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SOME GENERAL CONVERGENCE THEOREMS ON SEQUENCES OF FIXED POINTS

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Abstract. For each $n \in \mathbb{N}$(naturals), let $\{X_n\}$ be a sequence of nonempty subsets of a 2-metric space (X, ρ) and $\{T_n : X_n \to X_n\}$ a sequence of (ψ, ϕ)-weakly contractive mappings with respect to a different 2-metric ρ_n which converges in some general sense to a limit mapping. We obtain a number of stability results in this paper which extend certain well known results.

Keywords: (ψ, ϕ)-weakly contractive mappings; stability; 2-metric space; sequence of 2-metrics.

2000 AMS Subject Classification: 54H25, 47H10.

1. Introduction

Stability of fixed points of a sequence of mappings has been an interesting and continuing area of research in fixed point theory since its inception in 1962 when a result about the relationship between the convergence of a sequence of contraction mappings $\{T_n\}$ of a metric space X and their fixed points was obtained by Bonsall [5] (see also Nadler,Jr [24]). Subsequently, certain stability results for a sequence of contractive mappings were obtained by Fraser and Nadler [10]. Later on Rhoades [26] and Singh [28-30] studied the aspects of stability of fixed points in 2- metric spaces. In most of these results, pointwise

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and uniform convergence play a vital role in arriving at the desired conclusion. However, if the domain of definition of $\{T_n\}$ is different for each $n \in \mathbb{N}$, then the above notions do not work. An alternative to this problem has recently been presented by Barbet and Nachi [3-4], where some new notions of convergence have been introduced and utilized to obtain stability results in a metric space which extend the earlier results of Bonsall [5] and Nadler [24] over a variable domain. These results have been further generalized by Mishra et al. [16-22] in different settings. In this paper, we obtain a number of stability results in 2-metric spaces for a much wider class of (ψ, ϕ)-weakly contractive mappings (see[19, Remark1.1]) which include the well known contraction mappings, nonlinear contractions due to Boyd and Wong [7] and weakly contractive mappings (see [1], [27] for details). The results obtained herein thus compliment the results of Fraser and Nadler [10], Barbet and Nachi [4] and generalize the results of Mishra et al. [21] to 2-metric spaces.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ will denote the set of natural numbers while $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

For the sake of completeness and an easy reading, we recall some basic concepts on 2-metric spaces. For details we refer to Gähler [11] and Iséki [12-14].

Definition 2.1. Let X be a non-empty set. A real valued function $\rho : X \times X \times X \to \mathbb{R}$ is said to be a 2-metric on X if the following conditions are satisfied:

(a) To each pair of distinct points $a, b \in X$ there exists a point $c \in X$ such that $\rho(a, b, c) \neq 0$;

(b) If at least two of a, b, c are equal then $\rho(a, b, c) = 0$;

(c) $\rho(a, b, c) = \rho(b, c, a) = \rho(a, c, b)$ for all $a, b, c \in X$ (symmetry about three variables);

(d) $\rho(a, b, c) \leq \rho(a, b, d) + \rho(a, d, c) + \rho(d, b, c)$ for all $a, b, c, d \in X$ (triangular area inequality).

When ρ is a 2-metric on X, then the ordered pair (X, ρ) is called a 2-metric space. It is easily seen that ρ is non-negative and it abstracts the properties of the area function for

Euclidean triangles in the same manner as a metric abstracts the properties of the length function.

Definition 2.2. A sequence $\{x_n\}$ in a 2-metric space (X, ρ) is said to be convergent with limit $z \in X$ if $\lim_{n\to\infty} \rho(x_n, z, a) = 0$ for all $a \in X$. Notice that if the sequence $\{x_n\}$ converges to z, then $\lim_{n\to\infty} \rho(x_n, a, b) = \rho(z, a, b)$ for all $a, b \in X$. Further, the sequence $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{m,n\to\infty} \rho(x_m, x_n, a) = 0$ for all $a \in X$. If every Cauchy sequence in X is convergent, (X, ρ) is said to be a complete 2-metric space.

Definition 2.3. Let (X, ρ) be a 2-metric space and $T : X \to X$. Then T is called (ψ, ϕ)-weakly contractive if

$$\psi\left(\rho(Tx, Ty, a)\right) \leq \psi\left(\rho(x, y, a)\right) - \phi\left(\rho(x, y, a)\right) \tag{2.1}$$

for all $x, y, a \in X$ where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, ϕ is nondecreasing and ψ is increasing(strictly).

If we take $\psi(t) = t$ in (2.1), then it reduces to

$$\rho(Tx, Ty, a) \leq \rho(x, y, a) - \phi\left(\rho(x, y, a)\right) \tag{2.2}$$

for all $x, y, a \in X$. In this case, T is called weakly contractive mapping [27].

If we take $\phi(t) = t - \alpha(t)$ in (2.2), then it reduces to

$$\rho(Tx, Ty, a) \leq \alpha(\rho(x, y, a)) \tag{2.3}$$

for all $x, y \in X$, where $\alpha : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right and $\alpha(t) < t$ for $t > 0$. We note that $\alpha(0) = 0$. In this case, T is called a nonlinear contraction [7].

If we take $\phi(t) = (1 - k)t$ for $k \in [0, 1)$, $\psi(t) = t$ in (2.1), then it reduces to

$$\rho(Tx, Ty, a) \leq k\rho\left(x, y, a\right) \tag{2.4}$$

for all $x, y, a \in X$. In this case, T is called a k-contraction.

For corresponding definitions in a metric space, we refer to Mishra et al. [19] (see also [9]).

The class of (ψ, ϕ)- weakly contractive mappings was initially introduced and studied by Dutta and Choudhury [9] where both ϕ and ψ were assumed to be nondecreasing. However, recently it was observed by Bose and Roychowdhury [6] (see also [8]) that, in the above definition the function ψ must be assumed to be strictly increasing.

It is well known that a contraction mapping in a 2-metric space X has a unique fixed point. Initially, an additional requirement of boundedness was placed on X by Iséki et al. [14] and which was dispensed with subsequently by Rhoades [26] and Lal and Singh [15] independently. Note that, in a complete 2-metric space, a convergent sequence need not be Cauchy (see Naidu and Prasad [25, Example 0.1]), but every convergent sequence in a 2-mertic space (X, ρ) is Cauchy whenever ρ is continuous. However, the converse need not be true (see Naidu and Prasad [25, Example 0.2]). For some recent developments on fixed points in $2-$metric spaces, we refer to Aliouche and Simpson [2].

Throughout this paper, let (X, ρ) be a 2-metric space with a continuous 2-metric ρ.

3. Stability results in 2-metric spaces

First we recall the following definitions from Mishra et al. [20] (see also Barbet and Nachi [4] for a comparison).

Definition 3.1. Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a 2-metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$, a family of mappings. Then:

T_∞ is called a (G)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$, or equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G) if the following condition holds:

(G): $Gr(T_\infty) \subset \liminf Gr(T_n)$: for every $z \in X_\infty$, there exists a sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that

$$\lim_{n\to\infty} \rho(x_n, z, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n x_n, T_\infty z, a) = 0 \text{ for all } a \in X,$$

where $Gr(T)$ denotes the graph of T.

T_∞ is called a (H)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$, or equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (H) if the following condition holds:

(H): For all sequences $\{x_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$, there exists a sequence $\{y_n\}$ in X_∞ such that

$$\lim_{n\to\infty} \rho(x_n, y_n, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n x_n, T_\infty y_n, a) = 0 \text{ for all } a \in X.$$

The following classical results were obtained by Fraser and Nadler [10].

Theorem 3.2. [10, Theorem 2]. *Let (X, d) be a metric space and $\{d_n\}_{n\in\mathbb{N}}$ a sequence of metrics on X converging uniformly to d, where each d_n is equivalent to d. Let $\{T_n : X \to X\}_{n\in\mathbb{N}}$ be a sequence of contractive mappings on (X, d_n) converging pointwise to a mapping $T_\infty : X \to X$. If for each $n \in \mathbb{N}$, x_n is a fixed point of T_n, and if $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence converging to x_∞, then x_∞ is a fixed point of T_∞.*

Theorem 3.3. [10, Theorem 3]. *Let (X, d) be a metric space and $\{d_n\}_{n\in\mathbb{N}}$ a sequence of metrics on X converging uniformly to d. Let $\{T_n : X \to X\}_{n\in\mathbb{N}}$ be a sequence of $k-$contraction mappings on (X, d_n) converging pointwise to a mapping $T_\infty : X \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

The above theorems have recently been extended to nonlinear contraction mappings due to Boyd and Wong [7] by Mishra et al.[21].

Following Nachi [23](see also [21]), we have the following convergence properties in 2-metric spaces.

Definition 3.4. Let (X, ρ) be a 2-metric space. $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X and $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X. Then $\{\rho_n\}_{n\in\mathbb{N}}$ is said to satisfy property:

(A): For all $x \in X_\infty$, $a \in X$ and $\{x_n\}_{n\in\mathbb{N}} \in \prod\limits_{n\in\mathbb{N}} X_n$, $\lim\limits_{n\to\infty} \rho_n(x_n, x, a) = 0 \Leftrightarrow \lim\limits_{n\to\infty} \rho(x_n, x, a) = 0$.

($\mathbf{A}_0$): For all $x \in X_\infty$, $a \in X$ and $\{x_n\}_{n\in\mathbb{N}} \subset X$, $\lim\limits_{n\to\infty} \rho_n(x_n, x, a) = 0 \Leftrightarrow \lim\limits_{n\to\infty} \rho(x_n, x, a) = 0$.

(B): For all sequences $\{x_n\}_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$, there exists a sequence $\{y_n\}$ in X_∞ such that $\lim_{n\to\infty} \rho_n(x_n, y_n, a) = 0 \Leftrightarrow \lim_{n\to\infty} \rho(x_n, y_n, a) = 0$ for all $a \in X$.

(B_0): For all sequences $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset X$: $\lim_{n\to\infty} \rho_n(x_n, y_n, a) = 0 \Leftrightarrow \lim_{n\to\infty} \rho(x_n, y_n, a) = 0$.

4. Convergence of fixed points

In this section we present some results for a sequence $\{T_n\}_{n\in\mathbb{N}}$ of (ψ, ϕ)- weakly contractive mappings in 2-metric spaces. The domain of definition being different for each T_n, the convergence of $\{T_n\}_{n\in\mathbb{N}}$ under consideration will be in the sense of (G) and (H).

The following Proposition gives a sufficient condition for the existence of a unique G-limit.

Proposition 4.1. Let X be a 2-metric space, $\{X_n\}_{n\in\ \mathbb{N}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\mathbb{N}}$ a sequence of (ψ, ϕ)-weakly contractive mappings. If $T_\infty : X_\infty \to X$ is a (G)-limit of $\{T_n\}$, then T_∞ is unique.

Proof. Assume that $T_\infty : X_\infty \to X$ and $T^*_\infty : X_\infty \to X$ are (G)-limit mappings of the sequence $\{T_n\}$. Hence for any point $x \in X_\infty$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ converging to x such that $\{T_n x_n\}$ and $\{T_n y_n\}$ converge to T_∞ and T^*_∞ respectively. Therefore

$$\lim_{n\to\infty} \rho\,(T_n x_n, T_\infty x, a) = 0, \ \lim_{n\to\infty} \rho\,(\ T_n y_n, T^*_\infty x, a) = 0 \ \text{ for all } a \in X.$$

Since T_n is (ψ, ϕ)-weakly contractive for each $n \in \mathbb{N}$,

$$\psi(\rho(T_n x_n, T_n y_n, a)) \le \psi(\rho(x_n, y_n, a)) - \phi(\rho(x_n, y_n, a))$$

which implies that

$$\psi(\rho(T_n x_n, T_n y_n, a)) \le \psi(\rho(x_n, y_n, a)).$$

As ψ is increasing, from the above inequality we have

$$\rho(T_n x_n, T_n y_n, a) \le \rho(x_n, y_n, a). \tag{4.5}$$

By the triangular area inequality and condition (4.5), for all $n \in \mathbb{N}$ and for any $a \in X$, we have

$$\begin{aligned}\rho(T_\infty x, T^*_\infty x, a) &\leq \rho(T_\infty x, T^*_\infty x, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T^*_\infty x, a)\\ &\leq \rho(T_\infty x, T^*_\infty x, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T^*_\infty x, T_n y_n)\\ &\quad +\rho(T_n x_n, T_n y_n, a) + \rho(T_n y_n, T^*_\infty x, a)\\ &\leq \rho(T_\infty x, T^*_\infty x, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T^*_\infty x, T_n y_n)\\ &\quad +\rho(x_n, y_n, a) + \rho(T_n y_n, T^*_\infty x, a) \to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence we deduce that $\lim\limits_{n\to\infty} \rho(T_\infty x, T^*_\infty x, a) = 0$ and the unicity of the limit is established. □

When $\psi(t) = t$ and $\phi(t) = t - \alpha(t)$ in the above proposition, we get the following result.

Corollary 4.2. [18, Proposition 3.3]. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{ T_n : X_n \to X\}_{n\in\mathbb{N}}$ a sequence of nonlinear contraction mappings. If $T_\infty : X_\infty \to X$ is a (G)-limit of $\{T_n\}$, then T_∞ is unique.*

Now we present the following analogue of Theorem 3.2 to 2-metric spaces for the mappings satisfying condition (2.1).

Theorem 4.3. *Let (X, ρ) be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (A). Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of X and $\{T_n : X_n \to X_n\}_{n\in\mathbb{N}}$ a sequence of (ψ, ϕ)-weakly contractive mappings on (X_n, ρ_n) converging in the sense of (G) to a mapping $T_\infty : X_\infty \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n and if the sequence $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence converging to a point $x_\infty \in X_\infty$, then x_∞ is a fixed point of T_∞.*

Proof. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ converging to $x_\infty \in X_\infty$. Then by the property (G) there exists a sequence $\{y_n\} \in \prod\limits_{n\in\mathbb{N}} X_n$ such that:

$$\lim_{n\to\infty} \rho(y_n, x_\infty, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n y_n, T_\infty x_\infty, a) = 0 \text{ for all } a \in X.$$

Therefore by (A),

$$\lim_{n\to\infty}\rho_n(y_n,x_\infty,a)=0 \text{ and } \lim_{n\to\infty}\rho_n(T_ny_n,T_\infty x_\infty,a)=0.$$

If we define a sequence $\{z_n\}$ such that

$$\begin{aligned} z_{n_j} &= x_{n_j} \text{ for all } j\in\mathbb{N},\\ z_n &= y_n \text{ if } n\neq n_j, \text{ for any } j\in\mathbb{N}. \end{aligned}$$

Then $\lim_{n\to\infty}\rho(z_n,x_\infty,a)=0$ and so $\lim_{n\to\infty}\rho_n(z_n,x_\infty,a)=0$, by (A).

Now

$$\rho(z_n,y_n,a)\leq\rho(z_n,y_n,x_\infty)+\rho(z_n,x_\infty,a)+\rho(x_\infty,y_n,a)\to 0 \text{ as } n\to\infty,$$

and thus

$$\lim_{n\to\infty}\rho_n(z_n,y_n,a)=0.$$

Further, we have

$$\begin{aligned} \text{(4.6)}\quad \rho_{n_j}\left(T_{n_j}z_{n_j},T_\infty x_\infty,a\right) &\leq \rho_{n_j}\left(T_{n_j}z_{n_j},T_\infty x_\infty,T_{n_j}y_{n_j}\right)+\rho_{n_j}(T_{n_j}z_{n_j},T_{n_j}y_{n_j},a)\\ &\quad+\rho_{n_j}\left(T_{n_j}y_{n_j},T_\infty x_\infty,a\right). \end{aligned}$$

Since T_{n_j} is a (ψ,ϕ)-weakly contractive mapping on (X_{n_j},ρ_{n_j}) for each $j\in\mathbb{N}$, we have

$$\begin{aligned} \psi(\rho_{n_j}(T_{n_j}z_{n_j},T_{n_j}y_{n_j},a)) &\leq \psi\left(\rho_{n_j}(z_{n_j},y_{n_j},a)\right)-\phi\left(\rho_{n_j}(z_{n_j},y_{n_j},a)\right)\\ &\leq \psi\left(\rho_{n_j}(z_{n_j},y_{n_j},a)\right). \end{aligned}$$

By the monotonicity of ψ, we obtain

$$\text{(4.7)}\qquad \rho_{n_j}(T_{n_j}z_{n_j},T_{n_j}y_{n_j},a)\leq\rho_{n_j}(z_{n_j},y_{n_j},a).$$

From (4.6) and (4.7) we have

$$\rho_{n_j}\left(T_{n_j}z_{n_j},T_\infty x_\infty,a\right)\leq\rho_{n_j}\left(T_{n_j}z_{n_j},T_\infty x_\infty,T_{n_j}y_{n_j}\right)+\rho_{n_j}(z_{n_j},y_{n_j},a)+\rho_{n_j}\left(T_{n_j}y_{n_j},T_\infty x_\infty,a\right).$$

On taking the limit as $j\to\infty$, we obtain

$$\text{(4.8)}\qquad \lim_{j\to\infty}\rho_{n_j}\left(T_{n_j}z_{n_j},T_\infty x_\infty,a\right)=0.$$

Since $T_{n_j} z_{n_j} = T_{n_j} x_{n_j} = x_{n_j}$ and x_{n_j} converges to x_∞ as $j \to \infty$, (4.8) becomes $\rho_{n_j}(x_\infty, T_\infty x_\infty, a) = 0$ for all $a \in X$. Hence $T_\infty x_\infty = x_\infty$. □

Using Theorem 4.3, we can deduce the following results.

Corollary 4.4. *Let (X, ρ) be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (A). Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of X and $\{T_n : X_n \to X_n\}_{n\in\mathbb{N}}$ a sequence of k-contraction mappings on (X_n, ρ_n) converging in the sense of (G) to a mapping $T_\infty : X_\infty \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n and if the sequence $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence converging to a point $x_\infty \in X_\infty$, then x_∞ is a fixed point of T_∞.*

Proof. This comes from Theorem 4.3, when $\psi(t) = t$ and $\phi(t) = (1-k)t$, $k \in [0, 1)$. □

When $X_n = X$ for all $n \in \overline{\mathbb{N}}$ in Theorem 4.3, we have the following result:

Corollary 4.5. *Let X be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$, a sequence of 2-metrics on X satisfying the property (A_0). Let $\{T_n : X \to X\}_{n\in\mathbb{N}}$ be a sequence of (ψ, ϕ)-weakly contractive mappings on (X_n, ρ_n) converging pointwise to a mapping $T_\infty : X_\infty \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n and if the sequence $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence converging to a point $x_\infty \in X_\infty$, then x_∞ is a fixed point of T_∞.*

The following theorem which is an extension of Theorem 3.3 to 2-metric spaces is our first stability result.

Theorem 4.6. *Let (X, ρ) be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (A). Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of X and $\{T_n : X_n \to X_n\}_{n\in\mathbb{N}}$ a sequence of (ψ, ϕ)-weakly contractive mappings on (X_n, ρ_n) converging in the sense of (G) to a mapping $T_\infty : X_\infty \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. Let $x_\infty \in X_\infty$ and by the property (G), there exists a sequence $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that

$$\lim_{n\to\infty} \rho(y_n, x_\infty, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n y_n, T_\infty x_\infty, a) = 0$$

for all $a \in X$. By the property (A), we deduce that

$$\lim_{n\to\infty} \rho_n(y_n, x_\infty, a) = 0 \text{ and } \lim_{n\to\infty} \rho_n(T_n y_n, T_\infty x_\infty, a) = 0.$$

We have

$$\begin{aligned} \psi\left(\rho_n(x_n, x_\infty, a)\right) &= \psi\left(\rho_n(T_n x_n, T_\infty x_\infty, a)\right) \\ &\leq \psi\left(\rho_n(T_n x_n, T_\infty x_\infty, T_n y_n) + \rho_n(T_n x_n T_n y_n, a) + \rho_n(T_n y_n, T_\infty x_\infty, a)\right). \end{aligned}$$

Making $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned} \lim_{n\to\infty} \psi\left(\rho_n(x_n, x_\infty, a)\right) &\leq \lim_{n\to\infty} \psi\left(\rho_n(T_n x_n, T_\infty x_\infty, T_n y_n) + \rho_n(T_n x_n, T_n y_n, a) + \rho_n(T_n y_n, T_\infty x_\infty, a)\right) \\ &= \lim_{n\to\infty} \psi\left(\rho_n(T_n x_n, T_n y_n, a)\right) \\ &\leq \lim_{n\to\infty} \left[\psi\left(\rho_n(x_n, y_n, a)\right) - \phi\left(\rho_n(x_n, y_n, a)\right)\right] \\ &\leq \lim_{n\to\infty} \psi\left(\rho_n(x_n, y_n, x_\infty) + \rho_n(x_n, x_\infty, a) + \rho_n(x_\infty, y_n, a)\right) \\ &\quad - \lim_{n\to\infty} \phi\left(\rho_n(x_n, y_n, x_\infty) + \rho_n(x_n, x_\infty, a) + \rho_n(x_\infty, y_n, a)\right) \\ &= \lim_{n\to\infty} \psi\left(\rho_n(x_n, x_\infty, a)\right) - \lim_{n\to\infty} \phi\left(\rho_n(x_n, x_\infty, a)\right). \end{aligned}$$

Hence

$$\lim_{n\to\infty} \phi\left(\rho_n(x_n, x_\infty, a)\right) \leq 0.$$

By the property of ϕ we get

$$\lim_{n\to\infty} \rho_n(x_n, x_\infty, a) = 0$$

and the conclusion holds. □

Corollary 4.7. *Let (X, ρ) be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (A). Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of X and $\{T_n : X_n \to X_n\}_{n\in\mathbb{N}}$ a sequence of k-contraction mappings on (X_n, ρ_n) converging in the sense of (G) to a mapping $T_\infty : X_\infty \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. This comes from Theorem 4.6, when $\psi(t) = t$ and $\phi(t) = (1 - k)t$. □

If $X_n = X$ for all $n \in \overline{\mathbb{N}}$ in Theorem 4.6, then we have the following:

Corollary 4.8. *Let X be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (A_0). Let $\{T_n : X \to X\}_{n\in\mathbb{N}}$ be a sequence of (ψ, ϕ)-weakly contractive mappings on (X, ρ_n) converging pointwise to a mapping $T_\infty : X_\infty \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

The following theorem is our second stability result using the (H)-convergence in 2-metric spaces.

Theorem 4.9. *Let (X, ρ) be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (B). Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of X and $\{T_n : X_n \to X_n\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings on (X_n, ρ_n) converging in the sense of (H) to a (ψ, ϕ)-weakly contractive mapping $T_\infty : X_\infty \to X$. If for each $n \in \overline{\mathbb{N}}$, $x_n \in X_n$ is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. By the property (H), there exists a sequence $\{y_n\}$ in X_∞ such that

$$\lim_{n\to\infty} \rho(x_n, y_n, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n x_n, T_\infty y_n, a) = 0 \text{ for any } a \in X.$$

Therefore using the property (B), we have

$$\lim_{n\to\infty} \rho_n(x_n, y_n, a) = 0 \text{ and } \lim_{n\to\infty} \rho_n(T_n x_n, T_\infty y_n, a) = 0 \text{ for any } a \in X.$$

By the triangular area inequality,

$$\psi\left(\rho_n(x_n, x_\infty, a)\right) \leq \psi\left(\rho_n(T_n x_n, T_\infty x_\infty, T_\infty y_n) + \rho_n(T_n x_n, T_\infty y_n, a) + \rho_n(T_\infty y_n, T_\infty x_\infty, a)\right).$$

Taking the limit as $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} \psi\left(\rho_n(x_n, x_\infty, a)\right) &\leq \lim_{n\to\infty} \psi\left(\rho_n(T_\infty y_n, T_\infty x_\infty, a)\right) \\
&\leq \lim_{n\to\infty} \left[\psi\left(\rho_n(y_n, x_\infty, a)\right) - \phi\left(\rho_n(y_n, x_\infty, a)\right)\right] \\
&\leq \lim_{n\to\infty} \left[\psi\left(\rho_n(y_n, x_\infty, x_n) + \rho_n(y_n, x_n, a) + \rho_n(x_n, x_\infty, a)\right)\right] \\
&\quad - \lim_{n\to\infty} \left[\phi\left(\rho_n(y_n, x_\infty, x_n) + \rho_n(y_n, x_n, a) + \rho_n(x_n, x_\infty, a)\right)\right] \\
&= \lim_{n\to\infty} \left[\psi\left(\rho_n(x_n, x_\infty, a)\right) - \lim_{n\to\infty} \phi\left(\rho_n(x_n, x_\infty, a)\right)\right].
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \phi\left(\rho_n(x_n, x_\infty, a)\right) = 0$$

and the conclusion follows. □

Corollary 4.10. *Let (X, ρ) be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (B). Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of X and $\{T_n : X_n \to X_n\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings on (X_n, ρ_n) converging in the sense of (H) to a k-contraction mapping $T_\infty : X_\infty \to X$. If for each $n \in \overline{\mathbb{N}}$, $x_n \in X_n$ is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. This comes from Theorem 4.9, when $\psi(t) = t$ and $\phi(t) = (1 - k)t$. □

When $X_n = X$ for all $n \in \overline{\mathbb{N}}$ in Theorem 4.9, we obtain the following.

Corollary 4.11. *Let X be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (B_0). Let $\{T_n : X \to X\}$ be a sequence of mappings on (X, ρ_n) converging uniformly to a (ψ, ϕ)- weakly contractive mapping $T_\infty : X \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

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