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FIXED POINTS OF NONLINEAR CONTRACTION

P.P.MURTHY *, K.N.V.V.VARA PRASAD, AND RASHMI

Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavidyalaya, Bilaspur 495009, India

Abstract. The main purpose of this paper is to improve the results of Babu and Alemayehu [7] in metric spaces by replacing the containment condition and giving the shorter proof than of the authors in their main results.

Keywords: Coincidence points, point of coincidence, Property (E.A), common property (E.A), Occasionally weakly compatible maps, common fixed points.

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1. Introduction

Aamri and Moutawakil [1] introduced the concept of property (E.A) which was perhaps inspired by the condition of compatibility introduced by Jungck [3]. Numerous research publications can be seen using this small but pivoting condition for choices of sequences for a pair of self maps in metric spaces and their related spaces.

Throughout this paper (X,d) is a metric space which we denote simply by X; and A,B,S and T are selfmaps of X.

2. Preliminaries

*Corresponding author

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Definition 2.1[4]. Let A and S be selfmaps of a set X. If $Au = Su = w$ (say), $w \in X$, for some u in X,then u is called a ***coincidence point*** of A and S and the set of coincidence points of A and S in X is denoted by $C(A,S)$, and w is called a ***point of coincidence*** of A and S.

Definition 2.2 Let A,B,S and T be selfmaps of a set X. If $u \in C(A,S)$ and $v \in C(B,T)$ for some $u,v \in X$ and $Au = Su = Bv = Tv = z(say)$, then z is called a ***common point of coincidence of the pairs*** (A,S) and (B,T).

Definition 2.3 The pair (A,S) is said to be

(i) *satisfy property* $(E.A)$[1] *if there exists a sequence* $\{x_n\}$ *in* X *such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ *for some* t *in* X.

(ii) *be compatible* [3] *if* $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$*, for some* t *in* X *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$

(iii) *be weakly compatible* [5], *if the commute at their coincidence point.*

(iv) *be occasionally weakly compatible* (owc)[9,10], *if* $ASx = SAx$ *for some* $x \in C(A,S)$.

Remark 2.4 (i) Every compatible pair is weakly compatible but its converse need not be true [5].

(ii) Weak compatibility and property (E. A) are independent of each other [14].

(iii) Every weakly compatible pair is occasionally weakly compatible but its converse need not be true [11].

(iv) Occasionally weakly compatible and property (E. A) are independent of each other [16].

Definition 2.5 [13] Let (X,d) be a metric space and A,B,S and T be four selfmaps on X. The pairs (A,S) and (B,T) are said to satisfy common property $(E.A)$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n$ for some t in X.

Remark 2.6 Let A,B,S and T be self maps of a set X. If the pairs (A,S) and (B,T) have common point of coincidence in X then $C(A,S) \neq \phi$ and $C(B,T) \neq \phi$. But converse is not true.

Example 2.7 Let $X = [0,\infty)$ with usual metric and A,B,S and T self maps on X and defined by $Ax = 1 - x^2$; $Sx = 1 - x$; $Bx = \frac{1}{2} + x^2$; $Tx = \frac{1+x}{2}$ for all $x \in X$

It is easy to observe that $C(A,S)=\{0,1\}$ and $C(B,T)=\{0,\frac{1}{2}\}$ but the pairs (A,S) and (B,T) not having common point of coincidence.

Remark 2.8 The converse of the Remark 2.6 is true provided it satisfies inequality (3.1). This is given as in Proposition 3.1 in Section III.

Proposition 2.9 ([15]) Let A and S be two self maps of a set X and the pair (A,S) is satisfies occasionally weakly compatible(owc) condition. If the pair (A,S) have unique point of coincidence $Ax=Sx=z$ then z is the unique common fixed point of A and S.

Proof: To be given

$$Ax = Sx = \{z\}(\text{say }) \ \text{ for any } x \in C(A,S). \tag{2.1}$$

Since the pair (A,S) satisfies the property owc, therefore

$Az = ASx = SAx = Sz$ implies that $z \in C(A,S)$.

From (2.1), we get $Az=Sz=z$. Hence proposition follows.

In 1996, Tas et al.[8] proved the following .

Theorem 2.10 Let A,B,S and T be selfmaps of a complete metric space (X,d) such that $A(X)\subseteq T(X)$ and $B(X)\subseteq S(X)$ and satisfying the inequality

$$[d(Ax,By)]^2 \le c_1 \, max\{[d(Sx,Ax)]^2, [d(Ty,By)]^2, [d(Sx,Ty)]^2\}$$
$$+c_2 \, max\{d(Sx,Ax)d(Sx,By), \ d(Ty,Ax)d(Ty,By)\}$$
$$+c_3 d(Sx,By)d(Ty,Ax)$$

for all $x,y\in X$, where $c_1,c_2,c_3 \ge 0, \ c_1+2c_2<1, \ c_1+c_3<1$. Further, assume that the pairs (A,S) and (B,T) are compatible on X. If one of the mappings A,B,S and T is continuous then A,B,S and T have a unique common fixed point in X.

3. Main results

Proposition 3.1. *Let A, B, S and T be self maps of a metric space (X, d) and satisfying the inequality*

$$[d(Ax,By)]^2 \leq c_1 \; max\{[d(Sx,Ax)]^2, [d(Ty,By)]^2, [d(Sx,Ty)]^2\}$$
$$+c_2 \; max\{d(Sx,Ax)d(Sx,By), \; d(Ty,Ax)d(Ty,By)\}$$
$$+c_3 d(Sx,By)d(Ty,Ax) \tag{3.1}$$

for all $x, y \in X$, where $c_1, c_2, c_3 \geq 0$ and $c_1 + c_3 < 1$. Then the pairs (A, S) and (B, T) have common point of coincidence in X if and only if $C(A,S) \neq \phi$ and $C(B,T) \neq \phi$.

Proof. `If Part`: It is trivial.

`Only if part`: Assume $C(A,S) \neq \phi$ and $C(B,T) \neq \phi$.

Then there is a $u \in C(A,S)$ and $v \in C(B,T)$ such that

$$Au = Su = p \text{ (say)} \tag{3.2}$$

$$Bv = Tv = q \text{ (say)} . \tag{3.3}$$

On taking $x = u$ and $y = v$ in (3.1), we get

$$[d(Au,Bv)]^2 \leq c_1 \; max\{[d(Su,Au)]^2, [d(Tv,Bv)]^2, [d(Su,Tv)]^2\}$$
$$+c_2 \; max\{d(Su,Au)d(Su,Bv), \; d(Tv,Au)d(Tv,Bv)\}$$
$$+c_3 d(Su,Bv)d(Tv,Au).$$

Using (3.2) and (3.3), we get

$[d(p,q)]^2 \leq (c_1 + c_3)[d(p,q)]^2$, a contradiction. Thus $p = q$.

Therefore A, B, S and T have common point of coincidence in X.

In the Proposition (2.1) of Babu et al.([7]), we can obtain some more conclusions of in their paper. Therefore our result improves and strengthen Proposition 3.1 and subsequent theorems in metric spaces.

Proposition 3.2. *Let A,B,S and T be four self maps of a metric space (X,d) satisfying the inequality* (3.1). *Suppose that either*

- **(i):** *$B(X) \subseteq S(X)$, the pair (B,T) satisfies property (E.A.) and $T(X)$ is a closed subspace of X; or*
- **(ii):** *$A(X) \subseteq T(X)$, the pair (A,S) satisfies property (E.A.) and $S(X)$ is a closed subspace of X, holds.*

Then the pairs (A,S) and (B,T) are satisfies the common property (E.A.); also both the pairs (A,S) and (B,T) have common point of coincidence in X.

We have shorten the proof of Theorem 2.2 of ([7])by relaxing many lines:

Theorem 3.3 (Improved version of Theorem 2.2, [7]). *Let A,B,S and T are satisfying all the conditions given Proposition* 3.2 *with the following additional assumption:*

the pairs (A,S) and (B,T) are owc on X.

Then A,B,S and T have a unique common fixed point in X.

Proof. By Proposition 3.2 the pairs (A,S) and (B,T) have common point of coincidence. Therefore there is $u \in C(A,S)$ and $v \in C(B,T)$ such that

$$Au = Su = z(say) = Bv = Tv. \tag{3.4}$$

Now, we show that z is unique common point of coincidence of the pairs (A,S) and (B,T).

Let if possible z' is another point of coincidence of A,B,S and T. Then there is $u' \in C(A,S)$ and $v' \in C(B,T)$ such that

$$Au' = Su' = z'(say) = Bv' = Tv'. \tag{3.5}$$

Putting $x = u$ and $y = v'$ in inequality (3.1), we have

$$[d(Au,Bv)]^2 \leq c_1\ max\{[d(Su,Au)]^2,[d(Tv,Bv)]^2,[d(Su,Tv)]^2\}$$
$$+c_2\ max\{d(Su,Au)d(Su,Bv),\ d(Tv,Au)d(Tv,Bv)\}$$
$$+c_3 d(Su,Bv)d(Tv,Au)$$

Now, using (3.4) and (3.5), we get

$[d(z,z')]^2 \leq (c_1+c_3)[d(z,z')]^2$, and arrive at a contradiction. Hence $z=z'$ and we have $C(A,S) = \{z\} = C(B,T)$

By Proposition 2.9, z is the unique common fixed point of A,B,S and T in X.

Remark 3.4 Proposition 2.5 of [7] and Theorem 2.6 of [7] are remain true, if we replace completeness of $S(X)$ and $T(X)$ by the completeness of $S(X) \cap T(X)$ in X. For this we have given an Example 2.7 in the following manner without proof.

Now, we rewriting the Proposition 2.5 and Theorem 2.6 of [7].

Proposition 3.5 Let A,B,S and T be four self maps of a metric space (X,d) satisfying the inequality (3.1) of proposition 3.1. Suppose that (A,S) and (B,T) satisfy a common property $(E.A)$ and $S(X) \cap T(X)$ are closed subspace of X, then A,B,S and T have unique common point of coincidence.

Theorem 3.6 In addition to the above proposition 3.5 on A,B,S and T, if both the pairs (A,S) and (B,T) are owc maps on X, then the point of coincidence is a unique common fixed point of A,B,S and T.

Example 3.7 Let $X = [\frac{1}{3}, 1)$ with the usual metric. We define mappings A,B,S and T on X by

$$A(x) = \begin{cases} \frac{1}{3}, & \text{if, } x \in [\frac{1}{3}, \frac{2}{3}); \\ \frac{2}{3}, & \text{if, } x \in [\frac{2}{3}, 1) \end{cases} \qquad B(x) = \begin{cases} \frac{3}{4}, & \text{if, } x \in [\frac{1}{3}, \frac{2}{3}); \\ \frac{2}{3}, & \text{if, } x \in [\frac{2}{3}, 1) \end{cases}$$

$$S(x) = \begin{cases} \frac{1}{2}, & \text{if, } x \in [\frac{1}{3}, \frac{2}{3}); \\ \frac{1}{3}+\frac{x}{2}, & \text{if, } x \in [\frac{2}{3}, 1) \end{cases} \qquad T(x) = \begin{cases} \frac{5}{6}, & \text{if, } x \in [\frac{1}{3}, \frac{2}{3}); \\ 1-\frac{x}{2}, & \text{if, } x \in [\frac{2}{3}, 1) \end{cases}$$

We observe that $S(X) = \{\frac{1}{2}\} \cup [\frac{2}{3}, \frac{5}{6})$ and $T(X) = (\frac{1}{2}, \frac{2}{3}] \cup \{\frac{5}{6}\}$ are not closed and $S(X) \cap T(X) = \{\frac{2}{3}\}$ is a closed subspace of X.

The pairs (A,S) and (B,T) satisfies a common property (E.A) at the sequence $\{x_n\}$,
$x_n = \frac{2}{3} + \frac{1}{n+3}$, $n = 1,2,3,...$ in X.

Case (i): If $x,y \in [\frac{1}{3}, \frac{2}{3})$ then the inequality (3.1), we get

$$(\tfrac{5}{12})^2 \leq c_1\, max\{(\tfrac{1}{6})^2, (\tfrac{1}{12})^2, (\tfrac{1}{3})^2\} + c_2\, max\{\tfrac{1}{6}\cdot\tfrac{1}{4}, \tfrac{1}{2}\cdot\tfrac{1}{12}\} + c_3\, \tfrac{1}{4}\cdot\tfrac{1}{2}$$

i.e., $25 \leq 16c_1 + c_2 6 + c_3 18$

Case (ii): If $x, y \in [\frac{2}{3}, 1)$ the inequality (3.1) holds trivial.

Case (iii): If $x \in [\frac{1}{3}, \frac{2}{3})$ and $y \in [\frac{2}{3}, 1)$ then from inequality (3.1), we have

$$(\frac{1}{3})^2 \leq c_1 max\{\frac{1}{36}, (\frac{y}{2} - \frac{1}{3})^2, (\frac{1-y}{2})^2\} + c_2 max\{\frac{1}{36}, (\frac{2}{3} - \frac{y}{2})(\frac{y}{2} - \frac{1}{3})\} + c_3(\frac{2}{3} - \frac{y}{2}).$$

$$4 \leq c_1 + c_2 + c_3.(4 - 3y).$$

Case (iv): if $x \in [\frac{2}{3}, 1)$ and $y \in [\frac{1}{3}, \frac{2}{3})$

$$(\frac{1}{12})^2 \leq c_1 max\{(\frac{x}{2} - \frac{1}{3})^2, (\frac{1}{12})^2, (\frac{x-1}{2})^2\} + c_2 max\{\mid \frac{x}{2} - \frac{1}{3} \mid\mid \frac{x}{2} - \frac{5}{12} \mid, \frac{1}{6}.\frac{1}{12}\}$$

$$+ c_3 \frac{1}{6} \mid \frac{x}{2} - \frac{5}{12} \mid$$

In all cases the inequality (3.1) holds with $c_1 = \frac{1}{3}, c_2 = 5\frac{5}{6}$ and $c_3 = \frac{1}{2}$. The pairs (A, S) and (B, T) satisfies owc at the point $\frac{2}{3}$. The point $\frac{2}{3}$ is a unique fixed point of A, B, S and T.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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