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RESULTS ON CONDENSED KANNAN-TYPE 2-CYCLIC MAP IN b-METRIC SPACES

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Abstract. Some authors recently proposed a condensed Kannan-type map that can solve nonlinear problems with unique and non-unique solutions. However, these findings may not address all situations involving inexact spaces. This study presents a strategy for proving fixed points of Kannan-type 2-cyclic contractions by condensation in inexact spaces, commonly referred to as b-metric spaces. We further extend the findings to study fixed points of trivially cyclic mappings. With the aid of examples comprising cyclic and trivially cyclic mappings, we validate all hypotheses of this study. The results show that the condensed cyclic Kannan-type map is more elaborate than the previous Kannan-type cyclic maps in the literature, solves problems with the inexact structures, and ensures unique and non-unique fixed points.

Keywords: Fixed point; b-metric spaces; condensed Kannan-type map.

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1. Introduction

A b-metric space is one of the generalizations of a standard metric space, where a real constant $s \geq 1$ scales the triangle inequality. It is widely used in the study of fixed point theorems of

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differential equations, integral equations, integro-differential equations, and optimization models, in which the metric points behave irregularly.

Definition 1.1. [1] Let $\mathbb{X}$ be a non-empty set and $s \geq 1$ is a real constant. A function $k_b : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ is called a b-metric if for every $u, v, w \in \mathbb{X}$, the following axioms hold:

b_1: $k_b(u,v) = 0$ if and only if $u = v$;

b_2: $k_b(u,v) = k_b(v,u)$;

b_3: $k_b(u,w) \leq s[k_b(u,v) + k_b(v,w)]$.

If $\mathbb{X}$ is endowed with the b-metric, then the triple $(\mathbb{X}, k_b, s)$ is called a b-metric space.

Example 1.2. [2] Let $(\mathbb{X}, k)$ be a metric space. Consider a convex function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\phi(t) = t^q$, for $q > 1$ and $t \in \mathbb{R}^+$, then a function $k_b : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ with coefficient $s = 2^{q-1}$ defined by

$$k_b(x,y) = \phi(k(u,v)) = [k(u,v)]^q, \text{ for all } u, v \in \mathbb{X}$$

is a b-metric space but not the standard metric space.

Definition 1.3. Let $(\mathbb{X}, k_b, s)$ be a b-metric space with $s \geq 1$. A sequence $\{u_n\} \in \mathbb{X}$ is called:

(a) b-convergent if there exists $u \in \mathbb{X}$ such that $\lim_{n\to\infty} k_b(u_n, u) = 0$.

(b) b-Cauchy sequence in $\mathbb{X}$ if $\lim_{n\to\infty} k_b(u_n, u_m) = 0$.

(c) complete b-metric space if every b-Cauchy sequence in $\mathbb{X}$ is convergent.

In an attempt to study the existence properties of discontinuous operators, Kannan [3] introduced two symmetry distance functions to establish a fixed point theorem. Karapinar [4] defined a Kannan-type map, called interpolative Kannan-type map, by multiplying the two symmetry functions with fractional powers as follows:

$$k(\mathfrak{T}u, \mathfrak{T}v) \leq \mu k(u, \mathfrak{T}u)^{\alpha} k(v, \mathfrak{T}v)^{1-\alpha}, \ \forall u, v \in (\mathbb{X}, k), \tag{1}$$

for $\mu \in [0,1)$ and $\alpha \in (0,1)$. In [5], Karapinar et al. pointed out that condition (1) is only applicable to study non-unique fixed point problems. Since then, several authors have studied the interpolative Kannan-type map in various distance spaces, see Ref. [6, 7, 8, 9, 10]. See also Ref. [11, 12, 13] for improved versions of inequality (1) that apply to unique and non-unique problems.

Kirk et al. [14] introduced the extension of Banach fixed point theorem to the class of cyclic maps and the cyclic contractive map. They proved the existence of unique fixed points in different types of cyclical contractive maps in the standard metric spaces.

Definition 1.4. [14] Let $(\mathbb{X},k)$ be a metric space. Let $\mathbb{P}$ and $\mathbb{Q}$ be two nonempty subsets of $\mathbb{X}$. A mapping $\mathfrak{I}:\mathbb{P}\cup\mathbb{Q}\to\mathbb{P}\cup\mathbb{Q}$ is said to be a cyclic mapping provided,

$$\mathfrak{I}(\mathbb{P})\subseteq\mathbb{Q},\mathfrak{I}(\mathbb{Q})\subseteq\mathbb{P}.$$

The authors proved the fixed point theorem for cyclic contraction map as follows:

Theorem 1.5. [14] *Let $(\mathbb{X},k)$ be a complete metric space and let $\mathbb{P}$ and $\mathbb{Q}$ be two nonempty subsets of $\mathbb{X}$. Suppose that $\mathfrak{I}:\mathbb{P}\cup\mathbb{Q}\to\mathbb{P}\cup\mathbb{Q}$ is a cyclic contraction and there exists $a\in(0,1)$ such that*

$$k(\mathfrak{I}u,\mathfrak{I}v)\le ak(u,v) \tag{2}$$

for all $u\in\mathbb{P}$ and $v\in\mathbb{Q}$. Then, $\mathfrak{I}$ has a unique fixed point in $\mathbb{P}\cap\mathbb{Q}$.

Definition 1.6. [3, 15] Let $(\mathbb{X},k)$ be a metric space and let $\mathbb{P}$ and $\mathbb{Q}$ be two nonempty subsets of $\mathbb{X}$. A cyclic map $\mathfrak{I}:\mathbb{P}\cup\mathbb{Q}\to\mathbb{P}\cup\mathbb{Q}$ is said to be a Kannan-type cyclic contraction if there exists $c\in(0,\frac{1}{2})$ such that

$$k(\mathfrak{I}u,\mathfrak{I}v)\le c[k(\mathfrak{I}u,u)+k(\mathfrak{I}v,v)],\forall u\in\mathbb{P},v\in\mathbb{Q}. \tag{3}$$

Researchers proved some fixed point theorems by introducing the interpolative versions for cyclic contractions. For clarity, we state two previous results to clearly identify the improvements of this paper.

Theorem 1.7. [16] *Let $(\mathbb{X},k)$ be a complete metric space and let $\mathbb{P}$ and $\mathbb{Q}$ be two non-empty subsets of $\mathbb{X}$. If $\mathfrak{I}:\mathbb{P}\cup\mathbb{Q}\to\mathbb{P}\cup\mathbb{Q}$ is an interpolative Kannan-type cyclic contraction, then $\mathfrak{I}$ has a unique fixed point in $\mathbb{P}\cap\mathbb{Q}$.*

The following results by Devi and Debnath [15] extended Theorem 1.7.

Theorem 1.8. [15] *Let* $(\mathbb{X}, k_b, s)$ *be a complete b-metric space and let* $\mathbb{P}$ *and* $\mathbb{Q}$ *be two non-empty subsets of* $\mathbb{X}$. *If* $\mathfrak{I} : \mathbb{P} \cup \mathbb{Q} \to \mathbb{P} \cup \mathbb{Q}$ *is an interpolative Kannan-type cyclic contraction, then* $\mathfrak{I}$ *has a unique fixed point in* $\mathbb{P} \cap \mathbb{Q}$.

This paper aims to suggest a more elaborate hybrid Kannan-type map and to develop a strategy for proving fixed point results of cyclic and trivially cyclic Kannan-type contraction by means of condensation in b-metric spaces.

2. Main Result

This section is devoted to generalizing fixed point theorems of Kannan-type cyclic contractions in *b*-metric spaces. Throughout, neither a cyclic map $\mathfrak{I}$ nor a trivial cyclic map $\mathfrak{T}$ is continuous on $\mathbb{X}$. We start by defining the following:

Definition 2.1. Let $(\mathbb{X}, k_b, s)$ be a complete *b*-metric space and let $\mathbb{P}$ and $\mathbb{Q}$ be two non-empty subsets of $\mathbb{X}$. A self-map $\mathfrak{I} : \mathbb{P} \cup \mathbb{Q} \to \mathbb{P} \cup \mathbb{Q}$ is said to be a condensed Kannan-type cyclic contraction (CKCC) if

I. $\mathfrak{I}(\mathbb{P}) \subseteq \mathbb{Q}$ and $\mathfrak{I}(\mathbb{Q}) \subseteq \mathbb{P}$;

II. there exist real constants $\lambda \in [0, \frac{1}{2s})$, $\mu \in [0, \frac{1}{s})$, and $\alpha \in [0, 1)$ such that

$$k_b(\mathfrak{I}u, \mathfrak{I}v) \begin{cases} \leq \lambda \left[k_b(u, \mathfrak{I}u)^{2\alpha} + k_b(v, \mathfrak{I}v)^{2(1-\alpha)} \right], & \text{(CKCC1)} \\ \\ \geq \mu k_b(u, \mathfrak{I}u)^{\alpha} k_b(v, \mathfrak{I}v)^{1-\alpha}, & \text{(CKCC2)} \end{cases} \tag{4}$$

for all $(u, v) \in \mathbb{P} \times \mathbb{Q}$ and $u, v \in \mathbb{X} \backslash Fix(\mathfrak{I})$.

A fixed point theorem of an operator satisfying (4) is presented as follows:

Theorem 2.2. *Let* $(\mathbb{X}, k_b, s)$ *be a complete b-metric space and let* $\mathbb{P}$ *and* $\mathbb{Q}$ *be two non-empty subsets of* $\mathbb{X}$. *If* $\mathfrak{I} : \mathbb{P} \cup \mathbb{Q} \to \mathbb{P} \cup \mathbb{Q}$ *is a CKCC map, then* $\mathfrak{I}$ *possesses a unique fixed point in* $\mathbb{P} \cap \mathbb{Q}$.

Proof. Let $u_0 \in \mathbb{P} \cup \mathbb{Q}$ be a generic point and define a sequence $\{u_n\} \subset \mathbb{P} \cup \mathbb{Q}$ such that

$$\mathfrak{I}u_n = u_{n+1}, \ \forall n = 0, 1, 2, \ldots. \tag{5}$$

Since $\Im$ is a cyclic map, then for every $u_0 \in \mathbb{P}$ there is a point $u_1 = \Im(u_0) \in \mathbb{Q}$. By induction, $u_{2n} = \Im^{2n}u_0 \in \mathbb{P}$ for all even $n \in \mathbb{N} \cup \{0\}$ and $u_{2n+1} = \Im^{2n+1}u_0 \in \mathbb{Q}$ for all odd $n \in \mathbb{N} \cup \{0\}$. By the hypothesis on $\Im$, we have

$$\begin{aligned} k_b(u_{2n}, u_{2n+1}) &= k_b(\Im^{2n}u_0, \Im^{2n+1}u_0) \\ &\leq \lambda \left[k_b(\Im^{2n-1}u_0, \Im^{2n}u_0)^{2\alpha} + k_b(\Im^{2n}, \Im^{2n+1}u_0)^{2(1-\alpha)} \right] \\ &= \lambda \left[k_b(u_{2n-1}, u_{2n})^{2\alpha} + k_b(u_{2n}, u_{2n+1})^{2(1-\alpha)} \right] \\ &= \lambda \left\{ \left[k_b(u_{2n-1}, u_{2n})^{\alpha} + k_b(u_{2n}, u_{2n+1})^{(1-\alpha)} \right]^2 - 2k_b(u_{2n-1}, u_{2n})^{\alpha} k_b(u_{2n}, u_{2n+1})^{(1-\alpha)} \right\}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} k_b(u_{2n}, u_{2n+1}) &= k_b(\Im^{2n}u_0, \Im^{2n+1}u_0) \\ &\geq \mu k_b(\Im^{2n-1}u_0, \Im^{2n}u_0)^{\alpha} k_b(\Im^{2n}, \Im^{2n+1}u_0)^{1-\alpha} \\ &= \mu k_b(u_{2n-1}, u_{2n})^{\alpha} k_b(u_{2n}, u_{2n+1})^{1-\alpha}. \end{aligned} \tag{7}$$

By combining the inequalities (6) and (7), we have

$$4\lambda k_b(u_{2n-1}, u_{2n})^{\alpha} k_b(u_{2n}, u_{2n+1})^{1-\alpha} \leq \lambda \left[k_b(u_{2n-1}, u_{2n})^{\alpha} + k_b(u_{2n}, u_{2n+1})^{(1-\alpha)} \right]^2, \ \mu = 2\lambda \in (0, \frac{1}{s}),$$

or

$$k_b(u_{2n-1}, u_{2n})^{\frac{\alpha}{2}} k_b(u_{2n}, u_{2n+1})^{\frac{1-\alpha}{2}} \leq \frac{k_b(u_{2n-1}, u_{2n})^{\alpha} + k_b(u_{2n}, u_{2n+1})^{(1-\alpha)}}{2} \tag{8}$$

$$\leq \max \left\{ k_b(u_{2n-1}, u_{2n})^{\alpha}, k_b(u_{2n}, u_{2n+1})^{(1-\alpha)} \right\}. \tag{9}$$

Let $k_b(u_{2n-1}, u_{2n})^{\alpha} = \max \left\{ k_b(u_{2n-1}, u_{2n})^{\alpha}, k_b(u_{2n}, u_{2n+1})^{(1-\alpha)} \right\}$ for all $\alpha \in (0,1)$, then the inequality (8) becomes

$$k_b(u_{2n-1}, u_{2n})^{\alpha} \geq k_b(u_{2n}, u_{2n+1})^{1-\alpha}. \tag{10}$$

Also, if $k_b(u_{2n}, u_{2n+1})^{1-\alpha} = \max \left\{ k_b(u_{2n-1}, u_{2n})^{\alpha}, k_b(u_{2n}, u_{2n+1})^{(1-\alpha)} \right\}$, then inequality (8) reduces to

$$k_b(u_{2n-1}, u_{2n})^{\alpha} \leq k_b(u_{2n}, u_{2n+1})^{1-\alpha} \tag{11}$$

Thus, equality holds for both (10) and (11) at large n. So, let p be a non-negative number such that

$$p = \lim_{n\to\infty} k_b(u_{2n-1}, u_{2n})^{\alpha} = \lim_{n\to\infty} k_b(u_{2n}, u_{2n+1})^{1-\alpha} \tag{12}$$

Using (12) in (6), we obtain

$$p^{\frac{1}{1-\alpha}} \leq 2\lambda p^2 = \mu p^2$$

This further gives

$$p^{2\alpha-1} \leq \mu < \frac{1}{s}.$$

This is a contradiction for $s \geq 1$ and $\alpha \in (0,1)$. Thus, $p = 0$.

Obviously, the sequence $\{u_n\}$ is a b-Cauchy sequence. Assume on contrary that it is not, then for $\varepsilon > 0$ there exist integers $n_k, m_k \geq k$ for some k such that $u_{n_k} \in \mathbb{P}$ and $u_{m_k} \in \mathbb{Q}$,

$$k_b(u_{n_k}, u_{m_k}) \geq \varepsilon \ \text{ and } \ k_b(u_{n_k}, u_{m_k-1}) < \varepsilon$$

But then,

$$\begin{aligned}\varepsilon \leq k_b(u_{n_k}, u_{m_k}) &\leq s[k_b(u_{n_k}, u_{m_k-1}) + k_b(u_{m_k-1}, u_{m_k})] \\ &< s[\varepsilon + k_b(u_{m_k-1}, u_{m_k})]\end{aligned}$$

and by taking limit as $n \to \infty$ over the last inequalities, we get

$$\varepsilon < s\varepsilon.$$

This is a contradiction if $s = 1$. Hence, we can conclude that $\{u_n\}$ is Cauchy. However, suppose that $s > 1$ and $\varepsilon > 0$, then

$$\begin{aligned}\varepsilon \leq k_b(u_{n_k}, u_{m_k}) &\leq s[k_b(u_{n_k}, u_{n_k+1}) + k_b(u_{n_k+1}, u_{m_k})] \\ &= sk_b(u_{n_k}, u_{n_k+1}) + sk_b(\Im u_{n_k}, \Im u_{m_k-1}) \\ &\leq sk_b(u_{n_k}, u_{n_k+1}) + s\lambda[k_b(u_{n_k}, u_{n_k+1})^{2\alpha} + k_b(u_{m_k-1}, u_{m_k})^{2(1-\alpha)}] \end{aligned} \tag{13}$$

Taking limit as $n \to \infty$ across (13), this gives $\varepsilon \leq 0$ which is a contradiction.

Hence, $\{u_n\}$ is a b-Cauchy sequence.

By the completeness of $(\mathbb{X}, k_b, s)$, the sequence $\{u_n\}$ converges to a point $u^* \in \mathbb{P} \cup \mathbb{Q}$ for which

$$u_n \to u^* \text{ as } n \to \infty.$$

Next, we show that u^* is a fixed point of $\mathfrak{I}$, that is, $u^* = \mathfrak{I}u^*$. Assume that this is not true. Let $u^* \in \mathbb{P}$ and $\mathfrak{I}u^* \in \mathbb{Q}$, then by hypotheses

$$\begin{aligned} k_b(u^*, \mathfrak{I}u^*) &\leq s(k_b(u^*, u_{n+1}) + k_b(u_{n+1}, \mathfrak{I}u^*) \\ &\leq sk_b(u^*, u_{n+1}) + s\lambda[k_b(u_n, \mathfrak{I}u_n)^{2\alpha} + k_b(u^*, \mathfrak{I}u^*)^{2(1-\alpha)}] \\ &= sk_b(u^*, u_{n+1}) + s\lambda k_b(u_n, u_{n+1})^{2\alpha} + s\lambda k_b(u^*, \mathfrak{I}u^*)^{2(1-\alpha)} \end{aligned}$$

Taking limit as $n \to \infty$, we obtain

$$k_b(u^*, \mathfrak{I}u^*) \leq s\lambda k_b(u^*, \mathfrak{I}u^*)^{2(1-\alpha)}$$

This further implies that

$$k_b(u^*, \mathfrak{I}u^*)^{2\alpha - 1} \leq s\lambda < \frac{1}{2},$$

which lead to a contradiction for some $s \geq 1$ and $\alpha \in (0, 1)$.

Therefore, $k_b(u^*, \mathfrak{I}u^*) = 0 \Leftrightarrow u^* = \mathfrak{I}u^*$..

For uniqueness, let u^* and v^* be two fixed points of $\mathfrak{I}$ such that $u^* \neq v^*$. By using (CKCC1) and (CKCC2), we obtain, respectively,

$$k_b(u^*, v^*) = k_b(\mathfrak{I}u^*, \mathfrak{I}v^*) \leq \lambda[k_b(u^*, \mathfrak{I}u^*)^{2\alpha} + k_b(v^*, \mathfrak{I}v^*)^{2(1-\alpha)}] = 0,$$

and

$$k_b(u^*, v^*) = k_b(\mathfrak{I}u^*, \mathfrak{I}v^*) \geq \mu k_b(u^*, \mathfrak{I}u^*)^{2\alpha} k_b(v^*, \mathfrak{I}v^*)^{2(1-\alpha)} = 0,$$

which lead to $k_b(u^*, v^*) = 0$, a contradiction. Therefore, $u^* = v^* \in \mathbb{P} \cap \mathbb{Q} \subset \mathbb{X}$. □

Example 2.3. Let $\mathbb{X} = [0, 2]$ be divided into $\mathbb{P} = [0, 1]$ and $\mathbb{Q} = [1, 2]$. Define $k_b : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ as $k_b(u, v) = (u - v)^q$ for $q < 2$ and for all $u, v \in \mathbb{X}$.

Also, let $\mathfrak{I} : \mathbb{P} \cup \mathbb{Q} \to \mathbb{P} \cup \mathbb{Q}$ be a self-map defined by

$$\mathfrak{I}u = \begin{cases} 2 - u & \text{if } u \in \mathbb{P} \\ \frac{u-1}{2} & \text{if } u \in \mathbb{Q}. \end{cases}$$

Then, $\mathfrak{I}$ is a CKCC map and $Fix(\mathfrak{I}) = \mathbb{P} \cap \mathbb{Q} = \{1\}$.

Firstly, it should be observed that $\mathfrak{I}$ is not a continuous map.
Secondly, since $\mathfrak{I}(\mathbb{P}) \subseteq \mathbb{Q}$ and $\mathfrak{I}(\mathbb{Q}) = [0, \frac{1}{2}] \subset \mathbb{P}$, then $\mathfrak{I}$ is a cyclic map.
More so, for $u \in \mathbb{P} \backslash \{1\}$ and $v \in \mathbb{Q} \backslash \{1\}$, select $u = 0$ and $v = 1.2$ with $q = 1.15$, $\alpha = 0.85$, $\mu = 0.9$, and $\lambda = 0.45$, then $\mathfrak{I}$ satisfies all hypothesis of Theorem 2.2.
For the above selection, however, the cyclic map $\mathfrak{I}$ is neither a Kannan cyclic contraction nor an interpolative Kannan-type cyclic contraction.
That is, for

$$k_b(\mathfrak{I}u, \mathfrak{I}v) = (\mathfrak{I}u - \mathfrak{I}v)^{1.15} = (2 - 0.1)^{1.15} \approx 2.09202.$$

$$k_b(u, \mathfrak{I}u) = (u - \mathfrak{I}u)^{1.15} = (0 - 2)^{1.15} \approx 2.21914.$$

$$k_b(v, \mathfrak{I}v) = (v - \mathfrak{I}v)^{1.15} = (1.2 - 0.1)^{1.15} \approx 1.11584.$$

Kannan cyclic contraction map:

- $k_b(\mathfrak{I}u, \mathfrak{I}v) = 2.09202 > 1.50071 \approx 0.45 \cdot [k_b(u, \mathfrak{I}u) + k_b(v, \mathfrak{I}v)]$.

Interpolative Kannan-type cyclic contraction:

- $k_b(\mathfrak{I}u, \mathfrak{I}v) = 2.09202 > 1.80149 \approx 0.90 \cdot k_b(u, \mathfrak{I}u)^{0.85} k_b(v, \mathfrak{I}v)^{0.15}$.

Remark 2.4. If the self-map $\mathfrak{I} : \mathbb{P} \cup \mathbb{Q} \to \mathbb{P} \cup \mathbb{Q}$ is trivially cyclic, that is, $T(\mathbb{P}) \subseteq \mathbb{Q}$ and $T(\mathbb{Q}) \subseteq \mathbb{P}$ such that $\mathbb{P} = \mathbb{Q}$, then Definition 2.1 can be restated as presented in the next definition.

Definition 2.5. Let $(\mathbb{X}, k_b, s)$ be a b-metric space. A self-map $\mathfrak{T} : \mathbb{X} \to \mathbb{X}$ is a condensed Kannan-type contraction if there exist $\alpha \in (0, 1)$, $\lambda \in [0, \frac{1}{2s})$ and $\mu \in [0, \frac{1}{s})$, and $s \geq 1$ such that

$$k_b(\mathfrak{T}u, \mathfrak{T}v) \begin{cases} \leq \lambda \left[k_b(u, \mathfrak{T}u)^{2\alpha} + k_b(v, \mathfrak{T}v)^{2(1-\alpha)} \right], & \text{(CKC1)} \\ \\ \geq \mu k_b(u, \mathfrak{T}u)^{\alpha} k_b(v, \mathfrak{T}v)^{1-\alpha}, & \text{(CKC2)} \end{cases} \tag{14}$$

for all $u, v \in \mathbb{X} \backslash Fix(\mathfrak{T})$.

In the light of Definition 2.5, we present the following theorem:

Theorem 2.6. *Let $(\mathbb{X}, k_b, s)$ be a complete b-metric space and $\mathfrak{T} : \mathbb{X} \to \mathbb{X}$ be a condensed Kannan-type contraction. Then, T possesses a unique fixed point in $\mathbb{X}$.*

Proof. Let $u_0 \in \mathbb{X}$ be generic and define the sequence $\{u_n\} \subset \mathbb{X}$ by the Picard operator $\mathfrak{T}u_n = u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

If there exists a number $n_0 \in \mathbb{N} \cup \{0\}$ such that $u_{n_0+1} = \mathfrak{T}u_{n_0} = u_{n_0}$, then the proof is over. Now, let $u_{n+1} \neq \mathfrak{T}u_n$ for each $n \in \mathbb{N}_0$; using the (CKC1) of (14), we get

$$\begin{aligned} k_b(u_n, u_{n+1}) &= k(\mathfrak{T}u_{n-1}, \mathfrak{T}u_n) \\ &\leq \lambda \left[k_b(u_{n-1}, \mathfrak{T}u_{n-1})^{2\alpha} + k_b(u_n, \mathfrak{T}u_n)^{2(1-\alpha)} \right] \\ &= \lambda \left[k_b(u_{n-1}, u_n)^{2\alpha} + k_b(u_n, u_{n+1})^{2(1-\alpha)} \right] \end{aligned} \tag{15}$$

More so, by condition (CKC2), we get

$$k_b(u_n, u_{n+1}) \geq \mu k_b(u_{n-1}, u_n)^{\alpha} k_b(u_n, u_{n+1})^{1-\alpha} \tag{16}$$

From (15) and (16), the following inequality is obtained:

$$\mu k_b(u_{n-1}, u_n)^{\alpha} k_b(u_n, u_{n+1})^{1-\alpha} \leq \lambda [k_b(u_{n-1}, u_n)^{2\alpha}] + \lambda [k_b(u_n, u_{n+1})^{2(1-\alpha)}] \tag{17}$$

By resolving (17), we have the following two sub-inequalities:

$$\begin{aligned} \frac{\mu}{2} k_b(u_{n-1}, u_n)^{\alpha} k_b(u_n, u_{n+1})^{1-\alpha} &\leq \lambda k_b(u_{n-1}, u_n)^{2\alpha} \\ \Rightarrow k_b(u_n, u_{n+1})^{1-\alpha} &\leq k_b(u_{n-1}, u_n)^{\alpha} \end{aligned} \tag{18}$$

and

$$\begin{aligned} \frac{\mu}{2} k_b(u_{n-1}, u_n)^{\alpha} k_b(u_n, u_{n+1})^{1-\alpha} &\leq \lambda k_b(u_n, u_{n+1})^{2(1-\alpha)} \\ \Rightarrow k_b(u_{n-1}, u_n)^{\alpha} &\leq k_b(u_n, u_{n+1})^{1-\alpha} \end{aligned} \tag{19}$$

Combination of (18) and (19) gives the recurrence relation

$$k_b(u_{n-1}, u_n)^{\alpha} = k_b(u_n, u_{n+1})^{1-\alpha}, \text{ as } n \to \infty. \tag{20}$$

The rest of the proof follows from Theorem 2.2. □

Example 2.7. Let $\mathbb{X} = \{u_1, u_2, u_3\}$ be endowed by the function $k_b : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$. For all $u, v \in \mathbb{X}$, define $k_b : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ by

$$k_b(u_1, u_2) = 0.25, \; k_b(u_1, u_3) = 2.25, \text{ and } \; k_b(u_2, u_3) = 1.$$

Then, $k_b(u,v)$ is not a standard metric space but it is a b-metric space with $s = \frac{9}{5}$. Furthermore, define a sequence $\{u_n\} \subset \mathbb{X}$ by

$$u_n = \begin{cases} u_2 \text{ if } n \leq 5, \\ u_3 \text{ if } n > 5. \end{cases}$$

Then, $\{u_n\}$ is a Cauchy sequence since for every topology $\mathscr{U} \subset \mathbb{X}$ of $\{u_n\}$, there exists $m \in \mathbb{N}$ with $m > n$ such that $u_m \in \mathscr{U}$. More so, $\lim_{n\to\infty} u_n = u_3 \in \mathbb{X}$.
For the above reasons, $(\mathbb{X}, k_b, \frac{9}{5})$ is a complete b-metric space.

Now, let us consider the map

$$\mathfrak{T}u = \begin{cases} u_1 \text{ if } u \in \{u_1, u_2\} \\ u_2 \text{ if } u = u_3 \end{cases}$$

with $Fix(\mathfrak{T}) = \{u_1\}$.
For $u, v \in \mathbb{X} \setminus Fix(\mathfrak{T})$, select $\lambda = 0.27$, $\mu = 0.54$, and $\alpha = 0.7$, then $\mathfrak{T}$ (not continuous) fulfills all hypotheses of Theorem 2.6. Obviously, T is not an interpolative Kannan-type map on $\mathbb{X}$. This is because for $(u_2, u_3) \in \mathbb{X} \times \mathbb{X}$,

$$k_b(\mathfrak{T}u_2, \mathfrak{T}u_3) = 0.2500 > 0.2046 \approx \mu k_b(u_2, \mathfrak{T}u_2)^{\alpha} k_b(u_3, \mathfrak{T}u_3)^{1-\alpha}.$$

Some consequences of Theorem 2.2 (and Theorem 2.6) are presented as follows:

Corollary 2.8. *Let $(\mathbb{X}, k_b, s)$ be a complete b-metric space and let $\mathfrak{I} : \mathbb{P} \cup \mathbb{Q} \to \mathbb{P} \cup \mathbb{Q}$ be a CKCC map such that*

$$k_b(\mathfrak{I}u, \mathfrak{I}v) \begin{cases} \leq \lambda \left[k_b(u, \mathfrak{I}u) + k_b(v, \mathfrak{I}v)\right], \\ \geq \mu k_b(u, \mathfrak{I}u)^{\frac{1}{2}} k_b(v, \mathfrak{I}v)^{\frac{1}{2}} \end{cases} \tag{21}$$

where $\lambda \in [0, \frac{1}{2s})$ and $\mu \in [0, \frac{1}{s})$. Then, $\mathfrak{I}$ has a unique fixed point in $\mathbb{P} \cap \mathbb{Q}$.

Proof. By setting $\alpha = \frac{1}{2}$ in Theorem 2.2, the proof is a routine. □

Corollary 2.9. *Let $(\mathbb{X}, k_b, s)$ be a complete b-metric space and let $\mathfrak{I} : \mathbb{P} \cup \mathbb{Q} \to \mathbb{P} \cup \mathbb{Q}$ be a CKCC map such that*

$$k_b(\mathfrak{I}u, \mathfrak{I}v) \leq \lambda \left[k_b(u, \mathfrak{I}u)^{2\alpha} + k_b(v, \mathfrak{I}v)^{2(1-\alpha)}\right], \tag{22}$$

for all $u,v \in \mathbb{X}$, where $\lambda \in [0,\frac{1}{2s})$ and $\alpha \in (0,1)$. Then, $\mathfrak{I}$ possesses a unique fixed point in $\mathbb{P} \cap \mathbb{Q}$.

Proof. The proof follows by replacing the set $\mathbb{X} \backslash Fix(\mathfrak{I})$ with $\mathbb{X}$ in Theorem 2.2. □

Corollary 2.10. *Let $(\mathbb{X}, k_b, s)$ be a complete b-metric space. If $\mathfrak{I} : \mathbb{P} \cup \mathbb{Q} \to \mathbb{P} \cup \mathbb{Q}$ is a cyclic map satisfying*

$$k_b(\mathfrak{I}u, \mathfrak{I}v) \le \lambda \left[k_b(u, \mathfrak{I}u) + k_b(v, \mathfrak{I}v)\right], \tag{23}$$

for all $u,v \in \mathbb{X}$, where $\lambda \in [0,\frac{1}{2s})$. Then, $\mathfrak{I}$ possesses a unique fixed point in $\mathbb{P} \cap \mathbb{Q}$.

Proof. The proof follows from Corollary 2.9 with $\alpha = \frac{1}{2}$. □

Remark 2.11. Corollaries 2.8, 2.9, and 2.10 are valid if $\mathfrak{I} : \mathbb{P} \cup \mathbb{Q} \to \mathbb{P} \cup \mathbb{Q}$ is a trivial cyclic map.

3. Conclusion

This paper studied the fixed point theorems of cyclic and trivially cyclic maps of Kannan-type contraction using a technique of condensation in inexact spaces. Two practical examples were considered to validate the claims in this paper. Both examples satisfied the hypotheses of Theorem 2.2, Theorem 2.6, and Corollary 2.9, and also showed superiority over previous cyclic Kannan-type maps in the literature. This indicated that the CKCC and CKC maps were more versatile than some of the existing cyclic Kannan-type maps in inexact spaces. The reliability and performance rates of the CKCC and CKC maps concerning the fractional power α and parameters s, λ, and μ in inexact spaces could be examined for further studies.

Conflict of Interests

The authors declare that there is no conflict of interests.

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