Tfx - fx \| \leqslant k \| fx - x \|$for some $k \geqslant 0$.

Lemma 2.12 **[11]:** Let $a, b, c, u \in E$,the real Banach space.

(i) If $a \leqslant b$ and $b \ll c$, then $a \ll c$.

(ii) If $a \ll b$ and $b \ll c$, then $a \ll c$.

(iii) If $0 \leqslant u \ll c$ for each $c \in int\, P$, then $u = 0$.

If $c \in intP,\ 0 \leqslant a_n$ and $a_n \to 0$, then there exists n_o such that for all $n > n_{o,}$ if follows that $a_n \ll c$.

3 Main Results

First, we give definitions of T-Reich contraction mapping and T-Rhoades contraction mapping on cone metric spaces which are based on the ideas of Morales and Rojas [4].

Definition 3.1 Let (X, d)be a cone metric space and $T, S : X \to X$ two functions;

(i): A mapping S is said to be T-Reich contraction, if there is a $a + b + c < 1$ such that

$d(TSx, TSy) \leqslant ad(Tx, TSx) + bd(Ty, TSy) + cd(Tx, Ty)$**:** for all all $x, y \in X$ and $a, b, c \geqslant 0$.

(ii): A mapping S is said to be T-Rhoades contraction if there is $a + b + c < 1$ such that

$d(TSx, TSy) \leqslant ad(Tx, TSy) + bd(Ty, TSx) + cd(Tx, Ty)$**:** for all all $x, y \in X$ and $a, b, c \geqslant 0$.

Theorem 3.2: Let T, f and g be three continuous self mappings of a complete cone metric space (X,d). Assume that T is a injective mapping and P is a normal cone with normal constant. If the mappings T, f and g satisfy

$d(Tfx, Tgy) \leqslant a_1 d(Tx, Tfx) + a_2 d(Ty, Tgy) + a_3 d(Tx, Ty)$**:** (3.2.1)

for: all $x, y \in X$ where $a_i, i = 1, 2, 3$ are all non-negative constants such that $a_1 + a_2 + a_3 < 1$, then f and g have a unique common fixed point in X. Moreover, if $(T, f)$and (T, g) are Banach pairs, then T, f and g have a unique common fixed point in X.

Proof: Let $x_o \in X$ as an arbitrary element and define the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ in X such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for each $n = 0, 1, 2, - - - - - - - - - - - \infty$. Consider $d(Tx_{2n+1}, Tx_{2n}) = d(Tfx_{2n,}Tgx_{2n-1}) \leqslant a_1 d(Tx_{2n}, Tfx_{2n}) + a_2 d(Tx_{2n-1}, Tgx_{2n-1})$

$+a_3 d(Tx_{2n}, Tx_{2n-1})$

$= a_1 d(Tx_{2n}, Tx_{2n+1}) + a_2 d(Tx_{2n-1}, Tx_{2n})$

$+a_3 d(Tx_{2n}, Tx_{2n-1})$

$\leqslant (a_2 + a_3) d(Tx_{2n}, Tx_{2n-1}) + a_1 d(Tx_{2n}, Tx_{2n+1})$

$d(Tx_{2n}, Tx_{2n+1}) \leqslant \frac{a_2+a_3}{1-a_1} d(Tx_{2n,}Tx_{2n-1})$

$\leqslant kd(Tx_{2n}, Tx_{2n-1})$

$k = \frac{a_2+a_3}{1-a_1} < 1 \; as \, a_1 + a_2 + a_3 < 1$

Proceeding further,

$d(Tx_{2n}, Tx_{2n+1}) \leqslant k^{2n} d(Tx_o, Tx_1)$ (3.2.2)

Next, to claim that $(Tx_{2n}\}$ is a Cauchy sequence.

Consider $m, n \in N$ such that $m > n$,

$d(Tx_{2n}, Tx_{2m}) \leqslant d(Tx_{2n,} Tx_{2n+1}) + d(Tx_{2n+1,} Tx_{2n+2})$

$+ - - - - - - - + d(Tx_{2m-1}, Tx_{2m})$

$\leqslant (k^{2n} + k^{2n+1} + - - - - - + k^{2m-1}) d(Tx_1, Tx_o)$

$= \frac{k^{2n}}{1-k} d(Tx_{o,} Tx_1)$.

From (2.1.1), it follows that

$\| d(Tx_{2m}, Tx_{2n}) \| \leqslant \frac{k^{2n}}{1-k} \| d(Tx_{o,} Tx_1) \|$ (3.2.3)

Since $k \in (0, 1), k^{2n} \to 0$ as $n \to \infty$. Therefore $\| d(Tx_{2m}, Tx_{2n}) \| \to 0$ as $m, n \to \infty$. Thus $\{Tx_{2n}\}$ is a Cauchy sequence in X. As X is a complete cone metric space, there exists $z \in X$ such that $lim_{n \to \infty} Tx_{2n} = z$.

Since T is subsequentially convergent $\{x_{2n}\}$, has a convergent subsequence $\{x_{2m}\}$ such that $lim_{m \to \infty} x_{2m} = u$. As T is continuous,

$lim_{m \to \infty} Tx_{2m} = Tu$ (3.2.4)

By the uniqueness of the limit, $z = Tu$. Since f is continuous $lim_{m\to\infty} fx_{2m} = fu$. Again as T is continuous, $lim_{n\to\infty} Tfx_{2m} = Tfu$

$lim_{m\to\infty} Tx_{2m+1} = Tfu$ (3.2.5)

Now consider, $d(Tfu, Tu) \leqslant d(Tfu, Tx_{2m+1}) + d(Tx_{2m+1,}Tu)$

$= d(Tfu, Tfx_{2m}) + d(Tx_{2m+1,}Tu)$

$\leqslant a_1 d(Tu, Tfu) + a_2 d(Tx_{2m,}Tfx_{2m})$

$+a_3 d(Tu, Tx_{2m}) + d(Tx_{2m+1,}Tu)$

$= a_1 d(Tu, Tfu) + a_2 d(Tx_{2m}, Tx_{2m+1})$

$+a_3 d(Tu, Tx_{2m}) + d(Tx_{2m+1}, Tu)$

$(1 - a_1) d(Tfu, Tu) \leqslant a_2 [d(Tx_{2m}, Tu) + d(Tu, Tx_{2m+1})]$

$+a_3 d(Tu, Tx_{2m}) + d(Tx_{2m+1,}Tu)$

$d(Tfu, Tu) \leqslant \frac{a_2+a_3}{1-a_1} d(Tu, Tx_{2m}) + \frac{1+a_2}{1-a_1} d(Tu, Tx_{2m+1})$

$d(Tfu, Tu) \leqslant \frac{a_2+a_3}{1-a_1} d(Tu, Tx_{2m}) + \frac{1+a_2}{1-a_1} d(Tu, Tx_{2m+1})$ (3.2.6)

Let $0 \ll c$ be arbitrary. By (3.2.4)

$d(Tu, Tx_{2m}) \ll \frac{c(1-a_1)}{2(a_2+a_3)}$

Similarly by (3.2.5) it follows that

$d(Tu, Tx_{2m+1}) \ll \frac{c(1-a_1)}{2(1+a_2)}$

Then (3.2.6) becomes

$d(Tu, Tfu) \ll \frac{c}{2} + \frac{c}{2} = c$ for each $c \in intP$. Now using Lemma 2.12(iii), it follows that

$d(Tu, Tfu) = 0$

which implies that $Tu = Tfu$. As T is injective, $u = fu$. Thus u is the fixed point of f. Similarly it can be established that, u is also the fixed point of g. That means u is the common fixed point of f and g.

To prove uniqueness: If w is another common fixed point of f and g, then $fw = w = gw$.

$d(Tu, Tw) = d(Tfu, Tgw)$

$\leqslant a_1 d(Tu, Tfu) + a_2 d(Tw, Tgw)$

$+ a_3 d(Tu, Tw)$

$d(Tu, Tw) \leqslant a_3 d(Tu, Tw)$

$< (a_1 + a_2 + a_3) d(Tu, Tw)$

¡$d(Tu, Tw)$ as $a_1 + a_2 + a_3 < 1$

a contradiction. Hence $d(Tu, Tw) = 0$

which implies $Tu = Tw$. As T is injective, $u = w$, is the unique common fixed point of f and g.

Since we have assumed that (T, f) and (T, g) are Banach pairs; (T, f) and (T, g) commutes at the fixed point of f and g respectively. This implies that $Tfu = fTu$ for $u \in F(f)$. So $Tu = fTu$ which gives that Tu is another fixed point of f . It is true for g,too. By the uniqueness of fixed point of f, $Tu = u$. Hence $u = Tu = fu = gu$, u is the unique commong fixed point of T, f and g in X.

Theorem 3.3: Let T, f and g be three continuous self mappings of a complete cone metric space (X, d). Assume that T is a injective mapping and P is a normal cone with normal constant. If the mappings T, f and g satisfy

$$d(Tfx, Tgy) \leqslant a_1 d(Tx, Tgy) + a_2 d(Ty, Tfx) + a_3 d(Tx, Ty) \quad (3.3.1)$$

for all $x, y \in X$, where $a_i, i = 1, 2, 3$ are all nonnegative constants such that $a_1+a_2+a_3 < 1$,then f and g have a unique common fixed point in X. Moreover, if $(T, f)$and (T, g) are Banach paris, then T, f and g have a unique common fixed point in X.

Proof: The proof is similar to the proof of Theorem 3.2

References

[1] Huang L.G., Zhang X., Cone Metric Spaces and Fixed Point Theorems of Contractive mappings, J.Math.Anal.Appl.332(2007),1468-1476.

[2] Beiranvand A., Moradi S., Omid M. and Pazandeh H., Two Fixed Point Theorems for Special Mappings, arxiv:0903.1504v1[math.FA],(2009).

[3] Morales J., Rojas E., Cone Metric Spaces and Fixed Point Theorems of T-Contractive Mappings, Revista Notas de Matematica, Vol.4(2), No.269(2008),66-78.

[4] Morales J., Rojas E., Cone Metric Spaces and Fixed Point Theorems of T-Kannan Contractive Mappings, arxiv:0907,3949v1[math.FA],(2009).

[5] Morales J., Rojas E., T-Zamfirescu and T-Weak Contraction Mappings on Cone Metric Spaces, arxiv:0909.1255v1[math.FA](2009).

[6] Jankovic S., Kadelburg Z., Radenovic S., On Cone Metric Spaces : A Survey, Nonlinear Analysis, (2010),doi.10.1016/J.na.2010.12.014.

[7] Subrahmanyam P.V., Remarks on Some Fixed Point Theorems Related to Banach's Contraction Principle, J. Math. Phys.Sci.,8(1974)445-457.

[8] Chen J., Li Z., Common Fixed Points For Banach Operator Pairs in Best Approximation, J. Math. Anal.Appl.336(2007),1466-1475.

[9] Sumitra R., Uthariaraj V.R., Hemavathy R., Common Fixed Point Theorem for T-Hardy-Rogers Contraction Mapping in a Cone Metric Space, International Mathematical Forum 5(2010),1495-1506.

[10] Beg I., Azam A., Arshad M., Common Fixed Points For Maps on Topological Vector Space Valued Cone Metric Spaces, International Journal of Mathematics and Mathematical Sciences, (2009),1-8.

[11] Ilic D., Rokocevic V., Common Fixed Points for Maps on Cone Metric Space, J. Math. Anal. Appl., 341(2008),876-882.

[12] Ozturk Mahpeyker, Basarir Metin, On Some Common Fixed Point Theorems for f-Contraction Mappings in Cone Metric Spaces, Int. Journal of Math. Analysis, Vol.5, no.3, (2011),119-127.

[13] Dubey A.K., Narayan A., Cone Metric Spaces and Fixed Point Theorems for Pair of Contractive Maps, Mathematica Aeterna, Vol.2,2012,no.10,839-845.

[14] Dubey A.K., Shukla Rita, Dubey R.P., An Extension of the Paper "Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings", Int. Journal of Applied Mathematical Research, 2(1), (2013), 84-90.

[15] Reich S., Some Remarks concerning Contraction Mappings, Canad. Math. Bull., 14, (1971),121-124.