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A COMMON SOLUTION OF CONSTRAINED CONVEX MINIMIZATION PROBLEM, GENERALIZED EQUILIBRIUM PROBLEM AND FIXED POINT PROBLEM OF DIRECTED NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we introduce an iterative algorithm that approximates a common solution of a constrained minimization problem of convex function, a generalized equilibrium problem involving averaged mapping, and fixed point problem of a directed nonexpansive mapping. We prove the strong convergence of the proposed iterative algorithm to a common solution that satisfies a variational inequality under some suitable conditions on the parameters. It generalizes the familiar gradient-projection algorithm for convex minimization problem. This result improves and extends some recent results in the literature.

Keywords: generalized equilibrium problem; metric projection; α -inverse strongly monotone mapping; directed nonexpansive mapping.

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1. INTRODUCTION

Let H be a real Hilbert space and C be a nonempty closed convex subset of H . A mapping $T : C \rightarrow C$ is said to be L -Lipschitzian mapping if for some $L \geq 0$,

$$(1.1) \quad \|Tx - Ty\| \leq L\|x - y\|, \quad \text{for all } x, y \in C.$$

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If $L = 1$ in (1.1), we say that T is a nonexpansive mapping, and if $0 \leq L < 1$ in (1.1), we say that T is a contraction mapping. We say that T is firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently,

$$(1.2) \quad \langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2 \quad \forall x, y \in C.$$

We say that T is α -averaged mapping for some $\alpha \in (0, 1)$ (see Tian and Liu [25]), if there is a nonexpansive mapping $S : C \rightarrow C$ such that $T = (1 - \alpha)I + \alpha S$. We denote the fixed point set of T by $Fix(T)$; that is, $Fix(T) = \{x \in C : Tx = x\}$.

Definition 1.1. (see Xu [30], Browder and Petryshyn [7]) *A mapping $T : C \rightarrow C$ is said to be monotone if*

$$(1.3) \quad \langle x - y, Tx - Ty \rangle \geq 0 \text{ for all } x, y \in C;$$

and is called ν -inverse strongly monotone (for short, ν -ism) for some $\nu > 0$, if

$$(1.4) \quad \langle x - y, Tx - Ty \rangle \geq \nu \|Tx - Ty\|^2 \text{ for all } x, y \in C.$$

The monotone operators have been widely used to solve practical problems in various fields such as optimization problems, traffic assignment problems, equilibrium problems, radiation therapy, and so on (see Browder [6], Byrne [9], Combettes and Hirstoaga [12], Han and Lo [14], Moudafi and Thèra [17], Xu [27, 28, 30], Yazdi [31, 32, 33], Yazdi and Sababe[34, 35] and the references therein).

Let us discuss about the problems that motivated us to develop approximation techniques. Let $\phi : C \times C \rightarrow \mathbb{R}$ be a bi-function. In 1994, Blum and Oettli [3] introduced an equilibrium problem (EP) as the problem of finding $u \in C$ such that

$$(1.5) \quad \phi(u, v) \geq 0 \text{ for all } v \in C.$$

The set of solutions of (1.5) is denoted by $EP(\phi)$. An equilibrium problem theory has motivated the study of problems which arise from image restoration, computer tomography, radiation therapy treatment planning, economics, optimization, etc. In some systems, solutions of equilibrium problems are also solutions of the fixed point problems of a nonlinear mapping. Many researchers looked for common solutions to the equilibrium and fixed point problems

of a system. Several authors have studied existence and approximation of common solutions of equilibrium and fixed point problems based on different relaxed monotonicity notions and various compactness assumptions. To mention some, see Blum and Oettli[3], Bnouhachem [4], Byrne[8, 9], Censor and Elfving [11], Moudafi[18], Sangago et al. [24], Zegeye et al. [36], and the references therein.

Many researchers considered a generalized equilibrium problem (GEP) of finding $z \in C$ such that

$$(1.6) \quad \phi(z, y) + \langle Az, y - z \rangle \geq 0 \text{ for all } y \in C,$$

where $A : C \rightarrow H$ is a monotone mapping. The set of solutions of (1.6) is denoted by $EP(\phi, A)$; that is, $EP(\phi, A) = \{z \in C : \phi(z, y) + \langle Az, y - z \rangle \geq 0 \forall y \in C\}$. In the case when $A \equiv 0$, GEP reduces to EP . Numerous problems in physics, variational inequalities, optimization, minimax problems, the Nash equilibrium problem in non-cooperative games and economics reduce to finding a solution of the GEP (1.6) (See Moudafi and Thèra [17], Moudafi [18, 19], Xu [30], Yazdi [31], Yazdi and Sababe [35], and the references therein).

The second problem of our interest is a constrained convex minimization problem which is given by

$$(1.7) \quad \min\{g(x) : x \in C\},$$

where $g : C \rightarrow \mathbb{R}$ is a convex function. We denote the set of solutions of the problem (1.7) by U . The widely considered approximation method to solve these problems is the gradient projection algorithm(GPA). If g is (Frechet) differentiable, then the GPA generates a sequence $\{x_n\}$ via the following recursive formula:

$$(1.8) \quad x_{n+1} = P_C(x_n - \lambda \nabla g(x_n)) \text{ for all } n \geq 0,$$

or more generally,

$$x_{n+1} = P_C(x_n - \lambda_n \nabla g(x_n)) \text{ for all } n \geq 0,$$

where $x_0 \in C$ is an arbitrarily initial guess and the parameters λ or λ_n , $n = 1, 2, 3, \dots$, are positive real numbers satisfying certain conditions. The convergence of algorithm in (1.8) depends on the behavior of the gradient ∇g .

In 2010, Xu [29] proved the following:

Theorem 1.2. *If $g : C \rightarrow \mathbb{R}$ is a continuously differentiable convex function such that the gradient ∇g is Lipschitz continuous with Lipschitz constant $L > 0$, and if the constrained convex minimization problem is consistent, then for each $\lambda \in (0, \frac{2}{L})$, the sequence $\{x_n\}$ generated by the gradient-projection algorithm (1.8) converges weakly to a solution of (1.7).*

In 2011, Xu [30] proposed an explicit operator-oriented approach to the algorithm (1.8) using the concept of an averaged mapping. He proved weak convergence of his averaged mapping approach to the GPA (1.8) and the relaxed gradient-projection algorithm given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C(x_n - \lambda_n \nabla g(x_n)) \quad n = 0, 1, 2, \dots,$$

where the sequences of parameters $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy some mild conditions. Moreover, he constructed a counter example which showed that the algorithm (1.8) does not converge in norm in an infinite-dimensional space and also presented two modifications of GPA which were shown to have strong convergence (see Xu [27, 28, 30]).

Many mathematicians in the field discussed approximation of a common solution for the three problems: fixed point problem for nonlinear mappings, generalized equilibrium problem, and constrained convex minimization problems. Some of them considered approximation of a common solution for combination of any two of them. Also many authors tried to develop approximation techniques for individual problems by studying the characteristics of each problem (See Combettes and Hirstoaga [12], Jung [15], Peng and Yao [21], Plubtieng and Punpaeng [22], Razani and Yazdi [23], Wang et al. [26], Yazdi [31], and their citations).

Let us discuss some of these results that are in line with our point of interest in this paper. In 2007, Plibtieng and Punpaneng [22] introduced an iterative scheme for finding a common element of the set of solutions of (1.5) and the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$(1.9) \quad \begin{cases} x_1 \in H, \\ \phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \text{ for all } y \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S u_n, \quad n \geq 1, \end{cases}$$

where $\phi : H \times H \rightarrow \mathbb{R}$ is a bi-function, A is strongly positive bounded linear operator on H , S is a nonexpansive self-mapping of H such that $Fix(S) \cap EP(\phi) \neq \emptyset$, f is a contraction, $\gamma > 0$ is a constant, $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. They proved that the sequence $\{x_n\}$ defined in (1.9) converges strongly to the unique solution of a certain variational inequality problem. In 2010, Wang et al. [26] introduced the following composite iterative Scheme:

$$(1.10) \quad \begin{cases} x_1 \in H, \\ \phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \text{ for all } y \in H, \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T_n u_n, \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \geq 1, \end{cases}$$

where $\phi : H \times H \rightarrow \mathbb{R}$ is a bi-function, A is strongly positive bounded linear operator on H , f is a contraction, $\{T_n\}$ is a countable family of nonexpansive self-mappings of H such that $\bigcap_{n=1}^{\infty} Fix(T_n) \cap EP(\phi) \neq \emptyset$, $\gamma > 0$ is some constant, $x_1 \in H$, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. By imposing some strict conditions on the parameters, they proved that the sequence $\{x_n\}$ generated by (1.10) converges strongly to a point in $\bigcap_{n=1}^{\infty} Fix(T_n) \cap EP(\phi) \neq \emptyset$.

Let $\phi : C \times C \rightarrow \mathbb{R}$ be a bi-function. Let $g : C \rightarrow \mathbb{R}$ be a convex function such that ∇g is an L -Lipschitzian mapping with $L > 0$. Assume that $U \cap EP(\phi) \neq \emptyset$ and let $f : C \rightarrow C$ be a contraction. Let $\{\alpha_n\} \subset (0, 1)$, $\{r_n\} \subset (0, \infty)$, $P_C(I - \lambda_n \nabla g) = s_n I + (1 - s_n) T_n$, $s_n = \frac{2 - \lambda_n L}{4}$ and $\{\lambda_n\} \subset (0, \frac{2}{L})$.

In 2012, Tian and Liu [25] studied the following explicit composite iterative schemes by the viscosity approximation method to find a common solution of an equilibrium problem and a constrained convex minimization problem:

$$(1.11) \quad \begin{cases} x_1 \in C, \\ \phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \text{ for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n u_n, \quad n \geq 1, \end{cases}$$

and proved that the sequences $\{u_n\}$ and $\{x_n\}$ defined in (1.11) converge strongly to a point in $U \cap EP(\phi)$ by imposing some mild conditions on the parameters.

In 2020, Yazdi [32] introduced the following explicit composite iterative method for finding the common solution of a generalized equilibrium problem and a constrained convex minimization problem:

$$(1.12) \quad \begin{cases} x_1 \in C, \\ \phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle + \langle Ax_n, y - u_n \rangle \geq 0 \quad \text{for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n u_n, \quad n \geq 1, \end{cases}$$

The author proved that the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.12) converge strongly to $q \in U \cap EP(\phi, A)$ under certain conditions, and showed that q solves a variational inequality problem.

In 2024, Yazdi and Sababe [35] proposed the two-layer iteration process defined as

$$(1.13) \quad \begin{cases} x_1 \in C, \\ \phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T u_n, \quad n \geq 1, \end{cases}$$

where $T : C \rightarrow C$ is an α -strongly quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero, and $Fix(T) \cap EP(\phi) \neq \emptyset$. They proved that the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.13) converge strongly to $q \in Fix(T) \cap EP(\phi)$ under certain conditions on the parameters.

In 2025, Sangago et al. [24] proposed the following viscosity iterative algorithm to approximate a common solution of a constrained convex minimization problem, a generalized equilibrium problem, and fixed point problem of directed nonexpansive mapping:

$$(1.14) \quad \begin{cases} x_1 \in C, \\ \phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle + \langle y - u_n, Bx_n \rangle \geq 0 \quad \text{for all } y \in C, \\ v_n = \alpha_n x_n + (1 - \alpha_n) T_n u_n \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) S v_n, \quad n \geq 1. \end{cases}$$

where $S : C \rightarrow C$ is a directed nonexpansive mapping, $B : C \times C \rightarrow \mathbb{R}$ is an α -ism, $f : C \rightarrow C$ is a contraction. The authors proved that the sequences $\{x_n\}$, $\{v_n\}$, and $\{u_n\}$ generated by (1.14)

converge strongly to $q \in \text{Fix}(S) \cap EP(\phi, B) \cap U$, where $U = \{p \in C : g(p) = \min_{x \in C} g(x)\}$, under mild conditions on the parameters.

Motivated and inspired by the above results, we propose a viscosity iterative scheme to approximate a common solution of fixed point problem of directed nonexpansive mappings, a generalized equilibrium problem and a constrained convex minimization problem. Then, we prove a strong convergence theorem which improves and extends recent results in the literature.

2. PRELIMINARIES

Let H be a real Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Weak and strong convergences are denoted by \rightharpoonup and \rightarrow , respectively. We have the following well known facts from the definition of norm and inner product on Hilbert spaces:

Lemma 2.1 (Khamsi and Kirk [16]). *Let H be a real Hilbert space. Then for every $x, y \in H$ and $\lambda \in (0, 1)$*

$$(2.1) \quad \|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle.$$

$$(2.2) \quad \|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

$$(2.3) \quad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

Let C be a nonempty closed convex subset of H . For any $x \in H$, there exists a unique nearest point in C , denoted by $P_C(x)$, satisfying

$$(2.4) \quad \|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}.$$

We say that P_C is a metric projection of H onto C . Some of the useful properties of projections are gathered in the lemma below.

Lemma 2.2 (Khamsi and Kirk [16], Cai et al. [10]). *Let C be a nonempty closed convex subset of H and $P_C : H \rightarrow C$ be the metric projection. Then*

(a) For $x \in H$ and $z \in C$,

$$z = P_C(x) \iff \langle x - z, z - y \rangle \geq 0 \text{ for all } y \in C.$$

(b) P_C is a firmly nonexpansive mapping; that is, for all $x, y \in H$,

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y \rangle.$$

(c) For all $x \in H$, $y \in C$,

$$\|x - P_C(x)\|^2 \leq \|x - y\|^2 - \|y - P_C(x)\|^2.$$

Sangago et al.[24] introduced the following definition.

Definition 2.3 (Sangago et al. [24]). *Let C be a nonempty closed convex subset of a real Hilbert space H . A mapping $T : C \rightarrow C$ is said to be a **directed nonexpansive** mapping if it satisfies the following three conditions:*

(i) T is a nonexpansive mapping;

(ii) $\text{Fix}(T) \neq \emptyset$; and

(iii) $\|Tx - p\|^2 \leq \|x - p\|^2 - \|x - Tx\|^2$ for every $x \in C$, for every $p \in \text{Fix}(T)$.

Obviously, every firmly nonexpansive mapping with nonempty fixed point set is a directed nonexpansive mapping. There are nonexpansive mappings that are not directed nonexpansive. For instance the mapping $T : B_H \rightarrow B_H$ (where $B_H = \{x \in H : \|x\| \leq 1\}$) defined by $Tx = -x$ is nonexpansive, but not directed nonexpansive mapping. Lemma 2.2 implies that the metric projection P_C is a directed nonexpansive mapping.

The following lemmas are key in proving our main results.

Lemma 2.4 (Goebel and Kirk [13]). *Let H be a real Hilbert space, C be a closed convex subset of H and $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C that converges weakly to x , and if $\{x_n - Tx_n\}$ converges strongly to y , then $(I - T)x = y$.*

Definition 2.5 (Blum and Oettli [3]). *Let C be a nonempty closed convex subset of H . A bi-function $\phi : C \times C \rightarrow \mathbb{R}$ is said to satisfy "Condition A" if the following four conditions hold:*

(A₁) $\phi(x, x) = 0$ for all $x \in C$;

(A₂) ϕ is monotone; that is, $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$;

(A₃) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x, y);$$

(A₄) for each $x \in C$, $y \mapsto \phi(x, y)$ is convex and weakly lower semi continuous.

Lemma 2.6 (Blum and Oettli [3]). *Let C be a nonempty closed convex subset of H and $\phi : C \times C \rightarrow \mathbb{R}$ be a bi-function satisfying **Condition A**. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$(2.5) \quad \phi(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \text{ for all } y \in C.$$

Lemma 2.7 (Combettes and Hirstoaga[12]). *Assume that $\phi : C \times C \rightarrow \mathbb{R}$ satisfies **Condition A**. For $r > 0$ and $x \in H$, define a mapping $Q_r : H \rightarrow C$ as follows:*

$$Q_r x = \{z \in C : \phi(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \text{ for all } y \in C\}.$$

Then, the following hold:

- (i) Q_r is single-valued;
- (ii) Q_r is firmly nonexpansive; that is, for all $x, y \in H$

$$\|Q_r x - Q_r y\|^2 \leq \langle Q_r x - Q_r y, x - y \rangle;$$

- (iii) $F(Q_r) = EP(\phi)$;
- (iv) $EP(\phi)$ is closed and convex.

Lemma 2.8 (Brezis [5], Byrne [9], Xu [30]). *Let $S, T, V, T_1, T_2 : H \rightarrow H$ be given mappings.*

- (i) T is nonexpansive if and only if $I - T$ is $\frac{1}{2}$ -ism.
- (ii) If T is v -ism, then for every $\gamma > 0$, γT is $\frac{v}{\gamma}$ -ism.
- (iii) T is averaged if and only if $I - T$ is v -ism for some $v > \frac{1}{2}$. In fact, for $\alpha \in (0, 1)$, T is α -averaged if and only if $I - T$ is $\frac{1}{2\alpha}$ -ism.
- (iv) If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$, S is averaged and V is nonexpansive, then T is averaged.
- (v) T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.
- (vi) If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$, S is firmly nonexpansive and V is nonexpansive, then T is averaged.
- (vii) If T_1 is α_1 -averaged, and T_2 is α_2 -averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite $T_1 T_2$ is α -averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1 \alpha_2$.

Lemma 2.9 (Xu [29]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Assume that $g : H \rightarrow \mathbb{R}$ is a convex function whose gradient ∇g is an L -Lipschitzian mapping with $L > 0$. Assume that the constrained convex minimization problem in (1.7) is consistent. Then*

- (i) ∇g is an $\frac{1}{L}$ -inverse strongly monotone mapping (shortly $\frac{1}{L}$ -ism).
- (ii) For $\lambda > 0$, the mapping $I - \lambda \nabla g$ is $\frac{\lambda L}{2}$ -averaged.
- (iii) The composite $P_C(I - \lambda \nabla g)$ is $(\frac{2 + \lambda L}{4})$ -averaged for $0 < \lambda < \frac{2}{L}$.
- (iv) $x^* \in C$ solves the minimization problem (1.7) if and only if $x^* \in C$ solves the fixed point equation

$$x^* = P_C(I - \lambda \nabla g)x^*,$$

where $\lambda > 0$ is any fixed positive number.

Lemma 2.10 (Yazdi [32]). *Suppose C is a nonempty closed convex subset of a real Hilbert space H , B is an α -inverse-strongly monotone mapping on C and $0 < r < 2\alpha$. Then, $I - rB$ is a nonexpansive mapping.*

Lemma 2.11. (Yazdi [32]) *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\phi : C \times C \rightarrow \mathbb{R}$ be bi-function satisfying the conditions **Condition A** and B be an α -inverse-strongly monotone mapping. Suppose $\{x_n\}$ is a bounded sequence in C and $\{r_n\} \subset [a, b] \subset (0, 2\alpha)$ is a real sequence. If $u_n = Q_{r_n}(x_n - r_n Bx_n)$, then*

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + M|r_{n+1} - r_n|,$$

where $M = \sup\{\|Bx_n\| + \frac{1}{a}\|u_{n+1} - (x_{n+1} - r_{n+1}Bx_{n+1})\| : n \in \mathbb{N}\}$.

Lemma 2.12 (Aoyama et al. [1]). *Assume that $\{a_n\} \subseteq [0, \infty)$, $\{\gamma_n\} \subseteq [0, 1]$, $\{\mu_n\} \subseteq [0, \infty)$ and $\{v_n\} \subseteq \mathbb{R}$ such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n v_n + \mu_n.$$

Then the conditions $\sum_{n=1}^{\infty} \gamma_n = \infty$, $\limsup_{n \rightarrow \infty} v_n \leq 0$, and $\sum_{n=1}^{\infty} \mu_n < \infty$, imply

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Lemma 2.13 (Naidu and Sangago [20]). *Let C be a nonempty closed convex subset of a real Hilbert space H , let Π_C denote the family of all contraction self mappings of C and suppose $T : C \rightarrow C$ is a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Then there is a unique mapping $\Delta : \Pi_C \rightarrow \text{Fix}(T)$ such that*

$$\limsup_{n \rightarrow \infty} \langle (I - f)\Delta(f), \Delta(f) - x_n \rangle \leq 0,$$

for any given $f \in \Pi_C$ and a bounded approximate fixed point sequence $\{x_n\}$ of T in C .

3. MAIN RESULTS

Throughout this section we use the following assumptions.

(B1) Let H be a real Hilbert space and let C be a nonempty closed convex subset of H .

(B2) Assume that $g : C \rightarrow \mathbb{R}$ is a real-valued convex function whose ∇g is a Lipschitzian mapping with Lipschitz constant $L > 0$. The solution set of the minimization problem $\min\{g(x) : x \in C\}$ is denoted by U ; that is,

$$(3.1) \quad U = \{z \in C : g(z) = \min_{y \in C} g(y)\}.$$

Let $\{\lambda_n\}$ be a sequence of real numbers in $(0, \frac{2}{L})$ such that

$$(3.2) \quad P_C(I - \lambda_n \nabla g) = \gamma_n I + (1 - \gamma_n) T_n$$

where $T_n : C \rightarrow C$ is nonexpansive, and $\gamma_n = \frac{2 - \lambda_n L}{4}$. Assume that $\lim_{n \rightarrow \infty} \gamma_n = 0$ and

$$\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$$

(B3) Let $B : C \rightarrow C$ be an α -ism mapping for some $\alpha > 0$. Let $\phi : C \times C \rightarrow \mathbb{R}$ be a bi-function satisfying **Condition A**. The solution set of the generalized equilibrium problem is denoted by $EP(\phi, B)$; that is,

$$(3.3) \quad EP(\phi, B) = \{z \in C : \phi(z, y) + \langle y - z, Bz \rangle \geq 0 \forall y \in C\}.$$

(B4) Let $f : C \rightarrow C$ be a contraction mapping with contraction constant $k \in [0, 1)$.

(B5) Let $S : C \rightarrow C$ be a directed nonexpansive mapping with fixed point set $\text{Fix}(S)$.

(B6) Assume that $\Sigma = U \cap EP(\phi, B) \cap \text{Fix}(S) \neq \emptyset$.

Now, we introduce a mixed Mann-type viscosity approximation scheme to approximate a common solution of the fixed point problem of directed nonexpansive mapping S , the generalized equilibrium problem (3.3) and the constrained convex minimization problem (3.1) as follows:

$$(3.4) \quad \begin{cases} x_1 \in C, \\ \phi(z_n, y) + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle + \langle y - z_n, Bx_n \rangle \geq 0 \text{ for all } y \in C, \\ w_n = \alpha_n x_n + (1 - \alpha_n) P_C(z_n - \lambda_n \nabla g(z_n)) \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) S w_n, \quad n \geq 1. \end{cases}$$

Now we state and prove a strong convergence theorem with some mild conditions on the parameters.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let ϕ , g , ∇g , f , B , T_n and S be as defined in the assumptions (B1)-(B6). Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ are real sequences satisfying the following conditions:*

$$(3.5) \quad \{\alpha_n\} \subset (0, 1], \lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty;$$

$$(3.6) \quad \{\beta_n\} \subset (0, 1], \lim_{n \rightarrow \infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty \text{ and } \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty;$$

$$(3.7) \quad \{r_n\} \subset [a, b] \subset (0, 2\alpha) \text{ and } \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

Then the sequences $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ defined by (3.4) converge strongly to $q = P_{\Sigma} f(q)$, which solves the variational inequality:

$$(3.8) \quad \langle q - f(q), y - q \rangle \geq 0, \text{ for all } y \in C.$$

Proof. Borrowing the notation in Lemma 2.7, we have

$$z_n = Q_{r_n}(x_n - r_n Bx_n) \text{ for } n = 1, 2, \dots.$$

For $p \in \Sigma$, we note that

$$\phi(p, y) + \frac{1}{r_n} \langle y - p, p - (I - r_n B)(p) \rangle = \phi(p, y) + \langle y - p, Bp \rangle \geq 0, \quad \forall y \in C;$$

so that $Q_{r_n}(p - r_n Bp) = p$ for each $n = 1, 2, 3, \dots$. We also observe that $P_C(I - \lambda_n \nabla g)p = p$, $T_n p = p$, and $S p = p$. Now we prove the theorem by partitioning into the following steps.

Step 1. The sequences $\{x_n\}$, $\{w_n\}$, and $\{z_n\}$ are bounded.

Let $p \in \Sigma$. Then, for each $n \in \mathbb{N}$, it follows from (3.4), Lemma 2.2 and Lemma 2.7 that

$$\begin{aligned}
\|w_n - p\| &= \|\alpha_n x_n + (1 - \alpha_n)P_C(z_n - \lambda_n \nabla g(z_n)) - p\| \\
&\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|P_C(z_n - \lambda_n \nabla g(z_n)) - p\| \\
&= \alpha_n \|x_n - p\| + (1 - \alpha_n) \|P_C(I - \lambda_n \nabla g)z_n - P_C(I - \lambda_n \nabla g)p\| \\
&\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|z_n - p\| \\
&\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|Q_{r_n}(x_n - r_n Bx_n) - Q_{r_n}(p - r_n Bp)\| \\
&\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|x_n - p\| \\
(3.9) \quad &\leq \|x_n - p\|.
\end{aligned}$$

It follows from (3.4), (3.9) and triangle inequality that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n f(x_n) + (1 - \beta_n)S w_n - p\| \\
&= \|\beta_n(f(x_n) - f(p)) + \beta_n(f(p) - p) + (1 - \beta_n)(S w_n - p)\| \\
&\leq \beta_n \|f(x_n) - f(p)\| + \beta_n \|f(p) - p\| + (1 - \beta_n) \|S w_n - p\| \\
&\leq k\beta_n \|x_n - p\| + \beta_n \|f(p) - p\| + (1 - \beta_n) \|w_n - p\| \\
&\leq k\beta_n \|x_n - p\| + \beta_n \|f(p) - p\| + (1 - \beta_n) \|x_n - p\| \\
&= (1 - (1 - k)\beta_n) \|x_n - p\| + (1 - k)\beta_n \left[\frac{\|f(p) - p\|}{1 - k} \right] \\
(3.10) \quad &\leq \max \left(\|x_n - p\|, \frac{\|f(p) - p\|}{1 - k} \right).
\end{aligned}$$

By induction on n , it follows from (3.10) that

$$(3.11) \quad \|x_n - p\| \leq \max \left(\|x_1 - p\|, \frac{\|f(p) - p\|}{1 - k} \right) \quad \forall n \in \mathbb{N}.$$

Hence $\{x_n\}$ is bounded, and as a consequence both sequences $\{w_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{Sx_n\}$, $\{Sw_n\}$, $\{Sz_n\}$, $\{P_C(I - \lambda_n \nabla g)x_n\}$ and $\{P_C(I - \lambda_n \nabla g)z_n\}$ are bounded.

STEP 2. We prove that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

Without loss of generality assume that a real number M is an upper bound of each of the bounded sequences such as $\{\|f(x_n)\|\}$, $\{\|Sx_n\|\}$, $\{\|w_n\|\}$, $\{\|z_n\|\}$, $\{\|Sw_n\|\}$, $\{\|P_C(I - \lambda_n \nabla g)z_n\|\}$ and so on. For each $n \in \mathbb{N}$, it follows from (3.4) that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|\beta_{n+1}f(x_{n+1}) + (1 - \beta_{n+1})Sw_{n+1} - \beta_n f(x_n) - (1 - \beta_n)Sw_n\| \\
&\leq (1 - \beta_{n+1})\|Sw_{n+1} - Sw_n\| + |\beta_{n+1} - \beta_n|\|f(x_n)\| \\
&\quad + |\beta_{n+1} - \beta_n|\|Sw_n\| + \beta_{n+1}\|f(x_{n+1}) - f(x_n)\| \\
&\leq (1 - \beta_{n+1})\|w_{n+1} - w_n\| + 2M|\beta_{n+1} - \beta_n| \\
(3.12) \quad &\quad + k\beta_{n+1}\|x_{n+1} - x_n\|.
\end{aligned}$$

On the other hand, for each $n \in \mathbb{N}$ it follows from (3.4) that

$$\begin{aligned}
\|w_{n+1} - w_n\| &\leq \alpha_{n+1}\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|x_n\| \\
&\quad + (1 - \alpha_{n+1})\|P_C(I - \lambda_{n+1}\nabla g)z_{n+1} - P_C(I - \lambda_n\nabla g)z_n\| \\
&\quad + |\alpha_{n+1} - \alpha_n|\|P_C(I - \lambda_n\nabla g)z_n\| \\
&\leq \alpha_{n+1}\|x_{n+1} - x_n\| + 2M|\alpha_{n+1} - \alpha_n| \\
&\quad + (1 - \alpha_{n+1})\|P_C(I - \lambda_{n+1}\nabla g)z_{n+1} - P_C(I - \lambda_n\nabla g)z_n\| \\
&\leq \alpha_{n+1}\|x_{n+1} - x_n\| + 2M|\alpha_{n+1} - \alpha_n| \\
&\quad + (1 - \alpha_{n+1})(\|\gamma_{n+1}z_{n+1} + (1 - \gamma_{n+1})T_{n+1}z_{n+1} - \gamma_n z_n - (1 - \gamma_n)T_n z_n\|) \\
&\leq \alpha_{n+1}\|x_{n+1} - x_n\| + 2M|\alpha_{n+1} - \alpha_n| \\
&\quad + (1 - \alpha_{n+1})\{\gamma_{n+1}\|z_{n+1} - z_n\| + (1 - \gamma_{n+1})\|T_{n+1}z_{n+1} - T_n z_n\|\} \\
&\quad + (1 - \alpha_{n+1})\{|\gamma_{n+1} - \gamma_n|\|z_n\| + |\gamma_{n+1} - \gamma_n|\|T_n z_n\|\} \\
&\leq \alpha_{n+1}\|x_{n+1} - x_n\| + 2M|\alpha_{n+1} - \alpha_n| + 2M|\gamma_{n+1} - \gamma_n| \\
&\quad + (1 - \alpha_{n+1})\{\gamma_{n+1}\|z_{n+1} - z_n\| + (1 - \gamma_{n+1})\|T_{n+1}z_{n+1} - T_n z_n\|\} \\
&\leq \alpha_{n+1}\|x_{n+1} - x_n\| + 2M|\alpha_{n+1} - \alpha_n| + 2M|\gamma_{n+1} - \gamma_n| \\
&\quad + (1 - \alpha_{n+1})\gamma_{n+1}\|z_{n+1} - z_n\| + (1 - \gamma_{n+1})\|T_{n+1}z_{n+1} - T_{n+1}z_n\| \\
&\quad + (1 - \alpha_{n+1})\|T_{n+1}z_n - T_n z_n\|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_{n+1} \|x_{n+1} - x_n\| + 2M|\alpha_{n+1} - \alpha_n| + 2M|\gamma_{n+1} - \gamma_n| \\
&\quad + (1 - \alpha_{n+1})\{\gamma_{n+1} \|z_{n+1} - z_n\| + (1 - \gamma_{n+1}) \|z_{n+1} - z_n\|\} \\
&\quad + (1 - \alpha_{n+1}) \|T_{n+1}z_n - T_n z_n\| \\
&\leq \alpha_{n+1} \|x_{n+1} - x_n\| + (1 - \alpha_{n+1}) \|z_{n+1} - z_n\| \\
&\quad + 2M|\alpha_{n+1} - \alpha_n| + 2M|\gamma_{n+1} - \gamma_n| \\
(3.13) \quad &\quad + (1 - \alpha_{n+1}) \|T_{n+1}z_n - T_n z_n\|.
\end{aligned}$$

It follows from (3.2) and the proof in Sangago et al. [24] that

$$\begin{aligned}
\|T_{n+1}z_n - T_n z_n\| &= \left\| \frac{P_C(I - \lambda_{n+1}\nabla g) - \gamma_{n+1}I}{1 - \gamma_{n+1}} z_n - \frac{P_C(I - \lambda_n\nabla g) - \gamma_n I}{1 - \gamma_n} z_n \right\| \\
&\leq \left\| \frac{4P_C(I - \lambda_{n+1}\nabla g)}{2 + \lambda_{n+1}L} z_n - \frac{4P_C(I - \lambda_n\nabla g)}{2 + \lambda_n L} z_n \right\| \\
&\quad + \left\| \frac{2 - \lambda_{n+1}L}{2 + \lambda_{n+1}L} z_n - \frac{2 - \lambda_n L}{2 + \lambda_n L} z_n \right\| \\
&= \left\| \frac{4(2 + \lambda_n L)P_C(I - \lambda_{n+1}\nabla g)z_n - 4(2 + \lambda_{n+1}L)P_C(I - \lambda_n\nabla g)z_n}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \right\| \\
&\quad + \frac{4L|\lambda_{n+1} - \lambda_n|}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \|z_n\| \\
&\leq \frac{4L|\lambda_{n+1} - \lambda_n| \|P_C(I - \lambda_{n+1}\nabla g)z_n\|}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} + \frac{4L|\lambda_{n+1} - \lambda_n|}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \|z_n\| \\
&\quad + \frac{4(2 + \lambda_{n+1}L) \|P_C(I - \lambda_{n+1}\nabla g)z_n - P_C(I - \lambda_n\nabla g)z_n\|}{(2 + \lambda_{n+1}L) * (2 + \lambda_n L)} \\
&\leq |\lambda_{n+1} - \lambda_n| [L \|P_C(I - \lambda_{n+1}\nabla g)z_n\| + 4 \|\nabla g(z_n)\| + L \|z_n\|] \\
(3.14) \quad &\leq (2L + 4)M|\lambda_{n+1} - \lambda_n|.
\end{aligned}$$

It follows from Lemma 2.11 that

$$(3.15) \quad \|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + M|r_{n+1} - r_n|.$$

It follows from (3.13), (3.14) and (3.15) that

$$\begin{aligned}
\|w_{n+1} - w_n\| &\leq \alpha_{n+1} \|x_{n+1} - x_n\| + (1 - \alpha_{n+1})(\|x_{n+1} - x_n\| + M|r_{n+1} - r_n|) \\
&\quad + 2M|\alpha_{n+1} - \alpha_n| + 2M|\gamma_{n+1} - \gamma_n|
\end{aligned}$$

$$\begin{aligned}
& +(2L+4)M(1-\alpha_{n+1})|\lambda_{n+1}-\lambda_n| \\
\leq & \|x_{n+1}-x_n\|+M(1-\alpha_{n+1})|r_{n+1}-r_n| \\
& +2M|\alpha_{n+1}-\alpha_n|+2M|\gamma_{n+1}-\gamma_n| \\
& +(1-\alpha_{n+1})(2L+4)M|\lambda_{n+1}-\lambda_n| \\
\leq & \|x_{n+1}-x_n\|+M|r_{n+1}-r_n| \\
& +2M|\alpha_{n+1}-\alpha_n|+2M|\gamma_{n+1}-\gamma_n| \\
(3.16) \quad & +(2L+4)M|\lambda_{n+1}-\lambda_n|.
\end{aligned}$$

We get from (3.12) and (3.16) that

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| & \leq (1-\beta_{n+1})\|w_{n+1}-w_n\|+2M|\beta_{n+1}-\beta_n|+k\beta_{n+1}\|x_{n+1}-x_n\| \\
& \leq (1-(1-k)\beta_{n+1})\|x_{n+1}-x_n\|+2M|\beta_{n+1}-\beta_n| \\
& \quad +2M(1-\beta_{n+1})[|\alpha_{n+1}-\alpha_n|+|\gamma_{n+1}-\gamma_n|] \\
& \quad +M(1-\beta_{n+1})[|\lambda_{n+1}-\lambda_n|+|r_{n+1}-r_n|] \\
(3.17) \quad & \leq (1-\theta_n)\|x_{n+1}-x_n\|+\theta_n\mu_n+v_n,
\end{aligned}$$

where

$$\begin{aligned}
\theta_n & = (1-k)\beta_{n+1}, \\
\mu_n & = -\frac{2M}{1-k}\left(|\alpha_{n+1}-\alpha_n|+|\gamma_{n+1}-\gamma_n|+(L+2)|\lambda_{n+1}-\lambda_n|+\frac{1}{2}|r_{n+1}-r_n|\right), \\
v_n & = 2M|\alpha_{n+1}-\alpha_n|+2M|\beta_{n+1}-\beta_n|+2M|\gamma_{n+1}-\gamma_n| \\
& \quad +(2L+4)M|\lambda_{n+1}-\lambda_n|+M|r_{n+1}-r_n|.
\end{aligned}$$

It follows from (3.5), (3.6), (3.7) and **Assumption B2** that

$$(3.18) \quad (i) \sum_{n=1}^{\infty} \theta_n = (1-k) \sum_{n=1}^{\infty} \beta_{n+1} = \infty; \quad (ii) \sum_{n=1}^{\infty} v_n < \infty \quad (iii) \lim_{n \rightarrow \infty} \mu_n \leq 0.$$

Moreover, letting $a_n = \|x_{n+1}-x_n\|$, (3.17) is re-written as

$$(3.19) \quad a_{n+1} \leq (1-\theta_n)a_n + \theta_n\mu_n + v_n.$$

Then it follows from (3.18), (3.19), and Lemma 2.12 that

$$(3.20) \quad \lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

STEP 3. We prove that $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.

Let $p \in \Sigma$. Because Q_{r_n} is nonexpansive and B is α -ism, we have

$$\begin{aligned} \|z_n - p\|^2 &= \|Q_{r_n}(x_n - r_n Bx_n) - p\|^2 \\ &= \|Q_{r_n}(x_n - r_n Bx_n) - Q_{r_n}(p - r_n Bp)\|^2 \\ &\leq \|(x_n - p) - r_n(Bx_n - Bp)\|^2 \\ &\leq \|x_n - p\|^2 + r_n^2 \|Bx_n - Bp\|^2 - 2r_n \langle x_n - p, Bx_n - Bp \rangle \\ (3.21) \quad &\leq \|x_n - p\|^2 + r_n(r_n - 2\alpha) \|Bx_n - Bp\|^2. \end{aligned}$$

We get from (3.4), Lemma 2.1, and (3.21) that

$$\begin{aligned} \|w_n - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(P_C(z_n - \lambda_n \nabla g(z_n)) - p)\|^2 \\ &= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|P_C(I - \lambda_n \nabla g)z_n - p\|^2 \\ &\quad - \alpha_n(1 - \alpha_n) \|x_n - P_C(I - \lambda_n \nabla g)z_n\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 \\ &\quad - \alpha_n(1 - \alpha_n) \|x_n - P_C(I - \lambda_n \nabla g)z_n\|^2 \\ &\leq \|x_n - p\|^2 + (1 - \alpha_n)r_n(r_n - 2\alpha) \|Bx_n - Bp\|^2 \\ (3.22) \quad &\quad - \alpha_n(1 - \alpha_n) \|x_n - P_C(I - \lambda_n \nabla g)z_n\|^2. \end{aligned}$$

It follows from (3.4), Lemma 2.1, and (3.22) that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\beta_n(f(x_n - p) + (1 - \beta_n)(Sw_n - p))\|^2 \\ &\leq \beta_n \|f(x_n) - p\|^2 + (1 - \beta_n) \|w_n - p\|^2 \\ &\quad - \beta_n(1 - \beta_n) \|f(x_n) - Sw_n\|^2 \\ &\leq \|w_n - p\|^2 + \beta_n \|f(x_n) - p\|^2 \\ &\quad - \beta_n(1 - \beta_n) \|f(x_n) - Sw_n\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|x_n - p\|^2 + (1 - \alpha_n)r_n(r_n - 2\alpha) \|Bx_n - Bp\|^2 \\
&\quad - \alpha_n(1 - \alpha_n) \|x_n - P_C(I - \lambda_n \nabla g)z_n\|^2 \\
(3.23) \quad &\quad + \beta_n \|f(x_n) - p\|^2 - \beta_n(1 - \beta_n) \|f(x_n) - Sw_n\|^2.
\end{aligned}$$

Thus we get from (3.23) that

$$\begin{aligned}
(1 - \alpha_n)r_n(2\alpha - r_n) \|Bx_n - Bp\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad - \alpha_n(1 - \alpha_n) \|x_n - P_C(I - \lambda_n \nabla g)z_n\|^2 \\
&\quad + \beta_n \|f(x_n) - p\|^2 \\
&\quad - \beta_n(1 - \beta_n) \|f(x_n) - Sw_n\|^2 \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_{n+1} - x_n\| \\
&\quad - \alpha_n(1 - \alpha_n) \|x_n - P_C(I - \lambda_n \nabla g)z_n\|^2 \\
&\quad + \beta_n \|f(x_n) - p\|^2 \\
(3.24) \quad &\quad - \beta_n(1 - \beta_n) \|f(x_n) - Sw_n\|^2.
\end{aligned}$$

It follows from Step 1, (3.5), (3.6), (3.7), (3.20), and (3.24) that

$$(3.25) \quad \lim_{n \rightarrow \infty} \|Bx_n - Bp\| = 0.$$

By Lemma 2.1 and Lemma 2.7 we get

$$\begin{aligned}
\|z_n - p\|^2 &= \|Q_{r_n}(x_n - r_n Bx_n) - Q_{r_n}(p - r_n Bp)\|^2 \\
&\leq \langle z_n - p, (x_n - r_n Bx_n) - (p - r_n Bp) \rangle \\
&= \langle z_n - p, x_n - p \rangle - r_n \langle z_n - p, Bx_n - Bp \rangle \\
&\leq \frac{1}{2} \left[\|z_n - p\|^2 + \|x_n - p\|^2 - \|z_n - x_n\|^2 \right] \\
(3.26) \quad &\quad - r_n \langle z_n - p, Bx_n - Bp \rangle.
\end{aligned}$$

It follows from (3.26) that

$$(3.27) \quad \|z_n - p\|^2 \leq \|x_n - p\|^2 - \|z_n - x_n\|^2 - 2r_n \langle z_n - p, Bx_n - Bp \rangle.$$

On the other hand, from the calculations in (3.22) and using (3.27) we get that

$$\begin{aligned}
\|w_n - p\|^2 &\leq \|x_n - p\|^2 - (1 - \alpha_n) \|z_n - x_n\|^2 \\
&\quad - 2(1 - \alpha_n)r_n \langle z_n - p, Bx_n - Bp \rangle \\
(3.28) \quad &\quad - \alpha_n(1 - \alpha_n) \|x_n - P_C(I - \lambda_n \nabla g)z_n\|^2.
\end{aligned}$$

Now, it follows from (3.23) and (3.28) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 + \beta_n \|f(x_n) - p\|^2 \\
&\quad - \beta_n(1 - \beta_n) \|f(x_n) - Sw_n\|^2 \\
&\leq \|x_n - p\|^2 - (1 - \alpha_n) \|z_n - x_n\|^2 \\
&\quad - 2(1 - \alpha_n)r_n \langle z_n - p, Bx_n - Bp \rangle \\
&\quad - \alpha_n(1 - \alpha_n) \|x_n - P_C(I - \lambda_n \nabla g)z_n\|^2 \\
(3.29) \quad &\quad + \beta_n \|f(x_n) - p\|^2 - \beta_n(1 - \beta_n) \|f(x_n) - Sw_n\|^2.
\end{aligned}$$

(3.29) implies that

$$\begin{aligned}
(1 - \alpha_n) \|z_n - x_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad - 2(1 - \alpha_n)r_n \langle z_n - p, Bx_n - Bp \rangle \\
&\quad - \alpha_n(1 - \alpha_n) \|x_n - P_C(I - \lambda_n \nabla g)z_n\|^2 \\
&\quad + \beta_n \|f(x_n) - p\|^2 - \beta_n(1 - \beta_n) \|f(x_n) - Sw_n\|^2 \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_{n+1} - x_n\| \\
&\quad - 2(1 - \alpha_n)r_n \langle z_n - p, Bx_n - Bp \rangle \\
&\quad - \alpha_n(1 - \alpha_n) \|x_n - P_C(I - \lambda_n \nabla g)z_n\|^2 \\
(3.30) \quad &\quad + \beta_n \|f(x_n) - p\|^2 - \beta_n(1 - \beta_n) \|f(x_n) - Sw_n\|^2.
\end{aligned}$$

Thus, we get from Step 1, (3.5), (3.6), (3.7), (3.20), (3.25) and (3.30) that

$$(3.31) \quad \lim_{n \rightarrow \infty} \|z_n - x_n\| = 0.$$

STEP 4. We show that $\lim_{n \rightarrow \infty} \|w_n - Sw_n\| = 0$.

Because S is a directed nonexpansive mapping, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|Sw_n - p\|^2 + \beta_n^2 \|f(x_n) - Sw_n\|^2 \\
&\quad - 2\beta_n \langle Sw_n - p, f(x_n) - Sw_n \rangle \\
&\leq \|w_n - p\|^2 - \|w_n - Sw_n\|^2 + \beta_n^2 \|f(x_n) - Sw_n\|^2 \\
&\quad - 2\beta_n \langle Sw_n - p, f(x_n) - Sw_n \rangle.
\end{aligned}
\tag{3.32}$$

Closely looking at (3.22) we have $\|w_n - p\| \leq \|x_n - p\|$, and hence it follows from (3.32) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 - \|w_n - Sw_n\|^2 + \beta_n^2 \|f(x_n) - Sw_n\|^2 \\
&\quad - 2\beta_n \langle Sw_n - p, f(x_n) - Sw_n \rangle.
\end{aligned}
\tag{3.33}$$

Thus we get from (3.33) that

$$\begin{aligned}
\|w_n - Sw_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n^2 \|f(x_n) - Sw_n\|^2 \\
&\quad - 2\beta_n \langle Sw_n - p, f(x_n) - Sw_n \rangle \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| \\
&\quad + \beta_n^2 \|f(x_n) - Sw_n\|^2 - 2\beta_n \langle Sw_n - p, f(x_n) - Sw_n \rangle.
\end{aligned}
\tag{3.34}$$

It follows from (3.6), Step 1, (3.20) and (3.34) that

$$\lim_{n \rightarrow \infty} \|w_n - Sw_n\| = 0.
\tag{3.35}$$

STEP 5. Because $P_\Sigma f$ is a contraction mapping, there exists a unique $q \in \Sigma$ such that $q = P_\Sigma f(q)$ (see Banach [2]). We prove here that the sequences $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ converge strongly to $q = P_\Sigma f(q)$.

From the conditions (3.5), (3.6) and (3.7) on the parameters, and by the boundedness of involved sequences by Step 1, we have

$$\lim_{n \rightarrow \infty} \|w_n - P_C(I - \lambda_n \nabla g)z_n\| = \lim_{n \rightarrow \infty} \alpha_n \|x_n - P_C(I - \lambda_n \nabla g)z_n\| = 0, \text{ and}
\tag{3.36}$$

$$\lim_{n \rightarrow \infty} \|x_{n+1} - Sw_n\| = \lim_{n \rightarrow \infty} \beta_n \|f(x_n) - Sw_n\| = 0.
\tag{3.37}$$

Repeatedly applying the triangle inequality we get

$$(3.38) \quad \begin{aligned} \|z_n - P_C(I - \lambda_n \nabla g)z_n\| &\leq \|z_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - Sw_n\| \\ &\quad + \|Sw_n - w_n\| + \|w_n - P_C(I - \lambda_n \nabla g)z_n\|. \end{aligned}$$

We get from Step 1, Step 2, Step 3, Step 4, (3.36), (3.37) and (3.38) that

$$(3.39) \quad \lim_{n \rightarrow \infty} \|z_n - P_C(I - \lambda_n \nabla g)z_n\| = 0.$$

We note that

$$(3.40) \quad \begin{aligned} \left\| P_C\left(I - \frac{2}{L}\nabla g\right)z_n - z_n \right\| &\leq \left\| P_C\left(I - \frac{2}{L}\nabla g\right)z_n - P_C\left(I - \lambda_n \nabla g\right)z_n \right\| \\ &\quad + \|P_C(I - \lambda_n \nabla g)z_n - z_n\| \\ &\leq \left(\frac{2}{L} - \lambda_n\right) \|\nabla g(z_n)\| + \|P_C(I - \lambda_n \nabla g)z_n - z_n\|. \end{aligned}$$

We conclude from **Assumption B2**, (3.39) and (3.40) that

$$(3.41) \quad \lim_{n \rightarrow \infty} \left\| P_C\left(I - \frac{2}{L}\nabla g\right)z_n - z_n \right\| = 0.$$

Let $\{z_{n_j}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_j} \rightharpoonup u$ for some $u \in C$ and

$$(3.42) \quad \limsup_{n \rightarrow \infty} \langle q - f(q), q - z_n \rangle = \lim_{j \rightarrow \infty} \langle q - f(q), q - z_{n_j} \rangle.$$

We note that it follows from Step 3 that $x_{n_j} \rightharpoonup u$ as $j \rightarrow \infty$. Since ∇g is $\frac{1}{L}$ -ism, $P_C(I - \frac{2}{L}\nabla g)$ is a nonexpansive self mapping on C . Therefore, from (3.42) and Lemma 2.4, we obtain

$$u = P_C\left(I - \frac{2}{L}\nabla g\right)u,$$

and hence $u \in U$ by Lemma 2.9. Because $z_n = Q_{r_n}(x_n - r_n Bx_n)$, we can write

$$\phi(z_n, y) + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle + \langle Bx_n, y - z_n \rangle \geq 0 \text{ for all } y \in C.$$

Because ϕ satisfies **Condition A**, we get

$$\langle Bx_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq \phi(y, z_n) \text{ for all } y \in C.$$

In particular for the subsequence $\{z_{n_j}\}$, we have for each $j = 1, 2, 3, \dots$

$$(3.43) \quad \langle Bx_{n_j}, y - z_{n_j} \rangle + \frac{1}{r_{n_j}} \langle y - z_{n_j}, z_{n_j} - x_{n_j} \rangle \geq \phi(y, z_{n_j}) \text{ for all } y \in C.$$

For $t \in (0, 1]$ and $y \in C$, set $y_t = ty + (1-t)u$. We obtain from (3.43) that

$$\begin{aligned}
 \langle y_t - z_{n_j}, By_t \rangle &\geq \langle y_t - z_{n_j}, By_t \rangle - \langle Bx_{n_j}, y_t - z_{n_j} \rangle \\
 &\quad - \langle y_t - z_{n_j}, \frac{z_{n_j} - x_{n_j}}{r_{n_j}} \rangle + \phi(y_t, z_{n_j}) \\
 &= \langle y_t - z_{n_j}, By_t - Bz_{n_j} \rangle + \langle y_t - z_{n_j}, Bz_{n_j} - Bx_{n_j} \rangle \\
 (3.44) \quad &\quad - \langle y_t - z_{n_j}, \frac{z_{n_j} - x_{n_j}}{r_{n_j}} \rangle + \phi(y_t, z_{n_j}).
 \end{aligned}$$

On the other hand, from inverse strong monotonicity of the mapping B and Step 3, we get that

$$(3.45) \quad \lim_{j \rightarrow \infty} \|Bz_{n_j} - Bx_{n_j}\| = 0, \text{ and}$$

$$(3.46) \quad \langle y_t - z_{n_j}, By_t - Bz_{n_j} \rangle \geq 0 \text{ for all } j = 1, 2, 3, \dots$$

Using (3.45) and (3.46), and letting $j \rightarrow \infty$ in (3.44), we get

$$(3.47) \quad \langle y_t - u, By_t \rangle \geq \phi(y_t, u).$$

It follows from **Condition A** and (3.47) that

$$\begin{aligned}
 0 &= \phi(y_t, y_t) \leq t\phi(y_t, y) + (1-t)\phi(y_t, u) \\
 &\leq t\phi(y_t, y) + (1-t)\langle y_t - u, By_t \rangle \\
 (3.48) \quad &= t\phi(y_t, y) + (1-t)t\langle y - u, By_t \rangle.
 \end{aligned}$$

Simplifying (3.48) we get

$$(3.49) \quad 0 \leq \phi(y_t, y) + (1-t)\langle y - u, By_t \rangle.$$

Using **Condition A** and letting $t \rightarrow 0$ in (3.50), we have

$$(3.50) \quad 0 \leq \phi(u, y) + \langle y - u, Bu \rangle.$$

Since y is an arbitrary element of C in (3.50), we conclude that

$$(3.51) \quad \phi(u, y) + \langle y - u, Bu \rangle \geq 0 \text{ for all } y \in C;$$

and thus $u \in EP(\phi, B)$.

Moreover, $z_{n_i} \rightarrow u$ and $x_{n_i} \rightarrow u$ imply that $w_{n_i} \rightarrow u$ as $i \rightarrow \infty$. By Lemma 2.4 and from (3.35), we conclude that $u \in \text{Fix}(S)$. Therefore, $u \in \Sigma$.

Since $P_{\Sigma}f(q) = q$, by Lemma 2.2

$$(3.52) \quad \langle f(q) - q, u - q \rangle \leq 0.$$

It follows from (3.42) and (3.52) that

$$(3.53) \quad \limsup_{n \rightarrow \infty} \langle f(q) - q, z_n - q \rangle = \lim_{j \rightarrow \infty} \langle f(q) - q, z_{n_j} - q \rangle = \langle f(q) - q, u - q \rangle \leq 0.$$

Finally, it follows from (3.4) and Lemma 2.1 that

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\beta_n(f(x_n) - f(q)) + \beta_n(f(q) - q) + (1 - \beta_n)(Sw_n - q)\|^2 \\ &\leq \beta_n^2 \|f(x_n) - f(q)\|^2 + (1 - \beta_n)^2 \|Sw_n - q\|^2 \\ &\quad + 2\beta_n \langle f(q) - q, x_{n+1} - q \rangle - \beta_n^2 \|f(q) - q\|^2 \\ &\leq \beta_n^2 k^2 \|x_n - q\|^2 + (1 - \beta_n)^2 \|x_n - q\|^2 \\ &\quad + 2\beta_n \langle f(q) - q, x_{n+1} - q \rangle - \beta_n^2 \|f(q) - q\|^2 \\ (3.54) \quad &\leq (1 - \beta_n(1 - k)) \|x_n - q\|^2 + 2\beta_n \langle f(q) - q, x_{n+1} - q \rangle. \end{aligned}$$

Put $a_n = \|x_n - q\|^2$, $v_n = \frac{2}{1 - k} \langle f(q) - q, x_{n+1} - q \rangle$ and $\delta_n = \beta_n(1 - k)$. Then for each $n = 1, 2, 3, \dots$, (3.54) is re-written as

$$(3.55) \quad a_{n+1} \leq (1 - \delta_n)a_n + \delta_n v_n.$$

We note that

$$\langle f(q) - q, x_{n+1} - q \rangle = \langle f(q) - q, x_{n+1} - z_{n+1} \rangle + \langle f(q) - q, z_{n+1} - q \rangle.$$

Since $\lim_{n \rightarrow \infty} \|x_{n+1} - z_{n+1}\| = 0$, it follows from (3.6) and (3.53) that $\limsup_{n \rightarrow \infty} v_n \leq 0$, $\lim_{n \rightarrow \infty} \delta_n = 0$

and $\sum_{n=1}^{\infty} \delta_n = \infty$. We get from Lemma 2.12 and (3.55) that

$$(3.56) \quad \lim_{n \rightarrow \infty} \|x_n - q\| = 0.$$

It follows from (3.56) that

$$(3.57) \quad \lim_{n \rightarrow \infty} \|z_n - q\| = 0.$$

$$(3.58) \quad \lim_{n \rightarrow \infty} \|w_n - q\| = 0.$$

This completes the proof. □

Remark 3.2. $q = P_{\Sigma}f(q)$ is the unique fixed point of the mapping $P_{\Sigma}f$. Moreover, $q = \Delta(f)$ satisfies Lemma 2.13 for the nonexpansive mapping S and for the approximate fixed point sequence $\{x_n\}$.

4. CONCLUSION

In this article, we have proposed an algorithm for approximating a common solution of a generalized equilibrium problem involving α -averaged mapping, fixed point problem of a directed nonexpansive mapping and a constrained minimization problem of convex function in the Hilbert space setting. We also proved strong convergence of the proposed algorithm by imposing some mild conditions on the parameters involved. The main result in this article modifies the results of Sangago et al. [24], Xu [30], Yazdi [31], Yazdi and Sababe [35].

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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