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ON COMMON FIXED POINTS OF WEAKLY COMPATIBLE MAPPINGS USING GENERALIZED CONDITION (B)' IN PARTIAL METRIC SPACES

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Abstract. The purpose of this paper is to prove the existence of points of coincidence and then we apply the result to obtain common fixed points of two weakly compatible mappings satisfying generalized condition (B)' in partial metric spaces.

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1. Introduction and preliminaries

The concept of weakly compatible of two self mappings was introduced by Jungck [6]. Abass, Babu and Alemayehu [1] proved the existence of common fixed points of two weakly compatible mappings satisfying a generalized condition (B)' in metric spaces. In this paper some theorems in [1] were generalized by using partial metric spaces which was introduced by Mathews in 1994.

Definition 1.1. $[8, 9]$ A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$;

(a) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;

(b) $p(x, x) \leq p(x, y)$;

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(c) $p(x,y) = p(y,x)$;

(d) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

A pair (X,p) is called a partial metric space, where X is a nonempty set and p is a partial metric on X.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open $p-$balls $\{B_p(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

It is easily to show that if p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ defined by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.1}$$

is a usual metric on X.

We recall some definitions and known results.

Definition1.2. [5, 8, 9]

(a) A sequence $\{x_n\}$ in a partial metric space (X,p) is said to be convergent to a point $z \in X$ if and only if $p(z,z) = \lim_{n\to\infty} p(z,x_n)$.

(b) A sequence $\{x_n\}$ in a partial metric space (X,p) is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

(c) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $z \in X$ such that

$$p(z,z) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Lemma 1.3. [5, 8, 9] Let (X,p) be a partial metric space,

a) A sequence $\{x_n\}_{n=0}^{\infty}$ in a partial metric space is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X,p^s).

b) A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete. Moreover $\lim_{n\to\infty} p^s(z,x_n) = 0$ iff $\lim_{n\to\infty} p(z,x_n) = \lim_{n,m\to\infty} p(x_n,x_m) = p(z,z)$.

The following Lemmas will be used in the proof of main result.

Lemma 1.4. [2] Let (X, p) be a partial metric space. Then

(a) If $p(x, y) = 0$, then $x = y$,

(b) If $x \neq y$, then $p(x, y) > 0$.

Definition 1.5. [2] Let f and g be two selfmappings defined on a set X. A point $x \in X$ is said to be coincidence point of f and g if $fx = gx = y$, where $y \in X$ is called a point of coincidence of f and g.

Definition 1.6. [2] Two selfmappings f and g are said to be weakly compatible if f and g commute at their coincidence point, i.e. $fgx = gfx$, $x \in X$, whenever $fx = gx$.

Definition 1.7. [1] Let (X, p) be a partial metric space and f, g be two selfmappings on X such that $f(X) \subset g(X)$. For any $x_0 \in X$ construct a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $gx_{n+1} = fx_n$, $n = 0, 1, 2, ...$ The sequence $\{gx_n\}_{n=1}^{\infty}$ is called an f-sequence with initial point x_0.

In order to prove theorems and corollaries in this paper, generalized condition (B)' is used by taking $\delta, L > 0$, and $\delta + 2L < 1$.

Definition 1.8. [1] A mapping $f : X \to X$ on a partial metric space X is said to satisfy generalized condition (B)' associated with a mapping $g : X \to X$ if there exists $\delta, L > 0$, and $\delta + 2L < 1$ such that

$$p\ (fx, fy) \leq \delta M(x, y) + L \min\{p(gx, fx), p(gy, fy), p(gx, fy), p(gy, fx)\} \tag{1.2}$$

for every $x, y \in X$, where,

$$M(x, y) = \max\{p(gx, gy), p(gx, fx), p(gy, fy), \frac{1}{2}[p(gx, fy) + p(gy, fx)]\}.$$

Lemma 1.9. [2] Let f and g be two selfmappings on a nonempty set X, which have a unique point of coincidence y in X. If f and g are weakly compatible, then y is the unique common fixed point of f and g.

2. Main results

The proof of the following theorem has been taken from [1]

Theorem 2.1. *Let (X, p) be a partial metric space, and f, g be two selfmappings on X such that $f(X) \subseteq g(X)$. Assume that f satisfied generalized condition (B)' associated with g. If either $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence. Furthermore if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

Proof. Pick $x_0 \in X$. Let $\{gx_n\}$ be an $f-$sequence with initial point x_0. Now compute the following:

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{p(gx_n, gx_{n-1}), p(gx_n, fx_n), p(gx_{n-1}, fx_{n-1}), \\ &\quad \frac{1}{2}[p(gx_n, fx_{n-1}) + p(gx_{n-1}, fx_n)]\} \\ &= \max\{p(gx_n, gx_{n-1}), p(gx_n, gx_{n+1}), p(gx_{n-1}, gx_n), \\ &\quad \frac{1}{2}[p(gx_n, gx_n) + p(gx_{n-1}, gx_{n+1})]\} \\ &\leq \max\{p(gx_n, gx_{n-1}), p(gx_n, gx_{n+1}), \\ &\quad \frac{1}{2}[p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1})]\} \\ &\leq \max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\}. \end{aligned}$$

The second part of right-hand side of (1.2) is computed as follows:

$$\begin{aligned} &\min\{p(gx_n, fx_n), p(gx_{n-1}, fx_{n-1}), p(gx_n, fx_{n-1}), p(gx_{n-1}, fx_n)\} \\ &= \min\{p(gx_n, gx_{n+1}), p(gx_{n-1}, gx_n), p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\} \\ &= \min\{p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\} \end{aligned} \tag{2.1}$$

Now apply generalzied condition (B)' for x_n and x_{n-1}, it follows that

$$\begin{aligned} p(fx_n, fx_{n-1}) &\leq \delta \max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\} + \\ &\quad L \min\{p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\}. \end{aligned}$$

This implies that

$$p(gx_{n+1}, gx_n) \leq \delta \max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\} + L \min\{p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\}.$$

We have four cases,

(a) If $\max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\} = p(gx_{n-1}, gx_n)$, and $\min\{p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\} = p(gx_n, gx_n)$ then

$$\begin{aligned} p(gx_n, gx_{n+1}) &\leq \delta p(gx_{n-1}, gx_n) + Lp(gx_n, gx_n) \\ &\leq \delta p(gx_{n-1}, gx_n) + Lp(gx_n, gx_{n+1}) \\ &= \frac{\delta}{1-L} \ \ p(gx_{n-1}, gx_n) \\ &\leq (\frac{\delta}{1-L})^2 \ p(gx_{n-2}, gx_{n-1}) \\ &\leq \ldots\ldots\ldots\ldots \\ &\leq \ \ (\frac{\delta}{1-L})^n p(gx_0, gx_1) \\ &= \delta_1^n \ p(gx_0, gx_1). \end{aligned} \tag{2.2}$$

Where $\delta_1 = \frac{\delta}{1-L} < 1$.

b) If $\max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\} = p(gx_{n-1}, gx_n)$, and $\min\{p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\} = p(gx_{n-1}, gx_{n+1})$, then

$$\begin{aligned} p(gx_n, gx_{n+1}) &\leq \delta p(gx_{n-1}, gx_n) + Lp(gx_{n-1}, gx_{n+1}) \\ &\leq \delta p(gx_{n-1}, gx_n) + Lp(gx_{n-1}, gx_n) + Lp(gx_n, gx_{n+1}) - Lp(gx_n, gx_n) \\ &\leq \delta p(gx_{n-1}, gx_n) + Lp(gx_{n-1}, gx_n) + Lp(gx_n, gx_{n+1}) \\ &= \ \frac{\delta + L}{1-L} p(gx_{n-1}, gx_n) \\ &\leq (\frac{\delta + L}{1-L})^2 p(gx_{n-2}, gx_{n-1}) \\ &\leq \ldots\ldots\ldots\ldots \\ &\leq (\frac{\delta + L}{1-L})^n p(gx_0, gx_1) \\ &= \delta_2^n \ p(gx_0, gx_1) \end{aligned} \tag{2.3}$$

where $\delta_2 = \frac{\delta+L}{1-L} < 1$.

c) If $\max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\} = p(gx_n, gx_{n+1})$, and $\min\{p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\} = p(gx_n, gx_n)$, then

$$\begin{aligned} p(gx_n, gx_{n+1}) &\leq \delta p(gx_n, gx_{n+1}) + Lp(gx_n, gx_n) \\ &\leq \delta p(gx_n, gx_{n+1}) + Lp(gx_{n-1}, gx_n) \\ &= \frac{L}{1-\delta} p(gx_{n-1}, gx_n) \\ &\leq (\frac{L}{1-\delta})^2 p(gx_{n-2}, gx_{n-1}) \\ &\leq \ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq \ (\frac{L}{1-\delta})^n p(gx_0, gx_1) \\ &= \delta_3^n \ p(gx_0, gx_1) \end{aligned} \tag{2.4}$$

where $\delta_3 = \frac{L}{1-\delta} < 1$.

d) If $\max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\} = p(gx_n, gx_{n+1})$, and $\min\{p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\} = p(gx_{n-1}, gx_{n+1})$, then,

$$\begin{aligned} p(gx_n, gx_{n+1}) &\leq \delta p(gx_n, gx_{n+1}) + Lp(gx_{n-1}, gx_{n+1}) \\ &\leq \delta p(gx_n, gx_{n+1}) + Lp(gx_{n-1}, gx_n) + Lp(gx_n, gx_{n+1}) \\ &- Lp(gx_n, gx_n) \\ &\leq \delta p(gx_n, gx_{n+1}) + Lp(gx_{n-1}, gx_n) + Lp(gx_n, gx_{n+1}) \\ &= \frac{L}{1-(\delta+L)} \ p(gx_{n-1}, gx_n) \\ &\leq (\frac{L}{1-(\delta+L)})^2 \ p(gx_{n-2}, gx_{n-1}) \\ &\leq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq \ (\frac{L}{1-(\delta+L)})^n \ p(gx_0, gx_1) \\ &= \delta_4^n \ p(gx_0, gx_1) \end{aligned} \tag{2.5}$$

where $\delta_4 = \frac{L}{1-(\delta+L)} < 1$.

Put $\delta = \max\{\delta_1, \delta_2, \delta_3, \delta_4\}$. Then $0 < \delta < 1$.From (2.2), (2.3), (2.4) and (2.5), we have

$$p(gx_n, gx_{n+1}) \leq \delta^n \, p(gx_0, gx_1). \tag{2.6}$$

For any positive integers m and n with $m > n$, we have from (2.6),

$$\begin{aligned} p(gx_m, gx_n) &\leq p(gx_n, gx_{n+1}) + p(gx_{n+1}, gx_{n+2}) + ... + p(gx_{m-1}, gx_m) \\ &\quad - [p(gx_{n+1}, gx_{n+1}) + p(gx_{n+2}, gx_{n+2}) + ... + p(gx_{m-1}, gx_{m-1})] \\ &\leq p(gx_n, gx_{n+1}) + p(gx_{n+1}, gx_{n+2}) + ... + p(gx_{m-1}, gx_m) \\ &\leq [\delta^n + \delta^{n+1} + ... + \delta^{m-1}] \, p(gx_0, gx_1) \\ &< \frac{\delta^n}{1-\delta} \, p(gx_0, gx_1). \end{aligned} \tag{2.7}$$

This implies that

$$\lim_{n,m\to\infty} p(gx_m, gx_n) = 0. \tag{2.8}$$

By using (1.1), we get that,

$$\begin{aligned} p^s(gx_m, gx_n) &= 2p(gx_m, gx_n) - p(gx_m, gx_m) - p(gx_n, gx_n) \\ &\leq 2p(gx_m, gx_n). \end{aligned} \tag{2.9}$$

By using (2.8), we have $\lim_{n,m\to\infty} p^s(gx_m, gx_n) = 0$.This implies that $\{gx_n\}$ is a Cauchy sequence in $(g(X), p^s)$. Now if $(g(X), p)$ is complete, then by Lemma 1.3 $(g(X), p^s)$ is complete and so the sequence $\{gx_n\}$ converges to $z \in g(X)$. Hence we can find u in X such that $gu = z$. Again by Lemma 1.3, we have

$$p(z, z) = \lim_{n\to\infty} p(gx_n, z) = \lim_{n,m\to\infty} p(gx_m, gx_n). \tag{2.10}$$

From (2.8) and (2.10), we have $p(z,z)=0$. Now

$$\begin{aligned}
p(z,fu) &\leq p(z,gx_{n+1})+p(gx_{n+1},fu)-p(gx_{n+1},gx_{n+1})\\
&\leq p(z,gx_{n+1})+p(fx_n,fu)\\
&\leq p(z,gx_{n+1})+\delta\max\{p(gx_n,gu),p(gx_n,fx_n),p(gu,fu),\\
&\quad \frac{1}{2}[p(gx_n,fu)+p(gu,fx_n)]\}+\\
&\quad +L\min\{p(gx_n,fx_n),p(gu,fu),p(gx_n,fu),p(gu,fx_n)\}\\
&= p(z,gx_{n+1})+\delta\max\{p(gx_n,z),p(gx_n,gx_{n+1}),p(z,fu),\\
&\quad \frac{1}{2}[p(gx_n,fu)+p(z,gx_{n+1})\}+\\
&\quad +L\min\{p(gx_n,gx_{n+1}),p(z,fu),p(gx_n,fu),p(z,gx_{n+1})\}\\
&\leq p(z,gx_{n+1})+\delta\max\{p(gx_n,z),p(gx_n,z)+p(z,gx_{n+1})-p(z,z),p(z,fu),\\
&\quad \frac{1}{2}[p(gx_n,z)+p(z,fu)-p(z,z)+p(z,gx_{n+1})]\}+\\
&\qquad +L\min\{p(gx_n,z)+p(z,gx_{n+1})-p(z,z)\},\\
&\quad p(z,fu),p(gx_n,z)+p(z,fu)-p(z,z),p(z,gx_{n+1})\}\\
&\leq p(z,gx_{n+1})+\delta\max\{p(gx_n,z)+p(z,gx_{n+1}),p(z,fu),\\
&\quad \frac{1}{2}[p(gx_n,z)+p(z,fu)+p(z,gx_{n+1})]\}+\\
&\quad +L\min\{p(gx_n,z)+p(z,gx_{n+1}),p(z,fu),p(gx_n,z)+\\
&\quad +p(z,fu),p(z,gx_{n+1})\}
\end{aligned}$$

Letting $n\to\infty$, by (2.8) and (2.10) we have

$$\begin{aligned}
p(z,fu) &\leq \delta p(z,fu)+L\,p(z,fu),\\
&\leq (\delta+L)p(z,fu),\\
&< p(z,fu).
\end{aligned}$$

It follows that $p(z,fu)=0$. Hence By Lemma 1.4, $fu=gu=z$, i.e z is a point of coincidence of f and g.

On the other hand, if $(f(X), p)$ is complete, then by Lemma 1.3 $(f(X), p^s)$ is complete and thus the sequence $\{gx_{n+1}\} = \{fx_n\}$ converges to $w \in f(X)$. Hence we can find v in X such that $fv = w$. By Lemma 1.3,

$$p(w, w) = \lim_{n\to\infty} p(fx_n, z) = \lim_{n,m\to\infty} p(fx_m, fx_n). \tag{2.11}$$

Since $p(fx_m, fx_n) = p(gx_{m+1}, gx_{n+1})$ from (2.8), we have

$$\lim_{n,m\to\infty} p(fx_m, fx_n) = 0.$$

This implies that $p(w, w) = 0$.

Using the same above arguments, we get that w is a point of coincidence.

Finally, we prove the uniqueness of point of coincidence. Assume that there are $z_1$and z_2 in X such that $z_1 = fu_1 = gu_1$, and $z_2 = fu_2 = gu_2$ for some u_1, u_2 in X.

$$\begin{aligned} M(u_1, u_2) &= \max\{p(gu_1, gu_2), p(gu_1, fu_1), p(gu_2, fu_2), \frac{1}{2}[p(gu_1, fu_2) + p(gu_2, fu_1)]\} \\ &= \max\{p(z_1, z_2), p(z_1, z_1), p(z_2, z_2), \frac{1}{2}[p(z_1, z_2) + p(z_2, z_1)]\} \\ &= p(z_1, z_2). \end{aligned}$$

It is obvious that $\min\{p(gu_1, fu_1), p(gu_2, fu_2), p(gu_{1,}fu_2), p(gu_2, fu_1)\} = 0$. Using (1.2),

$$\begin{aligned} p(z_1, z_2) &= p(fu_1, fu_2) \\ &\le \delta p(z_1, z_2) \\ &< p(z_1, z_2). \end{aligned}$$

This implies that $p(z_1, z_2) = 0$. By Lemma 1.4, we have $z_1 = z_2$. Since the point of coincidence is unique, by Lemma 1.9 z is the unique common fixed point of f and g.

The next corollary was taken from [1].

Corollary 2.2. *Let (X, p) be a partial metric space and f, g be selfmappings on X such that $f(X) \subseteq g(X)$. Assume that there exists $\delta, L > 0$ and $\delta + 2L < 1$ such that*

$$p(fx, fy) \le \delta M(x, y) + L \min\{p(gx, fx), p(gy, fy), p(gx, fy), p(gy, fx)\}, \tag{2.12}$$

for all $x, y \in X,$ *where*

$$M(x,y) = \max\{p(gx,gy), \frac{1}{2}[p(gx,fx)+p(gy,fy)], \frac{1}{2}[p(gy,fx)+p(gx,fy)]\}.$$

If either $f(X)$ *or* $g(X)$ *is a complete subspace of* X, *then* f, g *have a point of coincidence. Furthermore if* f *and* g *are weakly compatible, then* f, g *have a unique common fixed point.*

Proof. Pick $x_0 \in X$. Let $\{gx_n\}$ be an $f-$sequence with initial point x_0. Now compute the following:

$$\begin{aligned}
M(x_n, x_{n-1}) &= \max\{p(gx_n, gx_{n-1}), \frac{1}{2}[p(gx_n, fx_n) + p(gx_{n-1}, fx_{n-1})], \\
&\quad \frac{1}{2}[p(gx_n, fx_{n-1}) + p(gx_{n-1}, fx_n)]\} \\
&= \max\{p(gx_n, gx_{n-1}), \frac{1}{2}[p(gx_n, gx_{n+1}) + p(gx_{n-1}, gx_n)], \\
&\quad \frac{1}{2}[p(gx_n, gx_n) + p(gx_{n-1}, gx_{n+1})]\} \\
&\leq \max\{p(gx_n, gx_{n-1}), \frac{1}{2}[p(gx_n, gx_{n+1}) + p(gx_{n-1}, gx_n)], \\
&\quad \frac{1}{2}[p(gx_n, gx_n) + p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1}) - (gx_n, gx_n)]\}, \\
&\leq \max\{p(gx_n, gx_{n-1}), \frac{1}{2}[p(gx_n, gx_{n+1}) + p(gx_{n-1}, gx_n)]\}.
\end{aligned}$$

From (2.1) and (2.12), we get

$$\begin{aligned}
p(fx_n, fx_{n-1}) &\leq \delta \max\{p(gx_{n-1}, gx_n), \frac{1}{2}[p(gx_n, gx_{n+1}) + p(gx_{n-1}, gx_n)]\} \\
&\quad + L \min\{p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\}.
\end{aligned}$$

Now we consider four cases. The first and second ones are similar to a) and b) in Theorem 2.1. For the third and fourth cases suppose that

$$M(x_n, x_{n-1}) = \frac{1}{2}[p(gx_n, gx_{n+1}) + p(gx_{n-1}, gx_n)].$$

Then we have the following:

If $\min\{p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\} = p(gx_n, gx_n)$, then,

$$\begin{aligned} p(gx_n, gx_{n+1}) &\leq \frac{\delta}{2}[p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1})] + Lp(gx_n, gx_n) \\ & \frac{\delta}{2}[p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1})] + Lp(gx_{n-1}, gx_n) \\ &\leq \frac{\delta + 2L}{2-\delta}\, p(gx_{n-1}, gx_n) \leq ... \leq (\frac{\delta + 2L}{2-\delta})^n\, p(gx_0, gx_1) \\ &= k_1^n\, p(gx_0, gx_1), \end{aligned} \tag{2.13}$$

where $k_1 = \frac{\delta+2L}{2-\delta}$

If $\min\{p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\} = p(gx_{n-1}, gx_{n+1})$, then

$$\begin{aligned} p(gx_n, gx_{n+1}) &\leq \frac{\delta}{2}[p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1})] + Lp(gx_{n-1}, gx_{n+1}) \\ &\leq \frac{\delta}{2}[p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1})] + \\ & L[p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1}) - p(gx_n, gx_n)] \\ &\leq \delta[p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1})] + L[p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1})] \\ &\leq \frac{\delta + 2L}{2-\delta-2L}\, p(gx_{n-1}, gx_n) \leq ... \leq (\frac{\delta + 2L}{2-\delta-2L})^n\, p(gx_0, gx_1) \\ &= k_2^n\, p(gx_0, gx_1), \end{aligned} \tag{2.14}$$

where $k_2 = \frac{\delta+2L}{2-\delta-2L}$ Put $k = \max\{k_1, k_2\}$. Then $k \in (0, 1)$. From (2.13) and (2.14) we get that

$$p(gx_n, gx_{n+1}) \leq k^n\, p(gx_0, gx_1).$$

To complete the proof follow the the same arguments of the proof of Theorem 2.1.

We can get the following corollary as in [1] by letting $g = I_X$ in the previous theorem.

Corollary 2.3. *Let (X, p) be a partial metric space,and $f : X \to X$. Assume that f satisfies generalized condition (B)'. If $f(X)$ is a complete subspace of X, then f has a unique fixed point.*

Theorem 2.4. *Let (X, p) be a partial metric space, and f, g be two selfmappings on X such that $f(X) \subseteq g(X)$. Assume that there exists a constant $\delta, L > 0$, and $\delta + 2L < 1$*

such that

$$p(fx, fy) \leq \delta m(x, y) \ + L \min\{p(gx, fx), p(gy, fy), p(gx, fy), p(gy, fx)\} \ , \qquad (2.15)$$

for every $x, y \in X$, *where*

$$M(x, y) = \max\{p(gx, gy), p(gx, fx), p(gy, fy), p(gx, fy), p(gy, fx)\}.$$

If either $f(X)$ *or* $g(X)$ *is a complete subspace of* X, *then* f *and* g *have a unique point of coincidence. Furthermore, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Let $x_0 \in X$, and $\{gx_n\}$ be an $f-$sequence with initial point x_0. Now we find $M(x, y)$

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{p(gx_n, gx_{n-1}), p(gx_n, fx_n), p(gx_{n-1}, fx_{n-1}), \\ &p(gx_n, fx_{n-1}), p(gx_{n-1}, fx_n)\} \\ &= \max\{p(gx_n, gx_{n-1}), p(gx_n, gx_{n+1}), p(gx_{n-1}, gx_n), \\ &p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\} \\ &= \max\{p(gx_n, gx_{n-1}), p(gx_n, gx_{n+1}), p(gx_{n-1}, gx_{n+1})\} \\ &\leq \max\{p(gx_n, gx_{n-1}), p(gx_n, gx_{n+1}), p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1})\} \\ &= p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1}). \qquad (2.16) \end{aligned}$$

Using (2.1), (2.15) and (2.16), we get

$$p(gx_n, gx_{n+1}) \leq \delta[p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1})] + L \min\{p(gx_n, gx_n), p(gx_{n-1}, gx_{n+1})\}.$$

To complete the proof of the theorem follow the same argument of the proof of the previous corollary.

We need the following definition to prove the next theorem.

Definition 2.5. Let (X, p) be a partial metric space. A mapping $f : X \to X$ is said to be continuous at $z \in X$ if for every sequence $\{x_n\}$ in X converges to z, then the sequence

$\{fx_n\}$ converges to fz, i.e

$$\lim_{n\to\infty} p(x_n, z) = p(z, z) \implies \lim_{n\to\infty} p(fx_n, fz) = p(fz, fz).$$

Theorem 2.6. *[1] Let (X, p) be a partial metric space,and f a selfmapping on X satisfy generalized condition (B)'associated with a selfmapping g on X. If the set $F(f, g) = \{z \in X : fz = gz = z, p(z, z) = 0\}$ of all common fixed points of f and g are nonempty , then f is continuous at $z \in F(f, g)$ whenever g is continuous at z.*

Proof. Let $z \in F(f, g)$ and $\{x_n\}$ be a sequence in X converges to z. Applying (1.2) for z and x_n, then

$$p(fz, fx_n) \leq \delta M(z, x_n) + L \min\{p(gz, fz), p(gx_n, fx_n), p(gz, fx_n), p(gx_n, fz)\}$$

where

$$M(z, x_n) = \max\{p(gz, gx_n), p(gz, fz), p(gx_n, fx_n), \frac{1}{2}[p(gz, fx_n) + p(gx_n, fz)\}$$

It follows that

$$\begin{aligned}
p(fz, fx_n) &\leq \delta \max\{p(gz, gx_n), p(z, z), p(gx_n, fx_n), \frac{1}{2}[p(fz, fx_n) + p(gx_n, gz)\} + \\
&\quad +L \min\{p(z, z), p(gx_n, fx_n), p(fz, fx_n), p(gx_n, gz)\} \\
&\leq \delta \max\{p(gz, gx_n), p(gx_n, gz) + p(fz, fx_n) - p(z, z), \frac{1}{2}[p(fz, fx_n) + p(gx_n, gz)\} \\
&\leq \delta \max\{p(gz, gx_n), p(gx_n, gz) + p(fz, fx_n), \frac{1}{2}[p(fz, fx_n) + p(gx_n, gz)\} \\
&= \delta[p(gx_n, gz) + p(fz, fx_n)]
\end{aligned}$$

It follows that

$$p(fz, fx_n) \leq \frac{\delta}{1-\delta}\, p(gx_n, gz) \tag{2.17}$$

Since g is continuous, $\lim\limits_{n\to\infty} p(gx_n, gz) = p(z, z) = 0$. By using (2.17), and the continuity of g we conclude that $\lim\limits_{n\to\infty} p(fx_n, fz) = p(z, z) = 0$.

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