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Adv. Fixed Point Theory, 2026, 16:11
<https://doi.org/10.28919/afpt/9660>
ISSN: 1927-6303

FUZZY ORDER THEORY: CHARACTERIZATION AND IMPLEMENTATION

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Abstract. This paper develops a new fuzzy extension of the classical Knaster–Tarski fixed point theorem and the converse result of Anne C. Davis. Working within the framework of r -fuzzy ordered sets, we introduce the notion of r -fuzzy complete lattices and establish necessary and sufficient conditions for the existence of fixed points of r -fuzzy monotone mappings. We prove that every r -fuzzy monotone self-map on a non-empty r -fuzzy complete lattice admits both a greatest and a least fixed point. Furthermore, we obtain a fuzzy version of Davis’s characterization of complete lattices by constructing an explicit r -fuzzy monotone operator that fails to have fixed points when completeness is absent. These results provide a unified approach to fixed-point theory in fuzzy environments and extend several known theorems in both classical and fuzzy order theory.

Keywords: fixed points; r -fuzzy; complete lattice.

2020 AMS Subject Classification: 06A15, 47H10.

1. INTRODUCTION

In fixed-point theory, the completeness of a set or space is an essential property that guarantees the existence of a fixed point for certain mappings. The Knaster-Tarski theorem is one of the results that provided a characterization related to an ordered set and Anne C. Davis proved an

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Received October 23, 2025

important converse theorem.. In contrast, for the Banach fixed-point theorem[1], the completeness of the space is simply a condition that guarantees the existence and uniqueness of a fixed point. Researchers have implemented this famous theorem in other directions (see [9, 11, 12]).

In 1965, Zadeh [3] generalized the definition of a crisp set by defining the fuzzy set that gives more efficient and accurate results. As fuzzy set addresses the uncertainty and give more accuracy compared to crisp set.

Fuzzy set theory is a powerful tool for modelling uncertainty and for processing vague or subjective information in mathematical models. several authors studied the existence of fixed point in fuzzy setting,[4, 6, 7, 5]. In fuzzy ordered sets, I. Beg [2] proved the existence of maximal fixed point of fuzzy monotone maps. The aim of this note is to give the following fuzzy version of Tarski's fix point Theorem [7] and its converse [13]. Now, suppose that $(X; r)$ is a non-empty r -fuzzy complete lattice and $f : X \rightarrow X$ is a r -fuzzy monotone map. Then the set $\text{Fix}(f)$ of all fixed points of f is a non-empty r -fuzzy complete lattice.

2. PRELIMINARIES

2.1. Notations and Definitions. In this note we shall use the following definition of order due to Claude Ponsard (see [8]).

Definition 2.1. *Let X be a crisp set. A fuzzy order relation on X is a fuzzy subset r of $X \times X$ satisfying the following three properties*

- i) *for all $x \in X$, $r(x, x) \in [0, 1]$ (r -reflexivity);*
- ii) *for all $x, y \in X$, $r(x, y) + r(y, x) > 1$ implies $x = y$ (r -antisymmetry);*
- iii) *for all $(x, y, z) \in X^3$, $[r(x, y) \geq r(y, x) \text{ and } r(y, z) \geq r(z, y)]$ implies $r(x, z) \geq r(z, x)$ (r -transitivity);*

Example 2.1. *Let $X = \{w, z, y, x\}$ and let $A : X \times X \rightarrow [0, 1]$ be a fuzzy relation such that (see (from [22]):*

$$A(x, x) = A(y, y) = A(z, z) = A(w, w) = 1,$$

$$A(x, y) = A(x, z) = A(x, w) = A(y, z) = A(y, w) = A(z, w) = 0,$$

$$A(y, x) = 0.3, A(z, x) = 0.5,$$

$$A(w, x) = 0.8, A(z, y) = 0.2,$$

$A(w,y) = 0.4$, and $A(w,z) = 0.1$.

Then it is easily to verify that A is a fuzzy order relation (further, it is total)

The following diagram show us the fuzzy order relation :

\times	w	z	y	x
w	1.0	0.1	0.4	0.8
z	0.0	1.0	0.2	0.5
y	0.0	0.0	1.0	0.3
x	0.0	0.0	0.0	1.0

The condition 2 (r-antisymmetry) in definition 2.1 prevents two distinct elements from being "too strongly" related in both directions simultaneously. The threshold 1 is often used in fuzzy logic systems where the standard negation is $n(a) = 1 - a$. The condition $a + b > 1$ is a way of expressing that a and b cannot both be ≤ 0.5 .

A nonempty set X with fuzzy order r defined on it, is called r-fuzzy ordered set. We denote it by $(X;r)$. An r-fuzzy order is total if for any pair of distinct elements x and y , we have either

$$r(x,y) > r(y,x)$$

or

$$r(y,x) > r(x,y)$$

An r-fuzzy ordered set with a total r-fuzzy order is called an r-fuzzy chain.

Let A be a non-empty subset of X . We say that $x \in X$ is a r-upper bound of A if

$$r(a;x) \geq r(x;a)$$

for all $a \in A$.

A r-upper bound x of A with $x \in A$ is called a greatest element of A . An element x in A is maximal if there is no other element y in A such that

$$r(x;y) \geq r(y;x).$$

Similarly, we can define r -lower bound, minimal and least element of A . We denote by, $sup_r(A)$ the least element of the set of r -upper bound of A , if it exists, it is unique. $max_r(A)$ the greatest element of A if it exists, it is unique. $inf_r(A)$ the greatest element of the set of r -lower bound of A if it exists, it is unique. $min_r(A)$ the least element of A if it exists, it is unique.

3. r -FUZZY LATTICE

We adopt the standard lattice definition, adjusting it for the fuzzy setting. Consequently, we refer to this structure as a r -fuzzy lattice.

Definition 3.1. *An r -fuzzy lattice, defined as a non-empty r -fuzzy ordered set where all finite subsets have an r -infimum and r -supremum, is called complete if this property extends to all non-empty subsets.*

If $(X; r)$ is a r -fuzzy complete lattice then the set $[a, b] = \{x \in X : r(a, x) \geq r(x, a) \text{ and } r(x, b) \geq r(b, x)\}$ is a r -fuzzy complete lattice too, indeed: Let $A \subset [a, b]$ since $(X; r)$ is a r -fuzzy complete lattice, $sup_r A$ and $inf_r A$ exist and we have $r(a, x) \geq r(x, a)$ and $r(x, b) \geq r(b, x)$ for all $x \in A$ that implies $r(a, inf_r A) \geq r(inf_r A, a)$ and $r(inf_r A, b) \geq r(b, inf_r A)$ because $inf_r A$ is the greatest lower bound of A . Since $sup_r A$ is the least upper bound A we have $r(sup_r A, b) \geq r(b, sup_r A)$ and $r(a, sup_r A) \geq r(sup_r A, a)$. Consequently, $inf_r A \in [a, b]$ and $sup_r A \in [a, b]$.

Let $(X; r)$ be a r -fuzzy lattice with least element 0 and greatest element 1 of X .

Lemma 3.1. *If any chain of $(X; r)$ has a r -supremum then, any chain of X admits an inf.*

Indeed: We denote by 0 and 1 the least element and greatest element of X . let $C = (c_i)$ be an increasing chain of X . So, $sup C = s$ exists and $inf C = c_0$. Now, let $C = (c_i)_{i \in I}$ be a decreasing chain of X . We consider L the set of lower bound of C , we have L is non empty ($0 \in L$). By hypothesis any chain $(m_j)_{j \in J}$ of L has a sup by hypothesis. So, we have $r(m_{j'}, c_i) \geq r(c_i, m_{j'})$ and $r(m_{j'}, sup(m_j)_{j \in J}) \geq r(sup(m_j)_{j \in J}, m_{j'})$ for all $i \in I, j' \in J$ then $r(sup(m_j)_{j \in J}, c_i) \geq r(c_i, sup(m_j)_{j \in J})$ because the sup is the smallest of the upper bound of $(m_j)_{j \in J}$. Which shows that $sup(m_j)_{j \in J} \in L$ and according to lemma Zorn, L admits maximal element.

For two maximal elements m_0, m_1 of L we have $\sup_r(m_0, m_1)$ exists because $(X; r)$ is a r -fuzzy lattice and $r(\sup_r(m_0, m_1), c_i) \geq r(c_i, \sup_r(m_0, m_1))$ for all $i \in I$ that implies $\sup_r(m_0, m_1) \in L$.

Consequently, $m_0 = m_1 = \sup_r(m_0, m_1) = \inf_r(c_i)_i$ by the maximality of m_0 and m_1 .

Then, $\inf_r C$ exists and $\inf_r C = m_0 = m_1$.

This result opens the door to further developments that will be useful in our work.

Let X be a r -fuzzy set that satisfies the conditions of the lemma 3.2. Let A a party of X and M be the set of upper bound of A . The set M is nonempty since $1 \in M$. Since, any chain of M admits an inf hence M satisfies the Zorn lemma that implies M has a minimal element. For two minimal elements m_0, m_1 of M we can easily show that $m_0 = m_1$ because $(X; r)$ is a r -fuzzy lattice and for the same context as discussed above. So, $m_0 = m_1 = \inf_r(m_0, m_1) = \sup_r A$.

From the same reasoning, we can demonstrate the existence of $\inf_r A$.

Theorem 3.1. *Let $(X; r)$ be a r -fuzzy lattice such that any chain has a r -supremum. Then, $(X; r)$ is a r -fuzzy complete lattice*

This theorem provides numerous results that we will explore later. The development of certain results from fixed-point theory, as seen in the classical case, is the objective of this work.

For a r -fuzzy lattice $(X; r)$, if $(X; r)$ is not complete then there exists a increasing chain has no sup. We show this by arguing that, otherwise, according to what we have seen above every chain has a sup, so $(X; r)$ is a r -fuzzy complete lattice, contradicting theorem 3.3.

This result highlights a key property of the r -fuzzy lattice, which is best understood through the concepts of increasing and decreasing sequences of intervals.

- An increasing sequence of intervals $([a_i, b_i])_{i \in I}$ is one that expands outwards, satisfying $r(a_{i+1}, a_i) \geq r(a_i, a_{i+1})$ and $r(b_i, b_{i+1}) \geq r(b_{i+1}, b_i)$.
- A decreasing sequence of intervals $([a_j, b_j])_{j \in J}$ is one that contracts inwards, satisfying $r(a_j, a_{j+1}) \geq r(a_{j+1}, a_j)$ and $r(b_{j+1}, b_j) \geq r(b_{j+1}, b_j)$.

Lemma 3.2. *Let $(X; r)$ be a r -fuzzy lattice, if X is not complete then there exist a decreasing sequence of intervals in X , $([a_i, b_i])_{i \in I}$ such that $\bigcap_{i \in I} [a_i, b_i] = \emptyset$.*

Indeed: According to theorem 3.3, the incompleteness of the r -fuzzy lattice $(X; r)$ implies the existence of an increasing chain $(a_i)_{i \in I}$ without a least upper bound. We consider a maximal

decreasing sequence $(b_j)_{j \in J}$ of its upper bounds, it is clear that $(\bigcap_{i \in I} a_i) \cap (\bigcap_{j \in J} b_j) = \emptyset$. Such a chain must exist, for if it did not, This would imply that every chain in the set of upper bounds of $(a_i)_{i \in I}$ has an upper bound within that set. By Zorn's Lemma, the set of upper bounds for $(a_i)_{i \in I}$ contains at least one minimal element; let's call one such element m . We now demonstrate that m is, in fact, the least upper bound (supremum) of $(a_i)_{i \in I}$. Let u be any other minimal upper bound for $(a_i)_{i \in I}$. Since $(X; r)$ be a r -fuzzy lattice, their infimum, $\inf(m, u)$, is also an upper bound for $(a_i)_{i \in I}$. By the minimality of m and u , we must have $\inf(m, u) = m = u$. Therefore, m is less than or equal to every other upper bound, making it the supremum of $(a_i)_{i \in I}$. This establishes the existence of a supremum, which contradicts our initial assumption that $(a_i)_{i \in I}$ has no supremum. Therefore, $([a_i, b_i])_{i \in I}$ is a decreasing sequence of intervals satisfy the condition of the lemma.

4. MAIN RESULTS

In this section, we establish a fuzzy version of Tarski's fixpoint Theorem [7]. Let $(X; r)$ be a r -fuzzy ordered set and let $f : X \rightarrow X$ be a map. We say that f is r -fuzzy monotone if for all $x, y \in X$ with $r(x, y) \geq r(y, x)$, then $r(f(x), f(y)) \geq r(f(y), f(x))$. We denote the set of all fixed points of f by $Fix(f)$.

Lemma 4.1. *Let $(X; r)$ be a non-empty r -fuzzy complete lattice and let $f : X \rightarrow X$ be a r -fuzzy monotone map. Then f has both a greatest and least fixed point.*

Proof. We denote by 0 and 1 the least element and greatest element of X .

Let

$$B = \{x \in X / r(x, f(x)) \geq r(f(x), x)\}$$

B is non-empty set since $0 \in B$ and let $p = \sup_r(B)$ (the existence of p is guaranteed since X is a r -fuzzy complete lattice).

Now, let $x \in B$, imply that

$$(4.1) \quad r(x, f(x)) \geq r(f(x), x)$$

We have

$$r(x, p) \geq r(p, x)$$

for all $x \in X$, and by the r -fuzzy monotone f , then

$$(4.2) \quad r(f(x), f(p)) \geq r(f(p), f(x))$$

From (1) and (2), we get $r(x, f(p)) \geq r(f(p), x)$, then $f(p)$ is a r -upper bound of B . As $p = \sup_r(B)$ imply that

$$(4.3) \quad r(p, f(p)) \geq r(f(p), p)$$

So, we get $p \in B$.

We will prove that p is the greatest fixed point, so

$$r(p, f(p)) \geq r(f(p), p)$$

By the r -fuzzy monotone of f , we have

$$r(f(p), f(f(p))) \geq r(f(f(p)), f(p))$$

Then $f(p) \in B$ and we have $f(p)$ is a r -upper bound of B that implies $f(p) = p$. So p is a fixed point of f .

Let a be a another fixed point of f , then $f(a) = a$ that implies $a \in B$. Therefore, we can $r(a, p) \geq r(p, a)$, that proves p is the greatest fixed point.

Then we can write

$$p = \sup_r(B) = \max_r[\text{Fix}(f)]$$

We now apply the same argument to the following set to demonstrate the existence of a least fixed point,

$$D = \{x \in X / r(f(x), x) \geq r(x, f(x))\}$$

□

Theorem 4.1. *Let $(X; r)$ be a non-empty r -fuzzy complete lattice and let $f : X \rightarrow X$ be a r -fuzzy monotone map. Then the set $\text{Fix}(f)$ of all fixed points of f is a non-empty r -fuzzy complete lattice.*

Proof. We denote by 0 and 1 the least element and greatest element of X .

The set $Fix(f)$ is non-empty set and has both a greatest element and least element according to lemma 4.1.

Now, for any subset A of $Fix(f)$, since $(X; r)$ be a non-empty r -fuzzy complete lattice, the supremum $sup_r A$ exists. Let

$$W = \{x \in X / r(sup_r A, x) \geq r(x, sup_r A)\} = [sup_r A, 1]$$

The set of upper bound of A .

We prove that f_W , the restriction of the monotone r -fuzzy map f to W , is a map from W to W .

For any element $x \in W$, we have $r(a, x) \geq r(a, x) \quad \forall a \in A$, by r -fuzzy monotone of f . We get

$$r(f(a), f(x)) \geq r(f(x), f(a))$$

Imply that

$$(4.4) \quad r(a, f(x)) \geq r(f(x), a) \quad \forall a \in A$$

Hence,

$$(4.5) \quad r(a, f_W(x)) \geq r(f_W(x), a) \quad \forall a \in A$$

Then $f_W(x) \in W$, we conclude that f_W is the maps W to W .

f_W is r -fuzzy monotone maps on the complete lattice W

So, recording to the above lemma f_W has a least fixed point $b \in W$.

So

$$r(a, b) \geq r(b, a) \quad \forall a \in A.$$

let c be another fixed point such that

$$r(a, c) \geq r(c, a) \quad \forall a \in A.$$

So, $c \in W$ that prove $r(b, c) \geq r(c, b)$ since b is least fixed point of W . Consequently, b is the least upper bound of A in $fix(f)$.

Similarly, we prove that A has a r -infimum in $fix(f)$.

Then $Fix(f)$ is a r -fuzzy complete lattice. □

5. APPLICATIONS

Fixed point theory stands as a highly developed field whose applications have broad reach, touching upon and enriching many scientific disciplines. As an application, we obtain a characterization of r -fuzzy set. This result can be seen as an extension of the Knaster-Tarski theorem.

Theorem 5.1. *If $(X; r)$ is a non-empty r -fuzzy lattice with the least element 0 and greatest element 1 such that any r -fuzzy monotone map $f : E \rightarrow E$ has a fixed point, then E is a complete lattice.*

Proof. Assume that X is not complete. According to Lemma 3.4 there exist a decreasing sequence of intervals in X , $[a_i, b_i]_{i \in I}$ such that $\bigcap_{i \in I} [a_i, b_i] = \emptyset$.

Specifically, we can select the chain of intervals that was previously constructed in the proof of Lemma 3.4.

Now we define a map $f : X \rightarrow X$ and proceed to verify that f is r -fuzzy monotone and has no fixed points. Let $x \in X$, let $f(x) = b_{j_0}$ if $x \in \bigcap_{i \in I} [a_i, 1]$ where

$j_0 = \inf\{j \in J / x \notin [0, b_j]\}$ i.e j_0 is least such that $r(x, b_{j_0}) \not\geq r(b_{j_0}, x)$. If $x \notin \bigcap_{i \in I} [a_i, 1]$,

let $f(x) = a_{i_0}$, where $i_0 = \inf\{i \in I / x \notin [a_i, 1]\}$ i.e i_0 is least such that $r(a_{i_0}, x) \not\geq r(x, a_{i_0})$.

By definition, either $r(x, f(x)) \not\geq r(f(x), x)$ or $r(f(x), x) \not\geq r(x, f(x))$, so f has no fixed points.

We conclude by showing that f is order-preserving, For this, suppose that $r(x, y) \geq r(y, x)$. If

$x \in \bigcap_{i \in I} [a_i, 1]$ then $y \in \bigcap_{i \in I} [a_i, 1]$, let $f(x) = b_{j_x}$ and $f(y) = b_{j_y}$, so $x \notin [0, b_{j_x}]$ and $y \notin [0, b_{j_y}]$,

it's easy to see that $r(b_{j_x}, b_{j_y}) \geq r(b_{j_y}, b_{j_x})$, so $r(f(x), f(y)) \geq r(f(y), f(x))$.

We can thus assume that $x \notin \bigcap_{i \in I} [a_i, 1]$. Let $f(x) = a_{i_x}$ so $x \notin [a_{i_x}, 1]$.

First, If there exist $i \in I$ such that $y \notin [a_i, 1]$ then we have $x \notin [a_i, 1]$. This would imply $r(a_{i_x}, a_i) \geq$

$r(a_i, a_{i_x})$ that show $r(f(x), f(y)) \geq r(f(y), f(x))$. Second, $y \in \bigcap_{i \in I} [a_i, 1]$, then $f(y)$ is defined as

an element from the chain of upper bounds, $(b_j)_{j \in J}$. By definition, any element of $(b_j)_{j \in J}$ is an upper bound for all elements of $(a_i)_{i \in I}$. Therefore, $r(f(x), f(y)) \geq r(f(y), f(x))$.

Theorem 5.2. *Let (P, r) be a non-empty r -fuzzy ordered set and let $f : P \rightarrow P$ be sequentially r -continuous, i.e., if $(c_n)_{n < \gamma}$ has a supremum c , then $\sup_r f(c_n)_{n < \gamma}$ exists and $r(\sup_r f(c_n)_{n < \gamma}, f(c)) \geq r(f(c), \sup_r f(c_n)_{n < \gamma})$. Assume that there is $b \in P$ such that $r(b, f(b)) \geq r(f(b), b)$ and the set $B = \{x : r(b, x) \geq r(x, b)\}$ is a r -fuzzy complete lattice. Let*

$b' = \sup_r(f^n(b))_n$, where f^n is the n th iterate of f . Then b' is a fixed point of f , and it is the least fixed point of f in $B = \{x : r(b, x) \geq r(x, b)\}$.

Proof. The obvious argument works. Just notice first that f is r -fuzzy monotone since if $r(x, y) \geq r(y, x)$ then $y = \sup_r\{x, y\}$, so $r(\sup_r\{f(x), f(y)\}, f(y)) \geq r(f(y), \sup_r\{f(x), f(y)\})$ by hypothesis and we have $r(f(x), \sup_r\{f(x), f(y)\}) \geq r(\sup_r\{f(x), f(y)\}, f(x))$ which implies $r(f(x), f(y)) \geq r(f(y), f(x))$ according to the r -transitivity of r -fuzzy.

Second, The restriction of the function f to B has values in B since f is r -fuzzy monotone and $r(b, f(b)) \geq r(f(b), b)$.

Theorem 5.3. *If $(X; r)$ is a non-empty r -fuzzy lattice with the least element 0 and greatest element 1 such that any order-preserving function $f : E \rightarrow E$ has a fixed point, then E is a complete lattice.*

Proof. Assume that X is not complete. According to Lemma 3.4 there exist Two sequences in X , $(a_\xi)_{\xi < \alpha}$ and $(b_\nu)_{\nu < \beta}$ such that

- $r(a_\xi, b_\nu) \geq r(b_\nu, a_\xi)$, for all $\xi < \alpha, \nu < \beta$.
- $(a_\xi)_{\xi < \alpha}$ is strictly increasing.
- $(b_\nu)_{\nu < \beta}$ is strictly decreasing.
- There is no $c \in E$ such that $r(a_\xi, c) \geq r(c, a_\xi)$ and $r(c, b_\nu) \geq r(b_\nu, c)$, for all $\xi < \alpha, \nu < \beta$.

Now we define a function $f : X \rightarrow X$ and proceed to verify that f is preserving order and has no fixed points. Let $x \in X$, let $f(x) = b_\nu$ if x is a upper bound of $(a_\xi)_{\xi < \alpha}$ where $\nu = \inf_{x \notin [a_\xi, b_{\nu'}]} \nu'$ i.e ν is least such that $r(x, b_\nu) \not\geq r(b_\nu, x)$. If x is not a upper bound of $(a_\xi)_{\xi < \alpha}$, let $f(x) = a_\xi$, where $\xi = \inf_{x \notin [a'_\xi, b_\nu]} \xi'$ i.e $\xi < \alpha$ is least such that $r(a_\xi, x) \not\geq r(x, a_\xi)$. By definition, either $r(x, f(x)) \not\geq r(f(x), x)$ or $r(f(x), x) \not\geq r(x, f(x))$, so f has no fixed points. We conclude by showing that f is order-preserving, For this, suppose that $r(x, y) \geq r(y, x)$. If x is a upper bound of $(a_\xi)_{\xi < \alpha}$ then y is also a upper bound of $(a_\xi)_{\xi < \alpha}$, let $f(x) = b_\nu$ and $f(y) = b_{\nu'}$, so $x \notin [a_\xi, b_\nu]$ and $y \notin [a_\xi, b_{\nu'}]$ for all $\xi < \alpha$, it's easy to see that $r(f(x), f(y)) \geq r(f(y), f(x))$. We can thus assume that x is not a upper bound of $(a_\xi)_{\xi < \alpha}$. Let $f(x) = a_{\xi_0}$ so $x \notin [a_{\xi_0}, b_\nu]$ for all $\nu < \beta$. We have $f(y) = b_{\nu_0}$ where $\nu_0 = \inf_{y \notin [a_\xi, b_{\nu'}]} \nu'$ if y is a upper bound of $(a_\xi)_{\xi < \alpha}$ or $f(y) = a_{\xi_1}$,

where $\xi_1 = \inf_{y \notin [a_{\xi_1}, b_v]} \xi'$ if not ($\xi_0 \leq \xi_1$ since if not we have $r(a_{\xi_1}, x) \not\geq r(x, a_{\xi_1})$) and in both cases we have $r(f(x), f(y)) \geq r(f(y), f(x))$.

Theorem 4.2. Let (P, r) be a non-empty r -fuzzy ordered set and let $f : P \rightarrow P$ be sequentially r -continuous, i.e., if $(c_n)_{n < \gamma}$ has a supremum c , then $r(\sup_n f(c_n), f(c)) \geq r(f(c), \sup_n f(c_n))$. Assume that there is $b \in P$ such that $r(b, f(b)) \geq r(f(b), b)$ and the set $\{x : r(b, x) \geq r(x, b)\}$ is a r -fuzzy complete lattice. Let $b' = \sup_n f^n(b)$, where f^n is the n th iterate of f . Then b' is a fixed point of f , and it is the least fixed point of f in $\{x : r(b, x) \geq r(x, b)\}$.

Proof. The obvious argument works. Just notice that f is order preserving since if $r(x, y) \geq r(y, x)$ then $y = \sup\{x, y\}$, so $r(f(x), \sup\{f(x), f(y)\}) \geq r(\sup\{f(x), f(y)\}, f(x))$ and $r(\sup\{f(x), f(y)\}, f(y)) \geq r(f(y), \sup\{f(x), f(y)\})$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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