Available online at http://scik.org
Adv. Fixed Point Theory, 2026, 16:13
https://doi.org/10.28919/afpt/9598
ISSN: 1927-6303

ON THE CONVERGENCE AND EFFICIENCY OF A NEW CONTRACTION PRINCIPLE IN FUZZY METRIC SPACES

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Abstract. In this paper, we introduce a new contraction principle that guarantees the existence and uniqueness of a fixed point in a complete fuzzy metric space. The efficiency of the proposed contraction is examined through the estimation of both prior and posterior errors, and its rate of convergence is illustrated with numerical examples.

Keywords: fuzzy contractive mapping; prior error; posterior error; rate of convergence; complete fuzzy metric space; fuzzy fixed point.

2020 AMS Subject Classification: 47H10, 54A40, 54H25, 03E72.

1. Introduction

Fixed point theory is a fundamental tool in mathematical analysis with broad applications across scientific and engineering disciplines. When uncertainty and imprecision arise, classical approaches become insufficient, leading to the development of fuzzy fixed point theory as a more suitable analytical framework.

The formal development of fuzzy mathematics began with Zadeh's introduction of fuzzy set theory in 1965 [11]. Subsequently, Kramosil and Michalek [6] extended the notion of distance by proposing fuzzy metric spaces. This concept was later refined by George and Veeramani [1],

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Received September 13, 2025

whose formulation ensured compatibility with classical topological structures and has since become the standard model for fuzzy metric spaces.

A significant advancement in this area was made by Grabiec [3], who extended the Banach contraction principle to fuzzy metric spaces. Since then, considerable attention has been devoted to the development of new types of contraction mappings and admissibility conditions, together with studies on their convergence properties. Various generalizations have been proposed in different settings, such as the uniform structure of metric spaces [4], M-complete non-Archimedean metric spaces [7], generalized fuzzy contractions [9], iterated contractions [10], and contractions involving strictly increasing functions [5]. These approaches aim to broaden the class of mappings for which fixed point results can be established. More recently, Gopal and Vetro [2] introduced the notion of α and β admissible self-mappings in fuzzy metric spaces, thereby providing greater flexibility and extending the scope of contraction mappings in fixed point theory. These contributions have significantly enhanced the development and applicability of fuzzy fixed point methods.

Despite the substantial literature in this area, comparatively limited attention has been devoted to the convergence analysis and error estimation of contractive mappings. This observation motivates the present study, whose objective is to establish fixed point results for the proposed contraction and to investigate its effectiveness through both theoretical and numerical convergence analysis and error estimation. The necessary definitions and preliminary results are presented in the following section.

2. PRELIMINARIES

For the remainder of this paper, the notation $\mathbb{N}$ represents the set of positive integers, $\mathbb{N}_0$ denotes the set of non-negative integers, $\mathbb{R}^+ = (0,\infty)$, and $I = [0,1]$, unless stated otherwise.

Definition 1. [8] *A binary operation $* : I \times I \longrightarrow I$ is said to be continuous t-norm (triangular norm), if $(I,*)$ is an abelian topological monoid. That is, $*$ holds the following conditions;*

(t-1) *$*$ is associative. That is, $(a*b)*c = a*(b*c)$ for all $a,b,c \in I$.*

(t-2) *$*$ is commutative. That is, $a*b = b*a$ for all $a,b \in I$.*

(t-3) *For any $a \in I$, $a*1 = a$.*

(t-4) *For any* $a,b,c,d \in I$ *with* $a \leq b$ *and* $c \leq d$, $a*c \leq b*d$.

(t-5) $*$ *is continuous.*

For example, $a*b = min\{a,b\}$, $a*b = ab$ and $a*b = max\{a+b-1,0\}$ are some continuous t-norms.

In this paper fuzzy metric space refers to the GV-Fuzzy Metric Space.

Definition 2. [1] *The 3-tuple* $(X,M,*)$ *is said to be a fuzzy metric space (referred to as GV-Fuzzy Metric Space), if* X *is an arbitrary non-empty set,* $*$ *is a continuous* t*-norm and* M *is a fuzzy set defined on* $X^2 \times \mathbb{R}^+$ *satisfying the following conditions;*

(GV-1) $M(x,y,t) > 0$ *for all* $x,y \in X$.

(GV-2) *For any* $t \in \mathbb{R}^+$, $M(x,y,t) = 1$ *if and only if* $x = y$.

(GV-3) $M(x,y,t) = M(y,x,t)$ *for any* $t \in \mathbb{R}^+$ *and* $x,y \in X$.

(GV-4) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$ *for all* $x,y,z \in X$ *and* $t,s \in \mathbb{R}^+$.

(GV-5) $M(x,y,\cdot) : \mathbb{R}^+ \longrightarrow I$ *is continuous.*

Lemma 1. [3] *For all* $x,y \in X$, $M(x,y,\cdot)$ *is non-decreasing.*

Remarks: 1. [1] *For any* $r \in (0,1)$, *whenever* $M(x,y,t) > 1-r$ *for* $x,y \in X$ *and* $t \in \mathbb{R}^+$, *there exists* $t_0 \in (0,t)$ *such that* $M(x,y,t_0) > 1-r$.

2. *For any* $r_1 > r_2$, *there exists* r_3 *such that* $r_1 * r_3 > r_2$, *and for any* r_4, *we can find a* r_5 *such that* $r_5 * r_5 > r_4$, *where* $r_1,r_2,r_3,r_4,r_5 \in (0,1)$.

Definition 3. *Let* $(X,M,*)$ *be a fuzzy metric space.*

(i) *For every* $t \in \mathbb{R}^+$, $B(x,r,t) = \{y \in X : M(x,y,t) > 1-r\}$ *is an open ball centered at* $x \in X$ *with radius* $r \in (0,1)$.

(ii) *A sequence* $\{x_n\}$ *converges to* $x \in X$, *if for every* $\delta \in (0,1)$ *and* $t > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $M(x_n,x,t) > 1-\delta$ *for all* $n \geq n_0$. *Moreover* $\lim_{n\to\infty} M(x_n,x,t) = 1$.

(iii) *A sequence* $\{x_n\}$ *is said to be a Cauchy, if for every* $\varepsilon > 0$ *and* $t \in \mathbb{R}^+$, *there exists* $n_0 \in \mathbb{N}$ *such that* $M(x_m,x_n,t) > 1-\varepsilon$ *for all* $m > n \geq n_0$.

(iv) *A fuzzy metric space in which every Cauchy sequence converges is called a complete fuzzy metric space.*

(v) *A sequence* $\{t_n\}$ *is said to be s-increasing, if there exists* $n_0 \in \mathbb{N}$ *such that* $t_n + 1 \leq t_{n+1}$ *for all* $n \geq n_0$.

Definition 4. [2] *Let* $(X, M, *)$ *be a fuzzy metric space. A mapping* $T : X \to X$ *is said to be an* α*-admissible, if there exists a function* $\alpha : X^2 \times \mathbb{R}^+ \to [0, \infty)$ *such that* $\alpha(Tx, Ty, t) \geq 1$, *whenever* $\alpha(x, y, t) \geq 1$ *for all* $x, y \in X$ *and* $t > 0$.

Definition 5. [2] *Let* $(X, M, *)$ *be a fuzzy metric space. A mapping* $T : X \to X$ *is said to be a* β*-admissible, if there exists a function* $\beta : X^2 \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\beta(Tx, Ty, t) \leq 1$, *whenever* $\beta(x, y, t) \leq 1$ *for all* $x, y \in X$ *and* $t > 0$.

3. MAIN RESULT

Motivated by the approach of Gregori and Sapena [4], we formulate and prove the following lemma, which plays a crucial role in establishing our main results.

Lemma 2. *Let* $(X, M, *)$ *be a complete fuzzy metric space, and* $\alpha : X^2 \times \mathbb{R}^+ \to [0, \infty)$ *be a function defined by* $\alpha(x, y, t) = 1$ *if* $x = y$, *otherwise* $\alpha(x, y, t) \geq 1$ *for all* $t > 0$. *Now, let* $\beta : X^2 \times \mathbb{R}^+ \to \mathbb{R}^+$ *be a function defined as* $\beta(x, y, t) = 1$ *if* $x = y$, *otherwise* $\beta(x, y, t) \leq 1$ *for all* $t > 0$. *For every* $a_k, b_k, c_k \in [0, \infty)$ *with* $a_k + b_k + c_k \leq 1$ *and* $a_k + b_k \leq c_k$, *if* $\left\{a_k\big(\alpha(x_{n_k}, y_{n_k}, t) + \beta(x_{n_k}, y_{n_k}, t)\big) + b_k\big(\alpha(x_{n_k}, y_{n_{k+1}}, t) + \beta(x_{n_k}, y_{n_{k+1}}, t)\big) + c_k\right\}$ *is an s-increasing sequence, then for every* $\varepsilon > 0$ *and* $t > 0$, *there exists* $N \in \mathbb{N}$ *such that*

$$\prod_{k=1}^{m} \left[a_k\big(\alpha(x_{n_k}, y_{n_k}, t) + \beta(x_{n_k}, y_{n_k}, t)\big) + b_k\big(\alpha(x_{n_k}, y_{n_{k+1}}, t) + \beta(x_{n_k}, y_{n_{k+1}}, t)\big) + c_k\right] > 1 - \varepsilon,$$

for all $n \geq N$. *Moreover*

$$\lim_{m \to \infty} \prod_{k=1}^{m} \left[a_k\big(\alpha(x_{n_k}, y_{n_k}, t) + \beta(x_{n_k}, y_{n_k}, t)\big) + b_k\big(\alpha(x_{n_k}, y_{n_{k+1}}, t) + \beta(x_{n_k}, y_{n_{k+1}}, t)\big) + c_k\right] = 1.$$

Proof: Since $(X, M, *)$ is a complete fuzzy metric space, for some $x, y \in X$ let $\{x_n\}, \{y_n\} \subset X$ be any two sequences such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

For any $t > 0$ and $k < m \in \mathbb{N}$, let

$$S_n(k) = a_k\big(\alpha(x_{n_k}, y_{n_k}, t) + \beta(x_{n_k}, y_{n_k}, t)\big) + b_k\big(\alpha(x_{n_k}, y_{n_{k+1}}, t) + \beta(x_{n_k}, y_{n_{k+1}}, t)\big) + c_k \tag{1}$$

and

$$P_n(m) = \prod_{k=1}^{m} \left[a_k\big(\alpha(x_{n_k}, y_{n_k}, t) + \beta(x_{n_k}, y_{n_k}, t)\big) + b_k\big(\alpha(x_{n_k}, y_{n_{k+1}}, t) + \beta(x_{n_k}, y_{n_{k+1}}, t)\big) + c_k\right]$$

$$P_n(m) = \prod_{k=1}^{m} S_n(k). \tag{2}$$

For any $x_{n_k}, y_{n_k} \in X$ with $x_{n_k} \neq y_{n_k}$, we have $\alpha(x_{n_k}, y_{n_k}, t) \geq 1$, $\beta(x_{n_k}, y_{n_k}, t) \leq 1$. There exists $a_k, b_k, c_k \in [0, \infty)$ and $N \in \mathbb{N}$ such that

$$a_k + b_k + c_k \leq S_n(k) \leq 1 \tag{3}$$

for all $n > N$, where $a_k + b_k + c_k \leq 1$ and $a_k + b_k \leq c_k$.

Now, from equation (2), we get

$$0 < P_n(m) = \prod_{k=1}^{m} S_n(k) \leq 1. \tag{4}$$

Since $S_n(k)$ is an s-increasing sequence and from equation (3), it is bounded above by 1, therefore $S_n(k)$ converges to a limit L_k. That is,

$$\lim_{n\to\infty} S_n(k) = L_k \leq 1. \tag{5}$$

For $c_k \leq \frac{1}{3}$, from equations (1) and (5), as $n \to \infty$,

$$S_n(k) \to a_k(1+1) + b_k(1+1) + c_k = 2(a_k + b_k) + c_k \leq 3c_k \leq 1,$$

$$\Longrightarrow \lim_{n\to\infty} S_n(k) = L_k = 1.$$

Now, from equation (2),

$$\begin{aligned} \log P_n(m) &= \log\left[\prod_{k=1}^{m} S_n(k)\right] \\ &= \sum_{k=1}^{m} \log S_n(k) \end{aligned} \tag{6}$$

For every k, $S_n(k) \to 1$ as $n \to \infty$.

$$\Longrightarrow \log S_n(k) \to 0$$

$$\Longrightarrow \sum_{k=1}^{m} \log S_n(k) \to 0$$

$$\Longrightarrow \log P_n(m) \to 0$$

$$\Longrightarrow P_n(m) \to 1.$$

$$\text{That is,} \quad \lim_{n\to\infty} P_n(m) = 1.$$

Hence for every $\varepsilon > 0$ and $t > 0$, there exists $N \in \mathbb{N}$ such that $P_n(m) < 1-\varepsilon$ for all $n \geq N$.

It is clear that, for any $k < m \in \mathbb{N}$, we have $n < n+k < n+m$, this implies that $n+m \to \infty$ as $n \to \infty$. Therefore, $\lim_{m,\, n\to\infty} P_n(m) = 1$.

$$\Longrightarrow \lim_{m\to\infty} \prod_{k=1}^{m} \left[a_k\big(\alpha(x_{n_k}, y_{n_k}, t) + \beta(x_{n_k}, y_{n_k}, t)\big) + b_k\big(\alpha(x_{n_k}, y_{n_{k+1}}, t) + \beta(x_{n_k}, y_{n_{k+1}}, t)\big) + c_k\right] = 1.$$

Now we can state the main result of the paper.

Theorem 1. *Let $(X, M, *)$ be a complete fuzzy metric space with $a * b = ab$ for all $a, b \in I$ and $T : X \longrightarrow X$ be α and β admissible mapping such that*

$$\begin{aligned} M(Tx, Ty, t) \geq & a(\alpha(x,y,t) + \beta(x,y,t))M(x,y,t) \\ & + b(\alpha(x,Ty,t) + \beta(x,Ty,t))M(x,Ty,t) \\ & + c|M(x,y,t) - M(x,Ty,t)|, \end{aligned} \tag{7}$$

where $a, b, c > 0$ with $a+b+c \leq 1$ and $a+b \leq c$, for all $x, y \in X$ and $t > 0$. Then

(i) T has a unique fixed point z in X.

(ii) the prior error estimation:

$$M(x_n, z, t) \geq \Big[\prod_{k=1}^{n} (a_k + b_k)\Big(\alpha(x_{n-k}, z, t) + \beta(x_{n-k}, z, t)\Big)\Big] M(x_0, z, t).$$

(iii) the posterior error:

$$M(x_{n+1}, z, t) \geq \Big[(a_0 + b_0)\Big(\alpha(x_0, z, t) + \beta(x_0, z, t)\Big)\Big]^n M(x_0, z, t).$$

(iv) the rate of convergence:

$$M(x_{n+1}, z, t) \geq (a+b)\Big(\alpha(x_n, z, t) + \beta(x_n, z, t)\Big) M(x_n, z, t).$$

Proof: Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0, t) \geq 1$ and $\beta(x_0, Tx_0, t) \leq 1$, and $\{x_n\} \subset X$ be sequence defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$. If there exists $n \in \mathbb{N}_0$ such that $x_n = x_{n+1} = Tx_n$, then x_n is a fixed point of T, and the proof is completed. So, assume that $x_n \neq x_{n+1} = Tx_n$

for all $n \in \mathbb{N}_0$. And, we have $\alpha(x_0, Tx_0, t) = \alpha(x_0, x_1, t) \geq 1$ and $\beta(x_0, Tx_0, t) = \beta(x_0, x_1, t) \leq 1$. By continuing this process, we get $\alpha(x_n, Tx_n, t) = \alpha(x_n, x_{n+1}, t) \geq 1$ and $\beta(x_n, Tx_n, t) = \beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}_0$ and $t > 0$. This implies that T is α and β admissible. Since $(X, M, *)$ is complete fuzzy metric space, for some $x \in X$, $x_n \to x$ as $n \to \infty$. That is, for given $\delta \in (0,1)$ and $t > 0$ there exists $N \in \mathbb{N}_0$ such that $M(x_n, x, t) > 1 - \delta$. This implies that $\lim_{n\to\infty} M(x_n, x, t) = 1$, $\lim_{n\to\infty} \alpha(x_n, x, t) = 1$, and $\lim_{n\to\infty} \beta(x_n, x, t) = 1$. Now, let T holds the contractive condition (7), that is there exists $a, b, c > 0$ with $a + b + c \leq 1$ and $a + b \leq c$, satisfying the Equation (7). Now, by taking $x = x_n$ and $y = x_{n+1}$ in the Equation (7), and for any $m \in \mathbb{N}$,

$$\begin{aligned}
M(x_{n+m+1}, x_{n+m+2}, t) =& M(Tx_{n+m}, Tx_{n+m+1}, 1) \\
\geq& a\big(\alpha(x_{n+m}, x_{n+m+1}, t) + \beta(x_{n+m}, x_{n+m+1}, t)\big) M(x_{n+m}, x_{n+m+1}, t) \\
&+ b\big(\alpha(x_{n+m}, Tx_{n+m+1}, t) + \beta(x_{n+m}, Tx_{n+m+1}, t)\big) M(x_{n+m}, Tx_{n+m+1}, t) \\
&+ c|M(x_{n+m}, x_{n+m+1}, t) - M(x_{n+m}, Tx_{n+m+1}, t)|
\end{aligned}$$

Now, by the triangular inequality and the remark (2), for every $t > 0$, we get

$$\begin{aligned}
M(x_{n+m+1}, x_{n+m+2}, t) =& M(Tx_{n+m}, Tx_{n+m+1}, t) \\
\geq& \Big[a\big(\alpha(x_{n+m}, x_{n+m+1}, t) + \beta(x_{n+m}, x_{n+m+1}, t)\big) \\
&+ b\big(\alpha(x_{n+m}, x_{n+m+2}, t) + \beta(x_{n+m}, x_{n+m+2}, t)\big) M(x_{n+m+1}, x_{n+m+2}, t) \\
&+ c\big(1 - M(x_{n+m+1}, x_{n+m+2}, t)\big)\Big] M(x_{n+m}, x_{n+m+1}, t)
\end{aligned}$$

There exists $b_m, c_m \in (0,1)$ such that $b_m = bM(x_{n+m+1}, x_{n+m+2}, t)$ and $c_m = c(1 - M(x_{n+m+1}, x_{n+m+2}, t))$. And by taking $a = a_m$, we have

$$\begin{aligned}
M(x_{n+m+1}, x_{n+m+2}, t) =& M(Tx_{n+m}, Tx_{n+m+1}, t) \\
\geq& \Big[a_m\big(\alpha(x_{n+m}, x_{n+m+1}, t) + \beta(x_{n+m}, x_{n+m+1}, t)\big) \\
&+ b_m\big(\alpha(x_{n+m}, x_{n+m+2}, t) + \beta(x_{n+m}, x_{n+m+2}, t)\big) \\
&+ c_m\Big] M(x_{n+m}, x_{n+m+1}, t)
\end{aligned}$$

Similarly,

$$\begin{aligned}M(x_{n+m},x_{n+m+1},t)=&M(Tx_{n+m-1},Tx_{n+m},t)\\&\geq\big[a_{m-1}\big(\alpha(x_{n+m-1},x_{n+m},t)+\beta(x_{n+m-1},x_{n+m},t)\big)\\&\quad+b_{m-1}\big(\alpha(x_{n+m-1},x_{n+m+1},t)+\beta(x_{n+m-1},x_{n+m+1},t)\big)\\&\quad+c_{m-1}\big]M(x_{n+m-1},x_{n+m},t).\end{aligned}$$

By continuing this we have,

$$\begin{aligned}M(Tx_{n+m},Tx_{n+m+1},t)=&M(x_{n+m+1},x_{n+m+2},t)\\&\geq\prod_{k=0}^{m}\big[a_k\big(\alpha(x_{n+k},x_{n+k+1},t)+\beta(x_{n+k},x_{n+k+1},t)\big)\\&\quad+b_k\big(\alpha(x_{n+k},x_{n+k+2},t)+\beta(x_{n+k},x_{n+k+2},t)\big)+c_k\big]M(x_n,x_{n+1},t).\end{aligned}$$

By the Lemma (2), for given $\varepsilon>0$ and $t>0$ there exists $N\in\mathbb{N}$ such that $M(x_{n+m},x_{n+m+1},t)>1-\varepsilon$ for all $n,m>N$. This implies that $Tx_n=x_{n+1}$ is a Cauchy. Since X is a complete metric space, there exists $z\in X$ such that $\lim_{n\to\infty}M(x_n,z,t)=1$. That is $\lim_{n\to\infty}x_n=z$. Now, we claim that z is a fixed point of T. It is easy to verify that $Tz=z$. From the given contraction, we have $M(Tz,z,t)=M(Tz,Tz,t)\geq a(\alpha(z,z,t)+\beta(z,z,t))+b(\alpha(z,Tz,t)+\beta(z,Tz,t))+c|M(z,z,t)-M(z,Tz,t)|=a(1+1)+b(1+1)+c(0)=2(a+b)=1$. Hence $Tz=z$. Now, to prove the uniqueness suppose that $Tz^*=z^*$ for some $z^*\in X$. Then from the contraction, we have $M(z,z^*,t)=M(Tz,Tz^*,t)\geq a(\alpha(z,z^*,t)+\beta(z,z^*,t))M(z,z^*,t)+b(\alpha(z,Tz^*,t)+\beta(z,Tz^*,t))+c|M(z,z^*,t)-M(z,Tz^*,t)|\geq(a+b)(\alpha(z,z^*,t)+\beta(z,z^*,t))M(z,z^*,t)$. Since $x_n\to z$ and $x_n\to z^*$ as $n\to\infty$, and α is bounded below by 1, also, β is bounded above by 1. This implies that $\lim_{n\to\infty}\alpha(x_n,z^*,t)=1=\alpha(z,z^*,t)$ and $\lim_{n\to\infty}\beta(x_n,z^*,t)=1=\beta(z,z^*,t)$. Now, we get $M(z,z^*,t)\geq 2(a+b)M(z,z^*,t)\implies M(z,z^*,t)\geq M(z,z^*,t)$. It is a contradiction, and hence it concludes the uniqueness.

(ii) Prior error estimation: For $n\in\mathbb{N}$,

$$\begin{aligned}M(x_n,z,t)=&M(Tx_{n-1},Tz,t)\\&\geq a_n(\alpha(x_{n-1},z,t)+\beta(x_{n-1},z,t))M(x_{n-1},z,t)\\&\quad+b_n(\alpha(x_{n-1},Tz,t)+\beta(x_{n-1},Tz,t))M(x_{n-1},Tz,t)\end{aligned}$$

$$+c_n|M(x_{n-1},z,t)-M(x_{n-1},Tz,t)|$$

$$M(x_n,z,t)\geq(a_n+b_n)(\alpha(x_{n-1},z,t)+\beta(x_{n-1},z,t))M(x_{n-1},z,t)$$

$$\begin{aligned}M(x_n,z,t)\geq&\big[(a_n+b_n)(\alpha(x_{n-1},z,t)+\beta(x_{n-1},z,t))\big]\\&\big[a_{n-1}(\alpha(x_{n-2},z,t)+\beta(x_{n-2},z,t))M(x_{n-2},z,t)\\&+b_{n-1}(\alpha(x_{n-2},Tz,t)+\beta(x_{n-2},Tz,t))M(x_{n-2},Tz,t)\\&+c_{n-1}|M(x_{n-2},z,t)-M(x_{n-2},Tz,t)|\big]\end{aligned}$$

$$\begin{aligned}M(x_n,z,t)\geq&\big[(a_n+b_n)(\alpha(x_{n-1},z,t)+\beta(x_{n-1},z,t))\big]\\&\big[(a_{n-1}+b_{n-1})(\alpha(x_{n-2},z,t)+\beta(x_{n-2},z,t))\big]M(x_{n-2},z,t)\\&\vdots\end{aligned}$$

$$M(x_n,z,t)\geq\Big[\prod_{k=1}^{n}(a_k+b_k)(\alpha(x_{k-1},z,t)+\beta(x_{k-1},z,t))\Big]M(x_0,z,t)$$

Hence, this establish an upper bound for the iterations, and provides a crude approximation of how far the current iteration is from the actual fixed point.

(iii) Posterior error estimation: For $n\in\mathbb{N}$, we have $M(x_n,z,t)\geq M(x_{n-1},z,t)\geq M(x_{n-2},z,t)\geq\ldots\geq M(x_0,z,t)$. And, by using contraction (7) recursively, we have

$$M(x_1,z,t)\geq(a_1+b_1)(\alpha(x_0,z,t)+\beta(x_0,z,t))M(x_0,z,t)$$

$$\begin{aligned}M(x_2,z,t)\geq&(a_2+b_2)(\alpha(x_1,z,t)+\beta(x_1,z,t))M(x_1,z,t)\\\geq&\Big((a_1+b_1)(\alpha(x_0,z,t)+\beta(x_0,z,t))\Big)^2M(x_0,z,t)\end{aligned}$$

$$\begin{aligned}M(x_3,z,t)\geq&(a_3+b_3)(\alpha(x_2,z,t)+\beta(x_2,z,t))M(x_2,z,t)\\\geq&\Big((a_1+b_1)(\alpha(x_0,z,t)+\beta(x_0,z,t))\Big)^3M(x_0,z,t)\\&\vdots\end{aligned}$$

$$\begin{aligned}M(x_n,z,t)\geq&(a_n+b_n)(\alpha(x_{n-1},z,t)+\beta(x_{n-1},z,t))M(x_{n-1},z,t)\\\geq&\Big((a_1+b_1)(\alpha(x_0,z,t)+\beta(x_0,z,t))\Big)^nM(x_0,z,t).\end{aligned}$$

This concludes the posterior error estimation, also it helps to track the error dynamically.

(iv) Rate of convergence: For $n \in \mathbb{N}$, and by contraction (7), we have

$$M(x_{n+1}, z, t) = M(Tx_n, Tz, t) \geq (a_{n+1} + b_{n+1})(\alpha(x_n, z, t) + \beta(x_n, z, t))M(x_n, z, t)$$

It is clear that, for any positive real numbers a_{n+1}, b_{n+1} with $a_{n+1} + b_{n+1} \leq \frac{1}{2}$, $0 < (a_{n+1} + b_{n+1})(\alpha(x_n, z, t) + \beta(x_n, z, t)) < 1$. There exists $p > 0$, such that

$$M(x_{n+1}, z, t) \approx p(a_{n+1} + b_{n+1})(\alpha(x_n, z, t) + \beta(x_n, z, t))M(x_n, z, t).$$

This implies that, $M(x_{n+1}, z, t) \approx qM(x_n, z, t)$, where $q = p(a_{n+1} + b_{n+1})(\alpha(x_n, z, t) + \beta(x_n, z, t))$. Hence the contraction principle order of convergence is first-order (linear), and hence

$$M(x_{n+1}, z, t) \approx qM(x_n, z, t) \geq (a_{n+1} + b_{n+1})(\alpha(x_n, z, t) + \beta(x_n, z, t))M(x_n, z, t).$$

The following example illustrates the above Theorem 1.

Example 1. *Let $X = [0,1]$ be a complete fuzzy metric space under the fuzzy metric $M(x,y,t) = \frac{kt}{kt+|x-y|}$ for all $x, y \in X$ and $t > 0$, and the continuous t-norm defined by $p * q = pq$ for all $p, q \in I$. And, let $T : X \longrightarrow X$ be α and β admissible mapping defined by $T(x) = 1 - e^{-x}$ for all $x \in X$. Where $\alpha : X^2 \times \mathbb{R}^+ \longrightarrow [0, \infty)$ defined as $\alpha(x,y,t) = 1 + \frac{|x-y|}{t}$ for all $x, y \in X$ and $t > 0$. And $\beta : X^2 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ defined as $\beta(x,y,t) = 1 - \frac{|x-y|}{t}$ for all $x, y \in X$ and $t > 0$.*

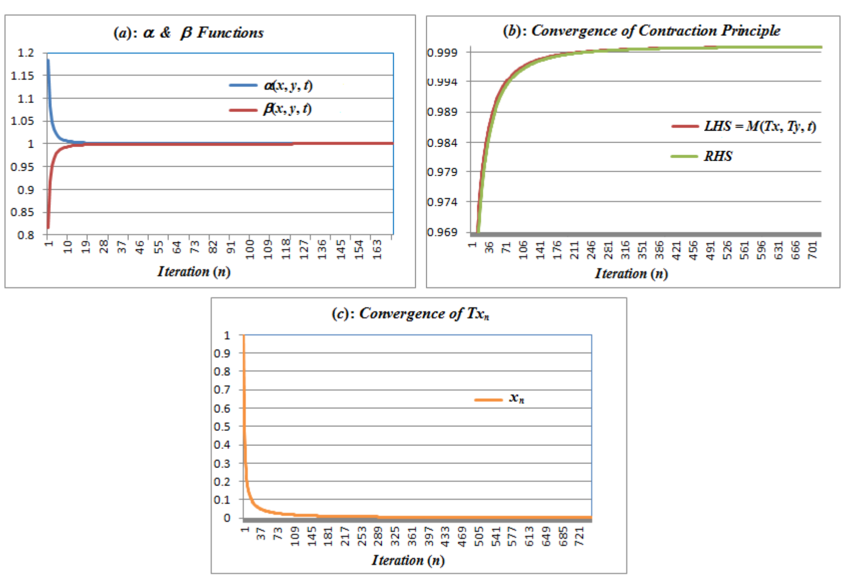

FIGURE 1. The Existence of Fixed Point.

Iteration (n)	x_n	$x_{n+1} = Tx_n$	$M(Tx_n, Tx_{n+1}, t)$	RHS	Prior Error	Posterior Error (e_n)	The Convergence of $M(Tx_n, z, t)$
1	1.00000000	0.63212056	0.07539972	0.07320293	0.02912621	0.02912621	0.02912621
2	0.63212056	0.46853639	0.15497135	0.14988950	0.02912621	0.00084834	0.04530897
3	0.46853639	0.37408231	0.24105275	0.23303700	0.02912621	0.00002471	0.06017615
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	0.03837667	0.03764962	0.97633831	0.97416383	0.02912621	0.00000000	0.43874614
51	0.03764962	0.03694968	0.97720076	0.97510052	0.02912621	0.00000000	0.44346150
52	0.03694968	0.03627537	0.97801715	0.97598749	0.02912621	0.00000000	0.44809774
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
9994	0.00020004	0.00020002	0.99999933	0.99999927	0.02912621	0.00000000	0.99337603
9995	0.00020002	0.00020000	0.99999933	0.99999927	0.02912621	0.00000000	0.99337668
9996	0.00020000	0.00019998	0.99999933	0.99999927	0.02912621	0.00000000	0.99337734
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$n \to \infty$	$x_n \to 0 = z$	$Tx_n \to 0 = Tz$	$M(Tx_n, Tx_{n+1}, t) \to 1$	$RHS \to 1$	$Pri.Err. \to Constant$	$Post.Err. \to 0$	$M(Tx_n, z, t) \to 1$

FIGURE 2. Table of Computation of Fixed Point for $t = 2, k = 0.015, a = 0.2, b = 0.3$ and $c = 0.5$, where $RHS = a(\alpha(x,y,t) + \beta(x,y,t))M(x,y,t) + b(\alpha(x,Ty,t) + \beta(x,Ty,t))M(x,Ty,t) + c|M(x,y,t) - M(x,Ty,t)|$.

From the graph in the Figure 1(a), it is clear that $\alpha(x,y,t) \geq 1$ *and* $\beta(x,y,t) \leq 1$, *that is,* $\alpha(x,y,t)$ *is bounded below by* 1 *and* $\beta(x,y,t)$ *is bounded above by* 1. *Also, it is noticed that as the number of iterations increases* $\alpha(x,y,t)$ *and* $\beta(x,y,t)$ *converges to* 1. *And, it is clear that, for every* $t > 0$ *and* $x, y \in X$ *with* $x \neq y$, T *is* α *and* β *admissible. The graph illustrated in the Figure 1(b), shows that the self mapping* T *satisfies the contraction principle (7) for all* $x, y \in X$ *and* $t > 0$. *Hence by the theorem (1),* T *has a unique fixed point* $z = 0$ *in* X. *The Figure 1(c) illustrates the existence of fixed point of* T, *and its convergence. However, varying the value of* t *influences the number of iterations required to approach the fixed point.*

To verify the above calculations, the reader is referred to the computational data up to eight places of decimals for $t = 2, k = 0.015, a = 0.2, b = 0.3$ *and* $c = 0.5$ *presented in the Table, which is shown in Figure 2. It is easy to verify that for every* n, *the contraction (7) holds,* $Tx_n \to 0$ *as* $n \to \infty$ *and* $T(0) = 0$. *Also, it is noticed that for every iteration* n, *the posterior error is constant, and the prior error converges to* 0, *as* $n \to \infty$. *For every* n, *from the posterior error estimation we get* $|e_{n+1}| \approx \lambda |e_n|$, *where* $\lambda = 0.029126213592233$. *This concludes that the contraction principle order of convergence is first-order (linear).*

4. Conclusion

Motivated by Gopal and Vetro [2], we established a new fixed point theorem by introducing a novel α-β-admissible contraction principle in a complete fuzzy metric space (GV), which works for large classes of mappings. The efficiency of the proposed contraction was demonstrated through the estimation of prior error, posterior error and the analysis of its convergence rate using numerical examples. Interpreting the admissible mappings α and β as uncertainty and reliability factors, respectively, the developed principle offers potential applications of fuzzy fixed point theory in forecasting and other uncertainty-based modeling contexts.

Conflict of Interests

The authors declare that there is no conflict of interests.

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