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CERTAIN APPLICATIONS OF (α,β)-(ξ,ϖ)-WEAKLY CYCLIC CONTRACTIONS IN C^*-ALGEBRA VALUED G-METRIC SPACES

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Abstract: This paper presents a novel class of generalized (α,β)-(ξ,ϖ)-weakly cyclic contraction mappings within the framework of C^*-algebra valued G-metric spaces ($\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$). We establish new common fixed-point theorems that broaden and unify several existing results in the literature. To illustrate the applicability and robustness of our approach, we provide concrete examples. Furthermore, the developed theory is employed to prove the existence and uniqueness of solutions to a functional Equations and to explore implications in homotopy theory.

Keywords: (α,β)-(ξ,ϖ)-weakly cyclic contraction mappings; ω-compatible; altering distance functions; $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$ and common fixed points.

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1. INTRODUCTION

Metric Fixed-Point Theory originated in 1922 with Banach's Contraction Mapping Theorem (CMT) [1] in a complete metric space $(\mathbb{V},\rho)$, offering stronger conditions than Brouwer's [2] theorem to ensure a unique fixed point. Over time, various generalizations have emerged in

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metric-like spaces such as dislocated, quasi, rectangular, *b*-metric, and fuzzy metric spaces. Mustafa and Sims [3] introduced *G*-metric spaces, while Zhenhua, Jiang, and Sun [4] developed C^*-algebra-valued metric spaces. Shen et al.[5] later combined these frameworks to study fixed points in complete C^*-algebra-valued *G*-metric spaces, with applications in differential equations.

In 1984, Khan, Swaleh, and Sessa [7] introduced the concept of an altering distance function, refining the conditions previously considered by Massa [8] within the framework of a complete metric space $(\mathbb{V}, \rho)$. This advancement led to further developments, including the work of Kirk et al. [9], who established fixed-point results using cyclic mappings. Since then, various forms of cyclic contractions have been studied in different topological settings [10, 11, 12, 13, 14, 15]. A notable contribution came during WCNA-2000, where Billy Rhoades [16] proposed the idea of weak contractive conditions. This concept has inspired a wide range of fixed-point results across diverse topological spaces, with extensive generalizations documented in the literature [17, 18, 19, 20, 21, 22, 23, 24, 25]. Later, Murthy et al. [26] extended the framework by considering weakly contractive conditions for pairs of mappings, further enriching the theory with new fixed-point theorems.

The primary aim of this paper is to develop unique common fixed point (UCFP) theorems within the framework of $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$ spaces, focusing on a class of generalized (α, β)-(ξ, ϖ)-weakly cyclic contraction mappings. In addition, we explore the applicability of these results to functional equations and homotopy theory, and provide insights into their mathematical significance and potential implications.

2. Preliminaries

This section provides a brief introduction to some fundamental aspects of C^*-algebra theory [27, 28]. Let $\mathscr{A}$ be a unital C^*-algebra with the unit element $1_{\mathscr{A}}$.

Define $\mathscr{A}_h = \{\mathfrak{e} \in \mathscr{A} : \mathfrak{e} = \mathfrak{e}^*\}$. An element $\mathfrak{e} \in \mathscr{A}$ is considered positive, denoted as $\mathfrak{e} \succeq 0_{\mathscr{A}}$, if $\mathfrak{e} = \mathfrak{e}^*$ and its spectrum $\eta(\mathfrak{e}) \subseteq [0, \infty)$. Here, $0_{\mathscr{A}}$ in $\mathscr{A}$ represents the zero element in $\mathscr{A}$, and $\eta(\mathfrak{e})$ denotes the spectrum of $\mathfrak{e}$. On $\mathscr{A}_h$, a natural partial ordering is defined by $\mathfrak{s} \preceq \mathfrak{v}$ if and only if $\mathfrak{v} - \mathfrak{s} \succeq 0_{\mathscr{A}}$. We denote $\mathscr{A}_+ = \{\mathfrak{e} \in \mathscr{A} : \mathfrak{e} \succeq 0_{\mathscr{A}}\}$ and $\mathscr{A}' = \{\mathfrak{e} \in \mathscr{A} : \mathfrak{e}\mathfrak{d} = \mathfrak{d}\mathfrak{e} \ \forall \mathfrak{d} \in \mathscr{A}\}$.

Definition 2.1:([5, 6]) Let $\mathbb{V}$ be a non-empty set and denote the associated C^*-algebra by $\mathscr{A}$. A mapping $\rho_{c^*}: \mathbb{V}\times\mathbb{V}\times\mathbb{V}\to\mathscr{A}$ that satisfies the required conditions is referred to as a C^*-algebra-valued G-metric.

(i) $\rho_{c^*}(\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3)=0_{\mathscr{A}}$ if $\mathfrak{s}_1=\mathfrak{s}_2=\mathfrak{s}_2$,

(ii) $0_{\mathscr{A}}\prec\rho_{c^*}(\mathfrak{s}_1,\mathfrak{s}_1,\mathfrak{s}_2)$ for all $\mathfrak{s}_1,\mathfrak{s}_2\in\mathbb{V}$ with $\mathfrak{s}_1\neq\mathfrak{s}_2$,

(iii) $\rho_{c^*}(\mathfrak{s}_1,\mathfrak{s}_1,\mathfrak{s}_2)\preceq\rho_{c^*}(\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3)$ for all $\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3\in\mathbb{V}$ with $\mathfrak{s}_1\neq\mathfrak{s}_3$,

(iv) $\rho_{c^*}(\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3)=\rho_{c^*}(P[\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3])$ where P is a permutation of $\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3$ (symmetry),

(v) $\rho_{c^*}(\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3)\preceq\rho_{c^*}(\mathfrak{s}_1,\mathfrak{s}_4,\mathfrak{s}_4)+\rho_{c^*}(\mathfrak{s}_4,\mathfrak{s}_2,\mathfrak{s}_3)$ for all $\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3,\mathfrak{s}_4\in\mathbb{V}$ (rectangle inequality)

Then the structure $(\mathbb{V},\mathscr{A},\rho_{c^*})$ is called a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$.

Example 2.2:([5, 6]) Let $\mathbb{V}=\mathbb{R}$ and define

$\rho_{c^*}: \mathbb{V}\times\mathbb{V}\times\mathbb{V}\to\mathscr{A}$ as $\rho_{c^*}(\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3)=||\mathfrak{s}_1-\mathfrak{s}_2||I_{\mathscr{A}}+||\mathfrak{s}_2-\mathfrak{s}_3||I_{\mathscr{A}}+||\mathfrak{s}_3-\mathfrak{s}_1||I_{\mathscr{A}}$ for all $\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3\in\mathbb{V}$ then $(\mathbb{V},\mathscr{A},\rho_{c^*})$ is a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}\mathbb{G}\mathbb{MS}$. ρ_{c^*} is a C^*-algebra valued G-metric.

Definition 2.3:([5, 6]) Assume that $(\mathbb{V},\mathscr{A},\rho_{c^*})$ is a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. According to $\mathscr{A}$ a sequence $\{\mathfrak{s}_k\}$ in $\mathbb{V}$ is defined as:

(1) C^*-algebra valued G-convergent to a point $\mathfrak{s}\in\mathbb{V}$ if, for each $0_{\mathscr{A}}\prec\varepsilon$, there exist $x,y\in\mathbb{N}$ such that $\rho_{c^*}(\mathfrak{s},\mathfrak{s}_x,\mathfrak{s}_y)\prec\varepsilon$. We can also use different presentations for that as follows:

$$\mathfrak{s}_x\to\mathfrak{s} \text{ or } \lim_{x\to\infty}\rho_{c^*}(\mathfrak{s},\mathfrak{s}_x,\mathfrak{s}_y)=0_{\mathscr{A}} \text{ or } \lim_{x\to\infty}\mathfrak{s}_x=\mathfrak{s}.$$

(2) C^*-algebra valued G-Cauchy sequence, if for $0_{\mathscr{A}}\prec\varepsilon$, there exists positive integer $x^*\in\mathbb{N}$ such that $\rho_{c^*}(\mathfrak{s}_x,\mathfrak{s}_y,\mathfrak{s}_z)\prec\varepsilon\ \forall\ x,y,z\geq x^*$ or $\rho_{c^*}(\mathfrak{s}_x,\mathfrak{s}_y,\mathfrak{s}_z)\to 0_{\mathscr{A}}$ as $x,y,z\to\infty$ or $||\rho_{c^*}(\mathfrak{s}_x,\mathfrak{s}_y,\mathfrak{s}_z)||\to 0$.

(3) It is referred to as being complete when a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}\mathbb{G}\mathbb{MS}$ $(\mathbb{V},\mathscr{A},\rho_{c^*})$ is present. If each Cauchy sequence in $\mathbb{V}$ converges to a point in $\mathbb{V}$.

Lemma 2.4: ([5, 6]) Let $\mathscr{A}$ be a C^*-algebra with the identity element $I_{\mathscr{A}}$ and $\mathfrak{v}$ be a positive element of $\mathscr{A}$.Then

(i) There is a unique element $\mathfrak{u}\in\mathscr{A}_+$ such that $\mathfrak{u}^2=\mathfrak{v}$,

(ii) The set $\mathscr{A}_+=\{\mathfrak{v}^*\mathfrak{v}/\mathfrak{v}\in\mathscr{A}\}$ with a conjugate-linear involution $*:\mathscr{A}\to\mathscr{A}$,

(iii) $\mathfrak{v},\mathfrak{u}\in\mathscr{A}$, and $0_{\mathscr{A}}\preceq\mathfrak{v}\preceq\mathfrak{u}$ then $||\mathfrak{v}||\leq||\mathfrak{u}||$,

(iv) If $\mathfrak{v} \in \mathscr{A}_+$ with $||\mathfrak{v}|| < \frac{1}{2}$ then $(I - \mathfrak{v})$ is invertible and $||\mathfrak{v}(I - \mathfrak{v})^{-1}|| < 1$.

3. Main Results

Definition 3.1: A function $\xi : \mathscr{A}_+ \to \mathscr{A}_+$ is called an altering distance function if the following properties are satisfied:

(a) ξ is continuous and monotonically increasing in $\mathscr{A}_+$;

(b) $\xi(\mathfrak{s}) = 0_{\mathscr{A}}$ if and only if $\mathfrak{s} = 0_{\mathscr{A}}$.

Definition 3.2: Let $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. A function $\varpi : \mathbb{V} \to \mathscr{A}_+$ is called *lower semi-continuous* (abbreviated $\mathbb{LSC}$) at a point $\mathfrak{s}_0 \in \mathbb{V}$ if for every sequence $\{\mathfrak{s}_\mathfrak{v}\} \subseteq \mathbb{V}$ converging to $\mathfrak{s}_0$, we have either $\liminf_{\mathfrak{s}_\mathfrak{v} \to \mathfrak{s}_0} \varpi(\mathfrak{s}_\mathfrak{v}) = \infty$ or $\varpi(\mathfrak{s}_0) \preceq \liminf_{\mathfrak{s}_\mathfrak{v} \to \mathfrak{s}_0} \varpi(\mathfrak{s}_\mathfrak{v})$, where $\preceq$ denotes the order on $\mathscr{A}_+$. If ϖ is $\mathbb{LSC}$ at every point in $\mathbb{V}$, then it is said to be $\mathbb{LSC}$ on $\mathbb{V}$.

example 3.3: Let $\mathbb{V} = \mathbb{R}$ and let $\mathscr{A} = M_2(\mathbb{C})$, the algebra of 2×2 complex matrices. Define $\mathscr{A}_+$ to be the set of positive semi-definite matrices in $\mathscr{A}$. Consider the function $\varpi : \mathbb{R} \to \mathscr{A}_+$ defined by $\varpi(\mathfrak{s}) = \begin{bmatrix} \max(0, \mathfrak{s}) & 0 \\ 0 & 1 \end{bmatrix}$. Then clearly, ϖ is lower semi-continuous on $\mathbb{R}$.

Definition 3.4: Let $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. A pair of self-mappings $\mathfrak{f}$ and $\mathfrak{g}$ on $\mathbb{V}$ are said to be weakly compatible if they commute at their coincidence points. In other words, if $\mathfrak{f}\mathfrak{s} = \mathfrak{g}\mathfrak{s}$ for some $\mathfrak{s} \in \mathbb{V}$, then $\mathfrak{f}\mathfrak{g}\mathfrak{s} = \mathfrak{g}\mathfrak{f}\mathfrak{s}$.

Definition 3.5: Let $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. Let $\mathfrak{r} \in \mathbb{N}$, $\{\mathfrak{h}_i\}_{i=1}^{\mathfrak{r}} \subseteq \mathbb{V}$ and $\Xi = \cup_{i=1}^{\mathfrak{r}} \mathfrak{h}_i$ and $\mathscr{G} : \Xi \to \Xi$. Then, $\mathscr{G}$ is the cyclic operator if:

(i) $\mathfrak{h}_i$, $i = 1, 2, \cdots, \mathfrak{r}$ are non-empty and closed,

(ii) $\mathscr{G}(\mathfrak{h}_1) \subseteq \mathfrak{h}_2 \cdots \mathscr{G}(\mathfrak{h}_{\mathfrak{r}-1}) \subseteq \mathfrak{h}_\mathfrak{r}, \mathscr{G}(\mathfrak{h}_\mathfrak{r}) \subseteq \mathfrak{h}_1$.

Definition 3.6: Let $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. Let $\mathfrak{r} \in \mathbb{N}$, $\{\mathfrak{h}_i\}_{i=1}^{\mathfrak{r}} \subseteq \mathbb{V}$ and $\Xi = \cup_{i=1}^{\mathfrak{r}} \mathfrak{h}_i$ and $\alpha, \beta : \Xi \times \Xi \to \mathscr{A}_+$. If $\mathscr{F}, \mathscr{G}, \mathfrak{f}, \mathfrak{g} \colon \Xi \to \Xi$ then the mappings pair $(\mathscr{F}, \mathscr{G})$ is $(\mathfrak{f}, \mathfrak{g})$-cyclic-$(\alpha, \beta)$-admissible if:

(i) $\alpha(\mathfrak{f}\mathfrak{s}, \mathfrak{g}\mathfrak{e}) \succeq 1_{\mathscr{A}}$ implies $\beta(\mathscr{F}\mathfrak{s}, \mathscr{G}\mathfrak{e}) \succeq 1_{\mathscr{A}}$ for some $(\mathfrak{s}, \mathfrak{e}) \in \mathfrak{h}_i \times \mathfrak{h}_{i+1}$ $i = 1, 2, \cdots \mathfrak{r}$ (with $\mathfrak{h}_{i+1} = \mathfrak{h}_1$);

(ii) $\beta(\mathfrak{g}\mathfrak{s}, \mathfrak{f}\mathfrak{e}) \succeq 1_{\mathscr{A}}$ implies $\alpha(\mathscr{G}\mathfrak{s}, \mathscr{F}\mathfrak{e}) \succeq 1_{\mathscr{A}}$ for some $(\mathfrak{s}, \mathfrak{e}) \in \mathfrak{h}_i \times \mathfrak{h}_{i+1}$ $i = 1, 2, \cdots \mathfrak{r}$ (with $\mathfrak{h}_{i+1} = \mathfrak{h}_1$)

Example 3.7: Let $\mathbb{V}=\{x_1,x_2,x_3\}$ and let $\mathscr{A}=M_2(\mathbb{C})$, the algebra of 2×2 complex matrices. Let $\mathscr{A}_+$ be the set of positive semi-definite matrices in $\mathscr{A}$. Define subsets $\mathfrak{h}_1=\{x_1\}$, $\mathfrak{h}_2=\{x_2\}$, $\mathfrak{h}_3=\{x_3\}$, so that $\Xi=\bigcup_{i=1}^{3}\mathfrak{h}_i=\{x_1,x_2,x_3\}$. Define the mappings:

$$\mathfrak{f}(x_1)=x_2,\quad \mathfrak{f}(x_2)=x_3,\quad \mathfrak{f}(x_3)=x_1,$$

$$\mathfrak{g}(x_1)=x_3,\quad \mathfrak{g}(x_2)=x_1,\quad \mathfrak{g}(x_3)=x_2,$$

$$\mathscr{F}(x_i)=x_i,\quad \mathscr{G}(x_i)=x_i\quad \text{(identity mappings)}.$$

Define control functions $\alpha,\beta:\Xi\times\Xi\to\mathscr{A}_+$ by:

$$\alpha(x_i,x_j)=\begin{cases}\begin{bmatrix}2&0\\0&2\end{bmatrix} & \text{if } i=j,\\[2ex] \begin{bmatrix}1&0\\0&1\end{bmatrix} & \text{if } i\neq j,\end{cases}\qquad \beta(x_i,x_j)=\begin{cases}\begin{bmatrix}1&0\\0&1\end{bmatrix} & \text{if } i=j,\\[2ex] \begin{bmatrix}3&0\\0&3\end{bmatrix} & \text{if } i\neq j.\end{cases}$$

Now consider $(x_1,x_2)\in\mathfrak{h}_1\times\mathfrak{h}_2$:

$$\mathfrak{f}(x_1)=x_2,\quad \mathfrak{g}(x_2)=x_1\Rightarrow\alpha(x_2,x_1)=\begin{bmatrix}1&0\\0&1\end{bmatrix}\succeq I_2 \text{ implies}$$

$$\mathscr{F}(x_1)=x_1,\quad \mathscr{G}(x_2)=x_2\Rightarrow\beta(x_1,x_2)=\begin{bmatrix}3&0\\0&3\end{bmatrix}\succeq I_2.$$

and

$$\mathfrak{g}(x_1)=x_3,\quad \mathfrak{f}(x_2)=x_3\Rightarrow\beta(x_3,x_3)=\begin{bmatrix}1&0\\0&1\end{bmatrix}\succeq I_2 \text{ implies}$$

$$\mathscr{G}(x_1)=x_1,\quad \mathscr{F}(x_2)=x_2\Rightarrow\alpha(x_1,x_2)=\begin{bmatrix}1&0\\0&1\end{bmatrix}\succeq I_2.$$

Similarly, for $(x_2,x_3)\in\mathfrak{h}_2\times\mathfrak{h}_3$:

$$\mathfrak{f}(x_2)=x_3,\quad \mathfrak{g}(x_3)=x_2\Rightarrow\alpha(x_3,x_2)=\begin{bmatrix}1&0\\0&1\end{bmatrix}\succeq I_2 \text{ implies}$$

$$\mathscr{F}(x_2) = x_2, \quad \mathscr{G}(x_3) = x_3 \Rightarrow \beta(x_2, x_3) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \succeq I_2.$$

and

$$\mathfrak{g}(x_2) = x_1, \quad \mathfrak{f}(x_3) = x_1 \Rightarrow \beta(x_1, x_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succeq I_2 \text{ implies}$$

$$\mathscr{G}(x_2) = x_2, \quad \mathscr{F}(x_3) = x_3 \Rightarrow \alpha(x_2, x_3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succeq I_2.$$

Hence, the pair $(\mathscr{F}, \mathscr{G})$ is $(\mathfrak{f}, \mathfrak{g})$-cyclic-$(\alpha, \beta)$-admissible in the C*-algebra valued G-metric space $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$.

Definition 3.8: Let $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. Let $\mathfrak{r} \in \mathbb{N}$, $\mathfrak{h}_1, \mathfrak{h}_2 \cdots \mathfrak{h}_\mathfrak{r}$ be non-void subsets of $\mathbb{V}$ and $\Xi = \cup_{i=1}^{\mathfrak{r}} \mathfrak{h}_i$. An operators $\mathscr{F}, \mathscr{G}, \mathfrak{f}, \mathfrak{g} : \Xi \to \Xi$ be satisfying $(\mathscr{F}, \mathscr{G})$ is a $(\mathfrak{f}, \mathfrak{g})$-cyclic-$(\alpha, \beta)$-admissible. Then $(\mathscr{F}, \mathscr{G})$ is a $(\mathfrak{f}, \mathfrak{g})$-cyclic (α, β)-(ξ, ϖ)-generalized weakly contraction type-I and type-II if

(i) $\Xi = \cup_{i=1}^{\mathfrak{r}} \mathfrak{h}_i$ is a cyclic representation of Ξ with respect to $\mathscr{F}, \mathscr{G}, \mathfrak{f}, \mathfrak{g}$ respectively,

(ii) for any $(\mathfrak{s}, \mathfrak{e}) \in (\mathfrak{h}_i, \mathfrak{h}_{i+1})$, $i = 1, 2, \cdots \mathfrak{r}$ (with $\mathfrak{h}_{i+1} = \mathfrak{h}_1$) and $a \in \mathscr{A}$ with $||a|| < 1$,

type-I:

(1) $\alpha(\mathfrak{f}\mathfrak{s}, \mathfrak{g}\mathfrak{e})\, \beta(\mathfrak{g}\mathfrak{s}, \mathfrak{f}\mathfrak{e}) \succeq 1_{\mathscr{A}} \quad \Rightarrow \xi(\rho_{c^*}(\mathscr{F}\mathfrak{s}, \mathscr{G}\mathfrak{e}, \mathscr{G}\mathfrak{e})) \preceq \xi(a\Delta_1(\mathfrak{s}, \mathfrak{e})a^*) - \varpi(a\Delta_2(\mathfrak{s}, \mathfrak{e})a^*)$

type-II:

(2) $\quad \alpha(\mathfrak{f}\mathfrak{s}, \mathfrak{g}\mathfrak{e})\, \beta(\mathfrak{g}\mathfrak{s}, \mathfrak{f}\mathfrak{e})\, \xi(\rho_{c^*}(\mathscr{F}\mathfrak{s}, \mathscr{G}\mathfrak{e}, \mathscr{G}\mathfrak{e})) \preceq \xi(a\Delta_1(\mathfrak{s}, \mathfrak{e})a^*) - \varpi(a\Delta_2(\mathfrak{s}, \mathfrak{e})a^*)$

where $\Delta_1(\mathfrak{s}, \mathfrak{e}) = \max\left\{ \rho_{c^*}(\mathfrak{f}\mathfrak{s}, \mathfrak{g}\mathfrak{e}, \mathfrak{g}\mathfrak{e}), \rho_{c^*}(\mathfrak{f}\mathfrak{s}, \mathscr{F}\mathfrak{s}, \mathscr{F}\mathfrak{s})\, \rho_{c^*}(\mathfrak{g}\mathfrak{e}, \mathscr{G}\mathfrak{e}, \mathscr{G}\mathfrak{e}) \right\}$,

$$\Delta_2(\mathfrak{s}, \mathfrak{e}) = \min\left\{ \begin{array}{c} \rho_{c^*}(\mathfrak{f}\mathfrak{s}, \mathfrak{g}\mathfrak{e}, \mathfrak{g}\mathfrak{e}), \\ \frac{1}{2}\left(\rho_{c^*}(\mathfrak{f}\mathfrak{s}, \mathscr{G}\mathfrak{e}, \mathscr{G}\mathfrak{e}) + \rho_{c^*}(\mathfrak{g}\mathfrak{e}, \mathscr{F}\mathfrak{s}, \mathscr{F}\mathfrak{s})\right) \end{array} \right\}$$

and $\varpi : \mathscr{A}_+ \to \mathscr{A}_+$ is lower semi-continuous, such that $\varpi(\mathfrak{s}) \succ 0_{\mathscr{A}}$ for all $\mathfrak{s} \succ 0_{\mathscr{A}}$ and discontinuous at $\mathfrak{s} = 0_{\mathscr{A}}$ with $\varpi(0_{\mathscr{A}}) = 0_{\mathscr{A}}$, $\xi : \mathscr{A}_+ \to \mathscr{A}_+$ is an altering distance function.

Theorem 3.9: Let $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. Let $\mathfrak{r} \in \mathbb{N}$, $\mathfrak{h}_1, \mathfrak{h}_2 \cdots \mathfrak{h}_\mathfrak{r}$ be non-void subsets of $\mathbb{V}$ and $\Xi = \cup_{i=1}^{\mathfrak{r}} \mathfrak{h}_i$ and $\alpha, \beta : \Xi \times \Xi \to \mathscr{A}_+$ be two mappings. Let $\mathscr{F}, \mathscr{G}, \mathfrak{f}, \mathfrak{g}$ be a self mappings on Ξ and pair $(\mathscr{F},\mathscr{G})$ is a $(\mathfrak{f},\mathfrak{g})$-cyclic-$(\alpha,\beta)$-admissible mappings such that $(\mathscr{F},\mathscr{G})$ is a $(\mathfrak{f},\mathfrak{g})$-cyclic (α,β)-(ξ,ϖ)-generalized weakly contraction type-I and type-II satisfying the following conditions:

(3.1) $\mathscr{F}(\Xi)\subseteq\mathfrak{g}(\Xi)$ and $\mathscr{G}(\Xi)\subseteq\mathfrak{f}(\Xi)$ with $\mathfrak{g}(\Xi)$ or $\mathfrak{f}(\Xi)$ is closed subspace of Ξ;

(3.2) there exists $(\mathfrak{s}_0,\mathfrak{s}_1)\in\mathfrak{h}_i\times\mathfrak{h}_{i+1}$ $i=1,2,\cdots\mathfrak{r}$ (with $\mathfrak{h}_{i+1}=\mathfrak{h}_1$) with $\alpha(\mathfrak{f}\mathfrak{s}_0,\mathfrak{g}\mathfrak{s}_1)\succeq 1_{\mathscr{A}}$ and $\beta(\mathfrak{g}\mathfrak{s}_0,\mathfrak{f}\mathfrak{s}_1)\succeq 1_{\mathscr{A}}$;

(3.3) if $\{\mathfrak{s}_\mathfrak{v}\}_{\mathfrak{v}=1}^{\infty}$ is a sequence in $\mathbb{V}$ with $\alpha(\mathfrak{s},\mathfrak{s}_{\mathfrak{v}+1})\succeq 1_{\mathscr{A}}$ for all $\mathfrak{v}$ and $\lim_{\mathfrak{v}\to\infty}\mathfrak{s}_\mathfrak{v}=\mathfrak{s}$ then $\beta(\mathfrak{g}\mathfrak{t},\mathfrak{s}_{\mathfrak{v}+1})\succeq 1_{\mathscr{A}}$ for some $\mathfrak{t}\in\Xi$;

(3.4) $\alpha(\mathfrak{f}\mathfrak{s},\mathfrak{s}_\mathfrak{v})\succeq 1_{\mathscr{A}}$ and $\beta(\mathfrak{g}\mathfrak{s},\mathfrak{s}_\mathfrak{v})\succeq 1_{\mathscr{A}}$ whenever, $\mathscr{F}\mathfrak{y}=\mathfrak{s}=\mathfrak{f}\mathfrak{y}$ and $\mathscr{G}\mathfrak{x}=\mathfrak{s}=\mathfrak{g}\mathfrak{x}$

(3.5) $(\mathscr{F},\mathfrak{f})$ and $(\mathscr{G},\mathfrak{g})$ are weakly compatible pairs.

Then, $\mathscr{F},\mathscr{G},\mathfrak{f}$ and $\mathfrak{g}$ have a unique common fixed point in $\cap_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$.

Proof Let $\mathfrak{s}_0\in\mathfrak{h}_1$ ($\mathfrak{h}_i$ is non-void for all i) be an arbitrary point. Since, $\mathscr{F}(\Xi)\subseteq\mathfrak{g}(\Xi)$ and $\mathscr{G}(\Xi)\subseteq\mathfrak{f}(\Xi)$ then consider the sequence $\{\mathfrak{s}_\mathfrak{v}\}$ and $\{\mathfrak{e}_\mathfrak{v}\}$ in $\mathbb{V}$ as

$$\mathscr{F}\mathfrak{s}_{2\mathfrak{v}}=\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1}=\mathfrak{e}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1}=\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}+2}=\mathfrak{e}_{2\mathfrak{v}+2},\text{ for }\mathfrak{v}\in\mathbb{N}\cup\{0\}.\tag{3}$$

Observes that in C^*-algebra, if $\kappa,b\in\mathscr{A}_+$ and $\kappa\preceq b$, then for any $x\in\mathscr{A}_+$ both $x^*\kappa x$ and x^*bx are positive. If $\mathfrak{e}_{2\mathfrak{v}}=\mathfrak{e}_{2\mathfrak{v}+1}$, then $\mathfrak{e}_{2\mathfrak{v}}$ is a point of coincidence of $\mathscr{F}$, $\mathscr{G}$ $\mathfrak{f}$ and $\mathfrak{g}$. Therefore, we assume that $\mathfrak{e}_{2\mathfrak{v}}\neq\mathfrak{e}_{2\mathfrak{v}+1}$ for all $\mathfrak{v}\geq 0$. Since, $\alpha(\mathfrak{f}\mathfrak{s}_0,\mathfrak{g}\mathfrak{s}_1)\succeq 1_{\mathscr{A}}$ and $(\mathscr{F},\mathscr{G})$ is a $(\mathfrak{f},\mathfrak{g})$-cyclic-$(\alpha,\beta)$-admissible mapping, we have

$$\beta(\mathfrak{g}\mathfrak{s}_1,\mathfrak{f}\mathfrak{s}_2)=\beta(\mathscr{F}\mathfrak{s}_0,\mathscr{G}\mathfrak{s}_1)\succeq 1_{\mathscr{A}}\implies\alpha(\mathscr{G}\mathfrak{s}_1,\mathscr{F}\mathfrak{s}_2)=\alpha(\mathfrak{f}\mathfrak{s}_2,\mathfrak{g}\mathfrak{s}_3)\succeq 1_{\mathscr{A}}$$

$$\text{and }\beta(\mathfrak{g}\mathfrak{s}_3,\mathfrak{f}\mathfrak{s}_4)=\beta(\mathscr{F}\mathfrak{s}_2,\mathscr{G}\mathfrak{s}_3)\succeq 1_{\mathscr{A}}\implies\alpha(\mathscr{G}\mathfrak{s}_3,\mathscr{F}\mathfrak{s}_4)=\alpha(\mathfrak{f}\mathfrak{s}_4,\mathfrak{g}\mathfrak{s}_5)\succeq 1_{\mathscr{A}}.$$

By continuing this procedure, we obtain that:

$$\alpha(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1})\succeq 1_{\mathscr{A}}\text{ and }\beta(\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}+2})\succeq 1_{\mathscr{A}}\ \forall\ \mathfrak{v}\in\mathbb{N}\cup\{0\}.\tag{4}$$

Similarly, Since, $\beta(\mathfrak{g}\mathfrak{s}_0,\mathfrak{f}\mathfrak{s}_1)\succeq 1_{\mathscr{A}}$ and $(\mathscr{F},\mathscr{G})$ is a $(\mathfrak{f},\mathfrak{g})$-cyclic-$(\alpha,\beta)$-admissible mapping, we have

$$\alpha(\mathfrak{f}\mathfrak{s}_1,\mathfrak{g}\mathfrak{s}_2)=\alpha(\mathscr{G}\mathfrak{s}_0,\mathscr{F}\mathfrak{s}_1)\succeq 1_{\mathscr{A}}\implies\beta(\mathscr{F}\mathfrak{s}_1,\mathscr{G}\mathfrak{s}_2)=\beta(\mathfrak{g}\mathfrak{s}_2,\mathfrak{f}\mathfrak{s}_3)\succeq 1_{\mathscr{A}}$$

$$\text{and }\alpha(\mathfrak{f}\mathfrak{s}_3,\mathfrak{g}\mathfrak{s}_4)=\alpha(\mathscr{G}\mathfrak{s}_2,\mathscr{F}\mathfrak{s}_3)\succeq 1_{\mathscr{A}}\implies\beta(\mathscr{F}\mathfrak{s}_3,\mathscr{G}\mathfrak{s}_4)=\beta(\mathfrak{g}\mathfrak{s}_4,\mathfrak{f}\mathfrak{s}_5)\succeq 1_{\mathscr{A}}.$$

By continuing this procedure, we obtain that:

$$\beta\left(\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}},\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}+1}\right)\succeq 1_{\mathscr{A}} \text{ and } \alpha\left(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}+1},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+2}\right)\succeq 1_{\mathscr{A}}\ \forall\ \mathfrak{v}\in\mathbb{N}\cup\{0\}. \tag{5}$$

From Eq.(4) and Eq.(5), it follows that:

$$\alpha\left(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1}\right)\beta\left(\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}},\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}+1}\right)\succeq 1_{\mathscr{A}}\ \forall\ \mathfrak{v}\in\mathbb{N}\cup\{0\}.$$

Then from (1), we have $i=i(\mathfrak{v})\in\{1,2,\cdots,\mathfrak{r}\}$ for all $\mathfrak{v}$ such that $(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+1})\in\mathfrak{h}_i\times\mathfrak{h}_{i+1}$ so that

$$\begin{aligned}\xi\left(\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2})\right) &= \xi\left(\rho_{c^*}(\mathscr{F}\mathfrak{s}_{2\mathfrak{v}},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1})\right)\\ &\preceq \xi\left(a\Delta_1(\mathfrak{s}_{2\mathfrak{v}},\mathfrak{s}_{2\mathfrak{v}+1})a^*\right)-\varpi\left(a\Delta_2(\mathfrak{s}_{2\mathfrak{v}},\mathfrak{s}_{2\mathfrak{v}+1})a^*\right)\end{aligned} \tag{6}$$

where

$$\begin{aligned}\Delta_1(\mathfrak{s}_{2\mathfrak{v}},\mathfrak{s}_{2\mathfrak{v}+1}) &= \max\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1}\right),\rho_{c^*}\left(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}},\mathscr{F}\mathfrak{s}_{2\mathfrak{v}},\mathscr{F}\mathfrak{s}_{2\mathfrak{v}}\right),\\ \rho_{c^*}\left(\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1}\right)\end{array}\right\}\\ &= \max\left\{\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1}\right),\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1}\right),\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)\right\}\\ &= \max\left\{\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1}\right),\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)\right\}\end{aligned}$$

and

$$\begin{aligned}\Delta_2(\mathfrak{s}_{2\mathfrak{v}},\mathfrak{s}_{2\mathfrak{v}+1}) &= \min\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1}\right),\\ \frac{1}{2}\left(\rho_{c^*}\left(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1}\right)+\rho_{c^*}\left(\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{F}\mathfrak{s}_{2\mathfrak{v}},\mathscr{F}\mathfrak{s}_{2\mathfrak{v}}\right)\right)\end{array}\right\}\\ &= \min\left\{\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1}\right),\frac{1}{2}\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)\right\}.\end{aligned}$$

If possible, let for some $\mathfrak{v}$, $\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1}\right)\prec\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)$ then,
$\Delta_1(\mathfrak{s}_{2\mathfrak{v}},\mathfrak{s}_{2\mathfrak{v}+1})=\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)$. Since,
$0_{\mathscr{A}}\prec\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)-\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1}\right)\preceq\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)$, we have
$\Delta_2(\mathfrak{s}_{2\mathfrak{v}},\mathfrak{s}_{2\mathfrak{v}+1})\succ 0_{\mathscr{A}}$. Then from Eq.(6) and the property off ξ and ϖ functions, we have

$$\begin{aligned}\xi\left(\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2})\right) &\preceq \xi\left(a\Delta_1(\mathfrak{s}_{2\mathfrak{v}},\mathfrak{s}_{2\mathfrak{v}+1})a^*\right)-\varpi\left(a\Delta_2(\mathfrak{s}_{2\mathfrak{v}},\mathfrak{s}_{2\mathfrak{v}+1})a^*\right)\\ &\preceq \xi\left(a\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)a^*\right)-\varpi\left(a\Delta_2(\mathfrak{s}_{2\mathfrak{v}},\mathfrak{s}_{2\mathfrak{v}+1})a^*\right)\\ &\prec \xi\left(a\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)a^*\right)\end{aligned}$$

Using the monotonically increasing property of ξ function, we have

$$||\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2})|| \leq ||a||^2||\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2})|| < ||\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2})|| \tag{7}$$

which is a contradiction. Hence, for all $\mathfrak{v} \geq 0$, we have

$$\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}) \preceq \rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1}).$$

Hence, we conclude that
$\Delta_1(\mathfrak{s}_{2\mathfrak{v}},\mathfrak{s}_{2\mathfrak{v}+1}) = \rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1})$ and $\Delta_2(\mathfrak{s}_{2\mathfrak{v}},\mathfrak{s}_{2\mathfrak{v}+1}) = \frac{1}{2}\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2})$.
Then from (6), we have

$$\xi(\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2})) \preceq \xi(a\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1})a^*) - \varpi\left(a\frac{1}{2}\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2})a^*\right)$$

Since $\|a\| < 1$, and both ξ and ϖ are continuous on $\mathscr{A}_+$, with $\varpi(\mathfrak{s}) \succ 0_{\mathscr{A}}$ for all $\mathfrak{s} \succ 0_{\mathscr{A}}$, we conclude:

$$\xi(\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2})) \preceq \xi(a\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1})a^*)$$

which implies that

$$\begin{aligned}
\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}) &\preceq a\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1})a^* \\
&\preceq (a)^2\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}-1},\mathfrak{e}_{2\mathfrak{v}},\mathfrak{e}_{2\mathfrak{v}})(a^*)^2 \\
&\vdots \\
&\preceq (a)^{2\mathfrak{v}+1}\rho_{c^*}(\mathfrak{e}_0,\mathfrak{e}_1,\mathfrak{e}_1)(a^*)^{2\mathfrak{v}+1} \to 0_{\mathscr{A}} \text{ as } \mathfrak{v} \to \infty.
\end{aligned}$$

Again from Eq.(4) and Eq.(5), it follows that:

$$\alpha(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}+1},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+2})\,\beta(\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}+2}) \succeq 1_{\mathscr{A}}\ \forall\ \mathfrak{v} \in \mathbb{N}\cup\{0\}.$$

Then from (1), we have $i = i(\mathfrak{v}) \in \{1,2,\cdots,\mathfrak{r}\}$ for all $\mathfrak{v}$ such that $(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2}) \in \mathfrak{h}_i \times \mathfrak{h}_{i+1}$ and arguing as above, we obtain

$$\begin{aligned}
\xi(\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+3},\mathfrak{e}_{2\mathfrak{v}+3})) &= \xi(\rho_{c^*}(\mathscr{F}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+2},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+2})) \\
&\preceq \xi(a\Delta_1(\mathfrak{s}_{2\mathfrak{v}+1},\mathfrak{s}_{2\mathfrak{v}+2})a^*) - \varpi(a\Delta_2(\mathfrak{s}_{2\mathfrak{v}+1},\mathfrak{s}_{2\mathfrak{v}+2})a^*) \to 0_{\mathscr{A}} \text{ as } \mathfrak{v} \to \infty.
\end{aligned}$$

Therefore, for all $\mathfrak{v} \in \mathbb{N}\cup\{0\}$ we have $\lim_{\mathfrak{v}\to\infty}||\rho_{c^*}(\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}+1},\mathfrak{e}_{\mathfrak{v}+1})|| = 0$ Next,we will show that $\{\mathfrak{e}_{\mathfrak{v}}\}$ is a Cauchy sequence in $\mathbb{V}$ with regard to $\mathscr{A}$. For this,it is enough to show that the sub-sequence $\{\mathfrak{e}_{2\mathfrak{v}}\}$ is a Cauchy sequence. To the contrary, suppose that $\{\mathfrak{e}_{2\mathfrak{v}}\}$ not be a Cauchy sequence,then for some $\varepsilon \succ 0_{\mathscr{A}}$ and the sequence of natural numbers $\{2\mathfrak{v}(j)\}$ and $\{2\mathfrak{u}(j)\}$ such that $2\mathfrak{u}(j) > 2\mathfrak{v}(j) \geq 2j$ for $j \in \mathbb{N}$ and

$$\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)}\right) \succeq \varepsilon \tag{8}$$

corresponding to $2\mathfrak{v}(j)$. We can choose $2\mathfrak{v}(j)$ to be the smallest such that(8) is satisfied. Then we have

$$\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)-1}, \mathfrak{e}_{2\mathfrak{u}(j)-1}\right) \prec \varepsilon. \tag{9}$$

Using Inequalities (8) and(9) and the rectangle inequity,we have

$$\begin{aligned} \varepsilon &\preceq \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)}\right) \\ &\preceq \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)-1}, \mathfrak{e}_{2\mathfrak{u}(j)-1}\right) + \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{u}(j)-1}, \mathfrak{e}_{2\mathfrak{u}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)}\right). \end{aligned}$$

Letting $j \to \infty$,we get

$$\lim_{j\to\infty} \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)}\right) = \varepsilon. \tag{10}$$

It follows from the rectangle inequity that

$$\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)}\right) \preceq \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)+}, \mathfrak{e}_{2\mathfrak{u}(j)+1}\right) + \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{u}(j)+1}, \mathfrak{e}_{2\mathfrak{u}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)}\right).$$

Letting $j \to \infty$, we get $\varepsilon \preceq \lim_{j\to\infty} \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)+1}, \mathfrak{e}_{2\mathfrak{u}(j)+1}\right)$ and

$$\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)+1}, \mathfrak{e}_{2\mathfrak{u}(j)+1}\right) \preceq \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)}\right) + \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{u}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)+1}, \mathfrak{e}_{2\mathfrak{u}(j)+1}\right).$$

Letting $j \to \infty$, we get

$$\lim_{j\to\infty} \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)+1}, \mathfrak{e}_{2\mathfrak{u}(j)+1}\right) = \varepsilon. \tag{11}$$

Similarly,we can show that

$$\lim_{j\to\infty} \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{u}(j)}, \mathfrak{e}_{2\mathfrak{v}(j)+1}, \mathfrak{e}_{2\mathfrak{v}(j)+1}\right) = \varepsilon. \tag{12}$$

Again using the triangular inequality, we get

$$\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{u}(j)+1}, \mathfrak{e}_{2\mathfrak{u}(j)+1}\right) \preceq \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)}, \mathfrak{e}_{2\mathfrak{v}(j)+1}, \mathfrak{e}_{2\mathfrak{v}(j)+1}\right) + \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)+1}, \mathfrak{e}_{2\mathfrak{u}(j)+1}, \mathfrak{e}_{2\mathfrak{u}(j)+1}\right).$$

Letting $j\to\infty$ and (11), we get $\varepsilon \preceq \lim\limits_{j\to\infty}\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)+1},\mathfrak{e}_{2\mathfrak{u}(j)+1},\mathfrak{e}_{2\mathfrak{u}(j)+1}\right)$ and

$$\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)+1},\mathfrak{e}_{2\mathfrak{u}(j)+1},\mathfrak{e}_{2\mathfrak{u}(j)+1}\right) \preceq \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)+1},\mathfrak{e}_{2\mathfrak{v}(j)},\mathfrak{e}_{2\mathfrak{v}(j)}\right)+\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)},\mathfrak{e}_{2\mathfrak{u}(j)+1},\mathfrak{e}_{2\mathfrak{u}(j)+1}\right).$$

Letting $j\to\infty$ and (11), we get

$$\lim_{j\to\infty}\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)+1},\mathfrak{e}_{2\mathfrak{u}(j)+1},\mathfrak{e}_{2\mathfrak{u}(j)+1}\right)=\varepsilon. \tag{13}$$

From Eq.(4) and Eq.(5), we obtain $\alpha\left(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}(j)},\mathfrak{g}\mathfrak{s}_{2\mathfrak{u}(j)}\right)\beta\left(\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}(j)},\mathfrak{f}\mathfrak{s}_{2\mathfrak{u}(j)}\right)\succeq 1_{\mathscr{A}}$.

Then from (1), we have

$$\begin{aligned}\xi\left(\rho_{c^*}(\mathfrak{e}_{2\mathfrak{v}(j)+1},\mathfrak{e}_{2\mathfrak{u}(j)+1},\mathfrak{e}_{2\mathfrak{u}(j)+1})\right) &= \xi\left(\rho_{c^*}(\mathscr{F}\mathfrak{s}_{2\mathfrak{v}(j)},\mathscr{G}\mathfrak{s}_{2\mathfrak{u}(j)},\mathscr{G}\mathfrak{s}_{2\mathfrak{u}(j)})\right)\\ &\preceq \xi\left(a\Delta_1(\mathfrak{s}_{2\mathfrak{v}(j)},\mathfrak{s}_{2\mathfrak{u}(j)})a^*\right)-\varpi\left(a\Delta_2(\mathfrak{s}_{2\mathfrak{v}(j)},\mathfrak{s}_{2\mathfrak{u}(j)})a^*\right)\end{aligned} \tag{14}$$

where

$$\begin{aligned}\Delta_1(\mathfrak{s}_{2\mathfrak{v}(j)},\mathfrak{s}_{2\mathfrak{u}(j)}) &= \max\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}(j)},\mathfrak{g}\mathfrak{s}_{2\mathfrak{u}(j)},\mathfrak{g}\mathfrak{s}_{2\mathfrak{u}(j)}\right),\rho_{c^*}\left(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}(j)},\mathscr{F}\mathfrak{s}_{2\mathfrak{v}(j)},\mathscr{F}\mathfrak{s}_{2\mathfrak{v}(j)}\right),\\ \rho_{c^*}\left(\mathfrak{g}\mathfrak{s}_{2\mathfrak{u}(j)},\mathscr{G}\mathfrak{s}_{2\mathfrak{u}(j)},\mathscr{G}\mathfrak{s}_{2\mathfrak{u}(j)}\right)\end{array}\right\}\\ &= \max\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)},\mathfrak{e}_{2\mathfrak{u}(j)},\mathfrak{e}_{2\mathfrak{u}(j)}\right),\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)},\mathfrak{e}_{2\mathfrak{v}(j)+1},\mathfrak{e}_{2\mathfrak{v}(j)+1}\right),\\ \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{u}(j)},\mathfrak{e}_{2\mathfrak{u}(j)+1},\mathfrak{e}_{2\mathfrak{u}(j)+1}\right)\end{array}\right\}\end{aligned}$$

and

$$\begin{aligned}\Delta_2(\mathfrak{s}_{2\mathfrak{v}(j)},\mathfrak{s}_{2\mathfrak{u}(j)}) &= \min\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}(j)},\mathfrak{g}\mathfrak{s}_{2\mathfrak{u}(j)},\mathfrak{g}\mathfrak{s}_{2\mathfrak{u}(j)}\right),\\ \frac{1}{2}\left(\rho_{c^*}\left(\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}(j)},\mathscr{G}\mathfrak{s}_{2\mathfrak{u}(j)},\mathscr{G}\mathfrak{s}_{2\mathfrak{u}(j)}\right)+\rho_{c^*}\left(\mathfrak{g}\mathfrak{s}_{2\mathfrak{u}(j)},\mathscr{F}\mathfrak{s}_{2\mathfrak{v}(j)},\mathscr{F}\mathfrak{s}_{2\mathfrak{v}(j)}\right)\right)\end{array}\right\}\\ &= \min\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)},\mathfrak{e}_{2\mathfrak{u}(j)},\mathfrak{e}_{2\mathfrak{u}(j)}\right),\\ \frac{1}{2}\left(\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}(j)},\mathfrak{e}_{2\mathfrak{u}(j)+1},\mathfrak{e}_{2\mathfrak{u}(j)+1}\right)+\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{u}(j)},\mathfrak{e}_{2\mathfrak{v}(j)+1},\mathfrak{e}_{2\mathfrak{v}(j)+1}\right)\right)\end{array}\right\}.\end{aligned}$$

Letting $j\to\infty$ in Eq. (14) with respect $\mathscr{A}$ and $||a||<1$ and using Eq.(10), (11), (12), (13) and with ξ, ϖ being continuous such that $\varpi(\mathfrak{s})\succ 0_{\mathscr{A}}$ for $\mathfrak{s}\succ 0_{\mathscr{A}}$, then we we obtain that

$$\begin{aligned}\xi(\varepsilon) &\leq \xi(a\varepsilon a^*)-\varpi(a\varepsilon a^*)\\ &\prec \xi(a\varepsilon a^*)\end{aligned}$$

which implies that $\varepsilon<||a||^2\varepsilon<\varepsilon$, which is a contradiction. Hence $\{\mathfrak{e}_{\mathfrak{v}}\}$ is Cauchy sequences in $\mathbb{V}$ with regard to $\mathscr{A}$. However, $(\mathbb{V},\mathscr{A},\rho_{c^*})$ is complete, so there exists $\mathfrak{z}\in\mathbb{V}$ such that

$\lim_{\mathfrak{v}\to\infty}\mathfrak{e}_{\mathfrak{v}}=\mathfrak{z}$. Consequently, the subsequences also converge to $\mathfrak{z}\in\mathbb{V}$ such that from Eq.(3), we have

$$(15)\quad \lim_{\mathfrak{v}\to\infty}\mathscr{F}\mathfrak{s}_{2\mathfrak{v}}=\lim_{\mathfrak{v}\to\infty}\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1}=\lim_{\mathfrak{v}\to\infty}\mathfrak{e}_{2\mathfrak{v}+1}=\mathfrak{z}\quad \lim_{\mathfrak{v}\to\infty}\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1}=\lim_{\mathfrak{v}\to\infty}\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}+2}=\lim_{\mathfrak{v}\to\infty}\mathfrak{e}_{2\mathfrak{v}+2}=\mathfrak{z}.$$

Now we show that $\mathfrak{z}\in\cap_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$. From the cyclic representation of Ξ concerning $\mathscr{F},\mathscr{G},\mathfrak{f},\mathfrak{g}$, and since $\mathfrak{e}_0\in\mathfrak{h}_1$, we have $\{\mathfrak{e}_{\mathfrak{v}\mathfrak{r}}\}_{\mathfrak{v}\geq 0}\subseteq\mathfrak{h}_1$. Since $\mathfrak{h}_1$ is closed, from (15),we obtain $\mathfrak{z}\in\mathfrak{h}_1$. Again using a cyclic representation of Ξ concerning $\mathscr{F},\mathscr{G},\mathfrak{f},\mathfrak{g}$, we get $\{\mathfrak{e}_{\mathfrak{v}\mathfrak{r}+1}\}_{\mathfrak{v}\geq 0}\subseteq\mathfrak{h}_2$. Since $\mathfrak{h}_2$ is closed, from (15),we obtain $\mathfrak{z}\in\mathfrak{h}_2$. Proceeding this way, we get $\mathfrak{z}\in\cap_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$.

Now we shall prove that $\mathfrak{z}$ is a common fixed point of $\mathscr{F},\mathscr{G},\mathfrak{f}$ and $\mathfrak{g}$. Since $\mathscr{G}(\Xi)\subseteq\mathfrak{f}(\Xi)$ and $\mathfrak{f}(\Xi)$ is closed subspace of Ξ, there exist $\mathfrak{y}\in\Xi$ such that $\mathfrak{z}=\mathfrak{f}\mathfrak{y}$. Now, we will show that $\mathscr{F}(\mathfrak{y})=\mathfrak{z}$. Let $\rho_{c^*}(\mathfrak{z},\mathscr{F}\mathfrak{y},\mathscr{F}\mathfrak{y})\neq 0_{\mathscr{A}}$, since for all $\mathfrak{v}$ there exists $i(\mathfrak{v})\in\{1,2,\cdots\mathfrak{r}\}$ such that $\mathfrak{e}_{\mathfrak{v}}\in\mathfrak{h}_{i(\mathfrak{v})}$. For this, since $\mathfrak{e}_{2\mathfrak{v}+1}\to\mathfrak{z}$, so from Eq. (3), it follows that $\alpha(\mathfrak{z},\mathfrak{e}_{2\mathfrak{v}+1})=\alpha(\mathfrak{f}\mathfrak{y},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1})\succeq 1_{\mathscr{A}}$ for all $\mathfrak{v}\in\mathbb{N}$. From Condition (3.3), we have $\alpha(\mathfrak{z},\mathfrak{e}_{2\mathfrak{v}+1})=\alpha(\mathfrak{f}\mathfrak{y},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1})\succeq 1_{\mathscr{A}}\implies\beta(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}+1})\succeq 1_{\mathscr{A}}$ and thus, $\alpha(\mathfrak{f}\mathfrak{y},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1})\beta(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}+1})\succeq 1_{\mathscr{A}}$ then, from Eq.(1), we have

$$\xi\left(\rho_{c^*}(\mathscr{F}\mathfrak{y},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1})\right)\preceq\xi\left(a\Delta_1(\mathfrak{y},\mathfrak{s}_{2\mathfrak{v}+1})a^*\right)-\varpi\left(a\Delta_2(\mathfrak{y},\mathfrak{s}_{2\mathfrak{v}+1})a^*\right)$$

where

$$\begin{aligned}\Delta_1(\mathfrak{y},\mathfrak{s}_{2\mathfrak{v}+1}) &= \max\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{f}\mathfrak{y},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1}\right),\rho_{c^*}\left(\mathfrak{f}\mathfrak{y},\mathscr{F}\mathfrak{y},\mathscr{F}\mathfrak{y}\right),\\ \rho_{c^*}\left(\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1}\right)\end{array}\right\}\\ &= \max\left\{\ \rho_{c^*}\left(\mathfrak{z},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1}\right),\rho_{c^*}\left(\mathfrak{z},\mathscr{F}\mathfrak{y},\mathscr{F}\mathfrak{y}\right),\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)\ \right\}\end{aligned}$$

and

$$\begin{aligned}\Delta_2(\mathfrak{y},\mathfrak{s}_{2\mathfrak{v}+1}) &= \min\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{f}\mathfrak{y},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1}\right),\\ \frac{1}{2}\left(\rho_{c^*}\left(\mathfrak{f}\mathfrak{y},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1}\right)+\rho_{c^*}\left(\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{F}\mathfrak{y},\mathscr{F}\mathfrak{y}\right)\right)\end{array}\right\}\\ &= \min\left\{\ \rho_{c^*}\left(\mathfrak{z},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1}\right),\frac{1}{2}\left(\rho_{c^*}\left(\mathfrak{z},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)+\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}+1},\mathscr{F}\mathfrak{y},\mathscr{F}\mathfrak{y}\right)\right)\ \right\}.\end{aligned}$$

Thus

$$\xi\left(\rho_{c^*}(\mathscr{F}\mathfrak{y},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2})\right)\preceq\xi\left(a\max\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{z},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1}\right),\\ \rho_{c^*}\left(\mathfrak{z},\mathscr{F}\mathfrak{y},\mathscr{F}\mathfrak{y}\right),\\ \rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)\end{array}\right\}a^*\right)$$

$$-\varpi\left(a\min\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{z},\mathfrak{e}_{2\mathfrak{v}+1},\mathfrak{e}_{2\mathfrak{v}+1}\right),\\ \frac{1}{2}\left(\rho_{c^*}\left(\mathfrak{z},\mathfrak{e}_{2\mathfrak{v}+2},\mathfrak{e}_{2\mathfrak{v}+2}\right)+\rho_{c^*}\left(\mathfrak{e}_{2\mathfrak{v}+1},\mathscr{F}\mathfrak{y},\mathscr{F}\mathfrak{y}\right)\right)\end{array}\right\}a^*\right).$$

Taking $\mathfrak{v}\to\infty$, we have

$$\xi\left(\rho_{c^*}(\mathscr{F}\mathfrak{y},\mathfrak{z},\mathfrak{z})\right) \preceq \xi\left(a\rho_{c^*}\left(\mathfrak{z},\mathscr{F}\mathfrak{y},\mathscr{F}\mathfrak{y}\right)a^*\right)-\varpi\left(a\frac{1}{2}\rho_{c^*}\left(\mathfrak{z},\mathscr{F}\mathfrak{y},\mathscr{F}\mathfrak{y}\right)a^*\right).$$

Using discontinuity of ϖ at $\mathfrak{s}=0_{\mathscr{A}}$ and $\varpi(\mathfrak{s})\succ 0_{\mathscr{A}}$ fort $\mathfrak{s}\succ 0_{\mathscr{A}}$, we observe that the last term on the right-hand side of the above inequality is non-zero. Therefore we obtain

$$\xi\left(\rho_{c^*}(\mathscr{F}\mathfrak{y},\mathfrak{z},\mathfrak{z})\right) \prec \xi\left(a\rho_{c^*}\left(\mathfrak{z},\mathscr{F}\mathfrak{y},\mathscr{F}\mathfrak{y}\right)a^*\right).$$

By the properties of ξ and since, $||a||^2<1$, then we have

$$||\rho_{c^*}(\mathscr{F}\mathfrak{y},\mathfrak{z},\mathfrak{z})||<||a||^2||\rho_{c^*}\left(\mathfrak{z},\mathscr{F}\mathfrak{y},\mathscr{F}\mathfrak{y}\right)||<||\rho_{c^*}(\mathscr{F}\mathfrak{y},\mathfrak{z},\mathfrak{z})||.$$

Hence we arrive at a contradiction. Therefore $||\rho_{c^*}(\mathscr{F}\mathfrak{y},\mathfrak{z},\mathfrak{z})||=0 \implies \mathscr{F}\mathfrak{y}=\mathfrak{z}$. Thus we conclude that $\mathscr{F}\mathfrak{y}=\mathfrak{z}=\mathfrak{f}\mathfrak{y}$. Since $(\mathscr{F},\mathfrak{f})$ is a weakly compatible pair of maps, so it commutes at their coincidence point $\mathfrak{y}$, *i.e* $\mathscr{F}\mathfrak{f}\mathfrak{y}=\mathfrak{f}\mathscr{F}\mathfrak{y}\implies\mathscr{F}\mathfrak{z}=\mathfrak{f}\mathfrak{z}$. Now we shall show that $\mathscr{F}\mathfrak{z}=\mathfrak{f}\mathfrak{z}=\mathfrak{z}$. Let $\rho_{c^*}\left(\mathscr{F}\mathfrak{z},\mathfrak{z},\mathfrak{z}\right)\neq 0_{\mathscr{A}}$, since $\mathscr{F}\mathfrak{y}=\mathfrak{z}=\mathfrak{f}\mathfrak{y}$ and for all $\mathfrak{v}$ there exists $i(\mathfrak{v})\in\{1,2,\cdots\mathfrak{r}\}$ such that $\mathfrak{e}_{\mathfrak{v}}\in\mathfrak{h}_{i(\mathfrak{v})}$. From Condition (3.4), we have $\alpha(\mathfrak{f}\mathfrak{z},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1})\succeq 1_{\mathscr{A}}$ and $\beta(\mathfrak{g}\mathfrak{z},\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}+1})\succeq 1_{\mathscr{A}}$ thus, $\alpha(\mathfrak{f}\mathfrak{z},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1})\beta(\mathfrak{g}\mathfrak{z},\mathfrak{f}\mathfrak{s}_{2\mathfrak{v}+1})\succeq 1_{\mathscr{A}}$ then, from Eq.(1), we have

$$\xi\left(\rho_{c^*}(\mathscr{F}\mathfrak{z},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1})\right)\preceq\xi\left(a\Delta_1(\mathfrak{z},\mathfrak{s}_{2\mathfrak{v}+1})a^*\right)-\varpi\left(a\Delta_2(\mathfrak{z},\mathfrak{s}_{2\mathfrak{v}+1})a^*\right)$$

$$\preceq \xi\left(a\max\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{f}\mathfrak{z},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1}\right),\\ \rho_{c^*}\left(\mathfrak{f}\mathfrak{z},\mathscr{F}\mathfrak{z},\mathscr{F}\mathfrak{z}\right),\\ \rho_{c^*}\left(\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1}\right)\end{array}\right\}a^*\right)$$

$$-\varpi\left(a\min\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{f}\mathfrak{z},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1}\right),\\ \frac{1}{2}\left(\rho_{c^*}\left(\mathfrak{f}\mathfrak{z},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{G}\mathfrak{s}_{2\mathfrak{v}+1}\right)+\rho_{c^*}\left(\mathfrak{g}\mathfrak{s}_{2\mathfrak{v}+1},\mathscr{F}\mathfrak{z},\mathscr{F}\mathfrak{z}\right)\right)\end{array}\right\}a^*\right).$$

Taking $\mathfrak{v}\to\infty$ and using $\mathscr{F}\mathfrak{z}=\mathfrak{f}\mathfrak{z}$, we get

$$\xi\left(\rho_{c^*}(\mathscr{F}\mathfrak{z},\mathfrak{z},\mathfrak{z})\right) \preceq \xi\left(a\rho_{c^*}\left(\mathscr{F}\mathfrak{z},\mathfrak{z},\mathfrak{z}\right)a^*\right)-\varpi\left(a\rho_{c^*}\left(\mathscr{F}\mathfrak{z},\mathfrak{z},\mathfrak{z}\right)a^*\right)$$

Using discontinuity of ϖ at $\mathfrak{s} = 0_{\mathscr{A}}$ and $\varpi(\mathfrak{s}) \succ 0_{\mathscr{A}}$ fort $\mathfrak{s} \succ 0_{\mathscr{A}}$,we observe that the last term on the right-hand side of the above inequality is non-zero. Therefore, we obtain

$$\xi\left(\rho_{c^*}(\mathscr{F}\mathfrak{z},\mathfrak{z},\mathfrak{z})\right) \prec \xi\left(a\rho_{c^*}\left(\mathscr{F}\mathfrak{z},\mathfrak{z},\mathfrak{z}\right)a^*\right).$$

By the properties of ξ and since, $||a||^2 < 1$, then we have

$$||\rho_{c^*}(\mathscr{F}\mathfrak{z},\mathfrak{z},\mathfrak{z})|| < ||a||^2||\rho_{c^*}(\mathscr{F}\mathfrak{z},\mathfrak{z},\mathfrak{z})|| < ||\rho_{c^*}(\mathscr{F}\mathfrak{z},\mathfrak{z},\mathfrak{z})||$$

which is contradiction. Therefore, $\rho_{c^*}(\mathscr{F}\mathfrak{z},\mathfrak{z},\mathfrak{z}) = 0_{\mathscr{A}}$ implies $\mathscr{F}\mathfrak{z} = \mathfrak{z}$. Hence, $\mathscr{F}\mathfrak{z} = \mathfrak{f}\mathfrak{z} = \mathfrak{z}$. Similarly, we can show that $\mathscr{G}\mathfrak{z} = \mathfrak{g}\mathfrak{z} = \mathfrak{z}$. Hence, $\mathscr{F}\mathfrak{z} = \mathfrak{f}\mathfrak{z} = \mathscr{G}\mathfrak{z} = \mathfrak{g}\mathfrak{z} = \mathfrak{z}$. For uniqueness, suppose that $\mathfrak{z}'$ is another fixed point with $\mathscr{F}\mathfrak{z}' = \mathfrak{f}\mathfrak{z}' = \mathscr{G}\mathfrak{z}' = \mathfrak{g}\mathfrak{z}' = \mathfrak{z}'$. Then,by the cyclic representation of Ξ concerning $\mathscr{F},\mathscr{G}$, $\mathfrak{f},\mathfrak{g}$ it implies that $\mathfrak{z}' \in \cap_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$. From Condition (3.4), we have $\alpha(\mathfrak{f}\mathfrak{z},\mathfrak{g}\mathfrak{z}')\beta(\mathfrak{g}\mathfrak{z},\mathfrak{f}\mathfrak{z}') \succeq 1_{\mathscr{A}}$, then, from Eq.(1), we have

$$\begin{aligned}\xi\left(\rho_{c^*}(\mathfrak{z},\mathfrak{z}',\mathfrak{z}')\right) = \xi\left(\rho_{c^*}(\mathscr{F}\mathfrak{z},\mathscr{G}\mathfrak{z}',\mathscr{G}\mathfrak{z}')\right) &\preceq \xi\left(a\Delta_1(\mathfrak{z},\mathfrak{z}')a^*\right) - \varpi\left(a\Delta_2(\mathfrak{z},\mathfrak{z}')a^*\right)\\ &\preceq \xi\left(a\rho_{c^*}\left(\mathfrak{f}\mathfrak{z},\mathfrak{g}\mathfrak{z}',\mathfrak{g}\mathfrak{z}'\right)a^*\right) - \varpi\left(a\rho_{c^*}\left(\mathfrak{f}\mathfrak{z},\mathfrak{g}\mathfrak{z}',\mathfrak{g}\mathfrak{z}'\right)a^*\right)\\ &\prec \xi\left(a\rho_{c^*}\left(\mathfrak{z},\mathfrak{z}',\mathfrak{z}'\right)a^*\right)\end{aligned}$$

which implies $||\rho_{c^*}(\mathfrak{z},\mathfrak{z}',\mathfrak{z}')|| < ||a||^2||\rho_{c^*}(\mathfrak{z},\mathfrak{z}',\mathfrak{z}')|| < ||\rho_{c^*}(\mathfrak{z},\mathfrak{z}',\mathfrak{z}')||$. Which is a contradiction.Hence, we have $\rho_{c^*}(\mathfrak{z},\mathfrak{z}',\mathfrak{z}') = 0_{\mathscr{A}} \Rightarrow \mathfrak{z} = \mathfrak{z}'$. Hence $\mathscr{F},\mathscr{G}$, $\mathfrak{f}$ and $\mathfrak{g}$ have a unique common fixed point in $\cap_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$. Let $\alpha\left(\mathfrak{f}\mathfrak{s},\mathfrak{g}\mathfrak{e}\right)\beta\left(\mathfrak{g}\mathfrak{s},\mathfrak{f}\mathfrak{e}\right) \succeq 1_{\mathscr{A}}$ then from Eq.(2), we get

$$\xi\left(\rho_{c^*}(\mathscr{F}\mathfrak{s},\mathscr{G}\mathfrak{e},\mathscr{G}\mathfrak{e})\right) \preceq \xi\left(a\Delta_1(\mathfrak{s},\mathfrak{e})a^*\right) - \varpi\left(a\Delta_2(\mathfrak{s},\mathfrak{e})a^*\right).$$

Thus, the Eq.(1) is satisfied, hence, the proof easily follows similar lines of above and $\mathscr{F},\mathscr{G}$, $\mathfrak{f}$ and $\mathfrak{g}$ have a unique common fixed point in $\cap_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$.

Corollary 3.10: Let $(\mathbb{V},\mathscr{A},\rho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. Let $\mathfrak{r} \in \mathbb{N}$, $\mathfrak{h}_1,\mathfrak{h}_2\cdots\mathfrak{h}_{\mathfrak{r}}$ be non-void subsets of $\mathbb{V}$ and $\Xi = \cup_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$ is a cyclic representation of Ξ with respect to $\mathscr{F},\mathscr{G}$ where $\mathscr{F},\mathscr{G} : \Xi \to \Xi$ be two self-mappings which satisfy the following inequality:

For any $(\mathfrak{s},\mathfrak{e}) \in (\mathfrak{h}_i,\mathfrak{h}_{i+1})$, $i = 1,2,\cdots\mathfrak{r}$ (with $\mathfrak{h}_{i+1} = \mathfrak{h}_1$) and $a \in \mathscr{A}$ with $||a|| < 1$,

$$\xi\left(\rho_{c^*}(\mathscr{F}\mathfrak{s},\mathscr{G}\mathfrak{e},\mathscr{G}\mathfrak{e})\right) \preceq \xi\left(a\max\left\{\begin{array}{c}\rho_{c^*}\left(\mathfrak{s},\mathfrak{e},\mathfrak{e}\right),\rho_{c^*}\left(\mathfrak{s},\mathscr{F}\mathfrak{s},\mathscr{F}\mathfrak{s}\right),\\ \rho_{c^*}\left(\mathfrak{e},\mathscr{G}\mathfrak{e},\mathscr{G}\mathfrak{e}\right)\end{array}\right\}a^*\right)$$

$$-\varpi\left(a\min\left\{\begin{array}{c}\rho_{c^*}(\mathfrak{s},\mathfrak{e},\mathfrak{e}),\\ \frac{1}{2}\left(\rho_{c^*}(\mathfrak{s},\mathscr{G}\mathfrak{e},\mathscr{G}\mathfrak{e})+\rho_{c^*}(\mathfrak{e},\mathscr{F}\mathfrak{s},\mathscr{F}\mathfrak{s})\right)\end{array}\right\}a^*\right)$$

and $\varpi:\mathscr{A}_+\to\mathscr{A}_+$ is lower semi-continuous, such that $\varpi(\mathfrak{s})\succ 0_{\mathscr{A}}$ for all $\mathfrak{s}\succ 0_{\mathscr{A}}$ and discontinuous at $\mathfrak{s}=0_{\mathscr{A}}$ with $\varpi(0_{\mathscr{A}})=0_{\mathscr{A}}$, $\xi:\mathscr{A}_+\to\mathscr{A}_+$ is an altering distance function.Then there exists a unique fixed point of $\mathscr{F},\mathscr{G}$ in $\cap_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$.

Proof The result follows directly from Theorem (3.9) by setting $\mathfrak{f}=\mathfrak{g}=1_{\mathscr{A}}$, and choosing the morphisms $\alpha(\mathfrak{f}\mathfrak{s},\mathfrak{g}\mathfrak{e})=1_{\mathscr{A}}$ and $\beta(\mathfrak{g}\mathfrak{s},\mathfrak{f}\mathfrak{e})=1_{\mathscr{A}}$.

Corollary 3.11: Let $(\mathbb{V},\mathscr{A},\rho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. Let $\mathfrak{r}\in\mathbb{N}$, $\mathfrak{h}_1,\mathfrak{h}_2\cdots\mathfrak{h}_{\mathfrak{r}}$ be non-void subsets of $\mathbb{V}$ and $\Xi=\cup_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$ is a cyclic representation of Ξ with respect to $\mathscr{F}$ where $\mathscr{F}:\Xi\to\Xi$ be a self-mapping which satisfy the following inequality: For any $(\mathfrak{s},\mathfrak{e})\in(\mathfrak{h}_i,\mathfrak{h}_{i+1})$, $i=1,2,\cdots\mathfrak{r}$ (with $\mathfrak{h}_{i+1}=\mathfrak{h}_1$) and $a\in\mathscr{A}$ with $||a||<1$,

$$\begin{aligned}\xi(\rho_{c^*}(\mathscr{F}\mathfrak{s},\mathscr{F}\mathfrak{e},\mathscr{F}\mathfrak{e})) \preceq\ & \xi\left(a\max\left\{\begin{array}{c}\rho_{c^*}(\mathfrak{s},\mathfrak{e},\mathfrak{e}),\rho_{c^*}(\mathfrak{s},\mathscr{F}\mathfrak{s},\mathscr{F}\mathfrak{s}),\\ \rho_{c^*}(\mathfrak{e},\mathscr{F}\mathfrak{e},\mathscr{F}\mathfrak{e})\end{array}\right\}a^*\right)\\ & -\varpi\left(a\min\left\{\begin{array}{c}\rho_{c^*}(\mathfrak{s},\mathfrak{e},\mathfrak{e}),\\ \frac{1}{2}\left(\rho_{c^*}(\mathfrak{s},\mathscr{F}\mathfrak{e},\mathscr{F}\mathfrak{e})+\rho_{c^*}(\mathfrak{e},\mathscr{F}\mathfrak{s},\mathscr{F}\mathfrak{s})\right)\end{array}\right\}a^*\right)\end{aligned}$$

and $\varpi:\mathscr{A}_+\to\mathscr{A}_+$ is lower semi-continuous, such that $\varpi(\mathfrak{s})\succ 0_{\mathscr{A}}$ for all $\mathfrak{s}\succ 0_{\mathscr{A}}$ and discontinuous at $\mathfrak{s}=0_{\mathscr{A}}$ with $\varpi(0_{\mathscr{A}})=0_{\mathscr{A}}$, $\xi:\mathscr{A}_+\to\mathscr{A}_+$ is an altering distance function.Then there exists a unique fixed point of $\mathscr{F}$ in $\cap_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$.

Proof The result follows directly from Corollary (3.10) by setting $\mathscr{F}=\mathscr{G}$ on Ξ.

Corollary 3.12: Let $(\mathbb{V},\mathscr{A},\rho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. Let $\mathfrak{r}\in\mathbb{N}$, $\mathfrak{h}_1,\mathfrak{h}_2\cdots\mathfrak{h}_{\mathfrak{r}}$ be non-void subsets of $\mathbb{V}$ and $\Xi=\cup_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$ is a cyclic representation of Ξ with respect to $\mathscr{F}$ where $\mathscr{F}:\Xi\to\Xi$ be a self-mapping which satisfy the following inequality: For any $(\mathfrak{s},\mathfrak{e})\in(\mathfrak{h}_i,\mathfrak{h}_{i+1})$, $i=1,2,\cdots\mathfrak{r}$ (with $\mathfrak{h}_{i+1}=\mathfrak{h}_1$) and $a\in\mathscr{A}$ with $||a||<1$ and $\lambda\in(0,1)$ such that

$$\xi(\rho_{c^*}(\mathscr{F}\mathfrak{s},\mathscr{F}\mathfrak{e},\mathscr{F}\mathfrak{e}))\preceq\lambda\xi(a\rho_{c^*}(\mathfrak{s},\mathfrak{e},\mathfrak{e})a^*)$$

where $\xi:\mathscr{A}_+\to\mathscr{A}_+$ is an altering distance function.Then there exists a unique fixed point of $\mathscr{F}$ in $\cap_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$.

Proof The result follows directly from Corollary (3.11) by setting $\Delta_1(\mathfrak{s},\mathfrak{e}) = \Delta_2(\mathfrak{s},\mathfrak{e})$ and $\varpi(\mathfrak{s}) = \frac{\lambda\xi(\mathfrak{s})}{2}$.

Example 3.13: Let $\mathbb{V} = \mathbb{C}$ and $\mathscr{A} = \mathsf{M}_3(\mathbb{C})$, the algebra of 3×3 complex matrices.

Define $\rho_{c^*} : \mathbb{V}^3 \to \mathscr{A}$ by $\rho_{c^*}(\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3) = \begin{pmatrix} G(\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3) & 0 & 0 \\ 0 & \iota G(\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3) & 0 \\ 0 & 0 & \kappa G(\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3) \end{pmatrix}$, where $\iota,\kappa > 0$ are constants and $G(\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3) = |\mathfrak{s}_1-\mathfrak{s}_2| + |\mathfrak{s}_2-\mathfrak{s}_3| + |\mathfrak{s}_3-\mathfrak{s}_1|$. then $(\mathbb{V},\mathscr{A},\rho_{c^*})$ is a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$.

Let $\mathfrak{h}_1 = \{z\in\mathbb{C} : \Re(z) \geq 0\}$ and $\mathfrak{h}_2 = \{z\in\mathbb{C} : \Re(z) < 0\}$, so $\Xi = \mathfrak{h}_1\cup\mathfrak{h}_2$ is a cyclic representation with $\mathfrak{r} = 2$. Define the mappings: $\mathscr{F}(z) = \frac{z}{2}$, $\mathscr{G}(z) = \frac{z}{2}$, $\mathfrak{f}(z) = z$, $\mathfrak{g}(z) = z$. Define $\alpha,\beta : \Xi\times\Xi \to \mathscr{A}_+$ by

$$\alpha(z_1,z_2) = \begin{cases} \begin{pmatrix} 1+|\Re(z_1-z_2)| & 0 & 0 \\ 0 & 1+|\Re(z_1-z_2)| & 0 \\ 0 & 0 & 1+|\Re(z_1-z_2)| \end{pmatrix} & \text{if } z_1,\in\mathfrak{h}_1, z_2,\in\mathfrak{h}_2 \\ 0_{\mathscr{A}} & \text{otherwise,} \end{cases}$$

$$\beta(z_1,z_2) = \begin{cases} \begin{pmatrix} 1+|\Im(z_1-z_2)| & 0 & 0 \\ 0 & 1+|\Im(z_1-z_2)| & 0 \\ 0 & 0 & 1+|\Im(z_1-z_2)| \end{pmatrix} & \text{if } z_1,\in\mathfrak{h}_1, z_2,\in\mathfrak{h}_2 \\ 0_{\mathscr{A}} & \text{otherwise,} \end{cases}$$

here, $\Re(z)$ denotes the real part of z and $\Im(z)$ denotes the imaginary part of z.

Define $\xi : \mathscr{A}_+ \to \mathscr{A}_+$ as $\xi(A) = A^2$ and $\varpi : \mathscr{A}_+ \to \mathscr{A}_+$ by $\varpi(A) = \varepsilon A$, for some fixed $\varepsilon > 0$. Let $a\in\mathscr{A}$ with $\|a\| < 1$, and $(\mathfrak{s},\mathfrak{e}) \in \mathfrak{h}_1\times\mathfrak{h}_2$. Then All conditions of Theorem 3.9 are satisfied. Therefore, the mappings $\mathscr{F},\mathscr{G},\mathfrak{f},\mathfrak{g}$ have a unique common fixed point in $\mathfrak{h}_1\cap\mathfrak{h}_2 = \{0\}$.

4. APPLICATIONS

4.1. Application to functional equations.

In this section we denote by $\mathbb{V}=\mathbb{C}([0,1],\mathbb{R}^+)\subseteq L^\infty([0,1],\mathbb{R}^+)$ and $\mathscr{A}=\mathbb{B}(L^2([0,1]))$ is C^* algebra with operator norm : $||\mathbb{A}||=\sup_{a\in[0,1]}||\langle\mathbb{A}a,\mathbb{A}a\rangle||$. We equip $\mathbb{V}$ with $\rho_{c^*}:\mathbb{V}^3\to\mathscr{A}$, which is ascertained by $\rho_{c^*}(\mathfrak{p},\mathfrak{r},\mathfrak{q})=\mathbb{M}_{|\mathfrak{p}-\mathfrak{q}|+|\mathfrak{r}-\mathfrak{q}|}$ for all $\mathfrak{p},\mathfrak{r},\mathfrak{q}\in\mathbb{V}$, where $\mathbb{M}_\phi:\mathscr{A}\to\mathscr{A}$ be ascertained by $\mathbb{M}_\phi(\alpha)=\phi\diamond\alpha$ composit of these operators where $\alpha\in L^2([0,1])$ and $\phi\in\mathscr{A}$. Therefore, $(\mathbb{V},\mathscr{A},\rho_{c^*})$ is a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-$\mathbb{G}$-$\mathbb{MS}$.

In this setting, we discuss the problem of dynamic programming related to multistage process[29, 30]. Indeed, this problem reduces to the problem of solving the functional equation

$$\mathfrak{s}(\mathfrak{v})=\sup_{\mathfrak{u}\in\mathscr{D}}\{\mathfrak{f}(\mathfrak{v},\mathfrak{u})+\mathscr{K}(\mathfrak{v},\mathfrak{u},\mathfrak{s}(\theta(\mathfrak{v},\mathfrak{u}))\},\ \mathfrak{v}\in[0,1]\ \cdot \tag{16}$$

where $\theta:[0,1]\times\mathscr{D}\to[0,1]$, $\mathfrak{f}:[0,1]\times\mathscr{D}\to\mathbb{R}^+$ and $\mathscr{K}:[0,1]\times\mathscr{D}\times\mathbb{R}^+\to\mathbb{R}^+$.

Specifically, we will prove the following theorem.

Theorem 4.1: Let $\mathscr{K}:[0,1]\times\mathscr{D}\times\mathbb{R}^+\to\mathbb{R}^+$ and $\mathfrak{f}:[0,1]\times\mathscr{D}\to\mathbb{R}^+$ be two bounded functions and let $\mathfrak{h}_1=\mathfrak{h}_2=\mathbb{V}=\mathbb{C}([0,1],\mathbb{R}^+)$. It is clear that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are closed subsets of $\mathbb{V}$. Consider the self mapping $\mathbb{Q}:\mathfrak{h}_1\cup\mathfrak{h}_2\to\mathfrak{h}_1\cup\mathfrak{h}_2$ be as $\mathbb{Q}(\mathfrak{s})(\mathfrak{v})=\sup_{\mathfrak{u}\in\mathscr{D}}\{\mathfrak{f}(\mathfrak{v},\mathfrak{u})+\mathscr{K}(\mathfrak{v},\mathfrak{u},\mathfrak{s}(\theta(\mathfrak{b},\mathfrak{u}))\}$ for all $\mathfrak{s}\in\mathfrak{h}_1\cup\mathfrak{h}_2$ and $\mathfrak{v}\in[0,1]$. Clearly, $\mathbb{Q}(\mathfrak{h}_1)\subseteq\mathfrak{h}_2$ and $\mathbb{Q}(\mathfrak{h}_2)\subseteq\mathfrak{h}_1$. Thus, $\mathbb{Q}$ is cyclic map on $\mathfrak{h}_1\cup\mathfrak{h}_2$. Suppose the following condition hold:
let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be a nonempty closed subsets of $\mathbb{V}$ such that $\mathfrak{s}\in\mathfrak{h}_1,\mathfrak{x}\in\mathfrak{h}_2$ and $\iota\in(0,1)$, then

$$|\mathscr{K}(\mathfrak{v},\mathfrak{u},\mathfrak{s}(\mathfrak{b}))-\mathscr{K}(\mathfrak{v},\mathfrak{u},\mathfrak{x}(\mathfrak{b}))|\ \leq\ \frac{\iota}{2}||\mathfrak{s}(\mathfrak{b})-\mathfrak{x}(\mathfrak{b})||\ \forall\mathfrak{v}\in[0,1],\mathfrak{s}(\mathfrak{b})\in\mathbb{R}^+.$$

Then the functional equation (16) have a bounded solution.

Proof Note that $(\mathbb{V},\mathscr{A},\rho_{c^*})$ is a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-$\mathbb{G}$-$\mathbb{MS}$. Let $\varepsilon\succ0_{\mathscr{A}}$ be an arbitrary and $\mathfrak{s}\in\mathfrak{h}_1$, $\mathfrak{x}\in\mathfrak{h}_2$, then there exist $\mathfrak{u}_1,\mathfrak{u}_1\in\mathscr{D}$ such that

$$\mathbb{Q}(\mathfrak{s})(\mathfrak{v})\prec\mathfrak{f}(\mathfrak{v},\mathfrak{u}_1)+\mathscr{K}(\mathfrak{v},\mathfrak{u}_1,\mathfrak{s}(\theta(\mathfrak{b},\mathfrak{u}_1)))+\varepsilon$$

$$\mathbb{Q}(\mathfrak{x})(\mathfrak{v})\prec\mathfrak{f}(\mathfrak{v},\mathfrak{u}_2)+\mathscr{K}(\mathfrak{v},\mathfrak{u}_2,\mathfrak{x}(\theta(\mathfrak{b},\mathfrak{u}_2)))+\varepsilon$$

$$\mathbb{Q}(\mathfrak{s})(\mathfrak{v})\succeq\mathfrak{f}(\mathfrak{v},\mathfrak{u}_2)+\mathscr{K}(\mathfrak{v},\mathfrak{u}_2,\mathfrak{s}(\theta(\mathfrak{b},\mathfrak{u}_2)))$$

$$\mathbb{Q}(\mathfrak{x})(\mathfrak{v})\succeq\mathfrak{f}(\mathfrak{v},\mathfrak{u}_1)+\mathscr{K}(\mathfrak{v},\mathfrak{u}_1,\mathfrak{x}(\theta(\mathfrak{b},\mathfrak{u}_1))).$$

Then, we conclude that

$$\begin{aligned}\mathbb{Q}(\mathfrak{s})(\mathfrak{v})-\mathbb{Q}(\mathfrak{x})(\mathfrak{v}) &\prec \mathscr{K}(\mathfrak{v},\mathfrak{u}_1,\mathfrak{s}(\theta(\mathfrak{b},\mathfrak{u}_1)))-\mathscr{K}(\mathfrak{v},\mathfrak{u}_1,\mathfrak{x}(\theta(\mathfrak{b},\mathfrak{u}_1)))+\varepsilon\\ &\leq |\mathscr{K}(\mathfrak{v},\mathfrak{u}_1,\mathfrak{s}(\theta(\mathfrak{b},\mathfrak{u}_1)))-\mathscr{K}(\mathfrak{v},\mathfrak{u}_1,\mathfrak{x}(\theta(\mathfrak{b},\mathfrak{u}_1)))|+\varepsilon\\ &\leq \frac{\iota}{2}||\mathfrak{s}(\mathfrak{b})-\mathfrak{x}(\mathfrak{b})||+\varepsilon\end{aligned}$$

and similarly,

$$\mathbb{Q}(\mathfrak{x})(\mathfrak{v})-\mathbb{Q}(\mathfrak{s})(\mathfrak{v}) \leq \frac{\iota}{2}||\mathfrak{s}(\mathfrak{b})-\mathfrak{x}(\mathfrak{b})||+\varepsilon$$

Since $\varepsilon \succ 0_{\mathscr{A}}$ is arbitrary, then

$$|\mathbb{Q}(\mathfrak{s})(\mathfrak{v})-\mathbb{Q}(\mathfrak{x})(\mathfrak{v})| \leq \frac{\iota}{2}||\mathfrak{s}(\mathfrak{b})-\mathfrak{x}(\mathfrak{b})||.$$

Now, consider

$$\rho_{c^*}(\mathbb{Q}(\mathfrak{s}),\mathbb{Q}(\mathfrak{x}),\mathbb{Q}(\mathfrak{x}))=\mathbb{M}_{|\mathbb{Q}(\mathfrak{s})-\mathbb{Q}(\mathfrak{x})|}.$$

We obtain that

$$\begin{aligned}||\rho_{c^*}(\mathbb{Q}(\mathfrak{s}),\mathbb{Q}(\mathfrak{x}),\mathbb{Q}(\mathfrak{x}))|| &= \left\|\mathbb{M}_{|\mathbb{Q}(s)-\mathbb{Q}(x)|}\right\| = \sup_{||h||=1}\left\langle \mathbb{M}_{(|\mathbb{Q}(\mathfrak{s})(\mathfrak{v})-\mathbb{Q}(\mathfrak{x})(\mathfrak{v})|)}h,h\right\rangle\\ &= \sup_{||h||=1}\int_{[0,1]}|\mathbb{Q}(\mathfrak{s})(\mathfrak{v})-\mathbb{Q}(\mathfrak{x})(\mathfrak{v})|h(\varsigma)\overline{h(\varsigma)}d\varsigma\\ &\leq \sup_{||h||=1}\int_{[0,1]}|h(\varsigma)|^2 d\varsigma.\frac{\iota}{2}||\mathfrak{s}(\mathfrak{b})-\mathfrak{x}(\mathfrak{b})||_\infty\\ &\leq \frac{\iota}{2}\sup_{||h||=1}\int_{[0,1]}|h(\varsigma)|^2 d\varsigma||\mathfrak{s}(\mathfrak{b})-\mathfrak{x}(\mathfrak{b})||_\infty.\end{aligned}$$

By setting $a=\iota 1_{\mathscr{A}}$, then $a\in\mathscr{A}$ so that $||a||=\iota<1$, then it follows that

$$||\rho_{c^*}(\mathbb{Q}(\mathfrak{s}),\mathbb{Q}(\mathfrak{x}),\mathbb{Q}(\mathfrak{x}))|| \leq \tfrac{1}{2}||a||^2||\rho_{c^*}(\mathfrak{s},\mathfrak{x},\mathfrak{x})||.$$

Let $\xi:\mathscr{A}_+\to\mathscr{A}$ as $\xi(x)=x\;\forall\, x\in\mathscr{A}_+$ and $\lambda\in(0,1)$. Consequently, we have

$$\xi(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}),\mathbb{Q}(\mathfrak{x}),\mathbb{Q}(\mathfrak{x})))\preceq\lambda\,\xi(a^\star\rho_{c^*}(\mathfrak{s},\mathfrak{x},\mathfrak{x})a)$$

Thus, by Corollary (3.12), $\mathbb{Q}$ has a fixed point, that is, the functional equation (16) has a bounded solution.

4.2. Applications to Homotopy.

In this section, we study the existence of an unique solution to Homotopy theory.

Theorem 4.2: Let $(\mathbb{V},\mathscr{A},\rho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. Let $\mathfrak{r}\in\mathbb{N}$, $\mathfrak{h}_1,\mathfrak{h}_2\cdots\mathfrak{h}_\mathfrak{r}$ be non-void subsets of $\mathbb{V}$ and $\Xi=\cup_{i=1}^{\mathfrak{r}}\mathfrak{h}_i$ is a cyclic representation and $(\Delta_1,\Delta_2,\cdots,\Delta_\mathfrak{r})$ and $(\overline{\Delta}_1,\overline{\Delta}_2,\cdots,\overline{\Delta}_\mathfrak{r})$ be an open and closed subset of Ξ such that $(\Delta_1,\Delta_2,\cdots,\Delta_\mathfrak{r})\subseteq(\overline{\Delta}_1,\overline{\Delta}_2,\cdots,\overline{\Delta}_\mathfrak{r})$ with $\cap_{i=1}^{\mathfrak{r}}\Delta_i\neq\emptyset$. Suppose $\mathscr{H}:\cup_{i=1}^{\mathfrak{r}}\overline{\Delta}_i\times[0,1]\to\Xi$ be an operator with following conditions are satisfying,

i) $\mathfrak{s}\neq\mathscr{H}(\mathfrak{s},\iota)$, for each $\mathfrak{s}\in\partial\cup_{i=1}^{\mathfrak{r}}\Delta_i$ and $\iota\in[0,1]$ (Here $\partial\cup_{i=1}^{\mathfrak{r}}\Delta_i$ is boundary of $\cup_{i=1}^{\mathfrak{r}}\Delta_i$ in Ξ);

ii) for all $(\mathfrak{s},\mathfrak{x})\in(\overline{\Delta}_i,\overline{\Delta}_{i+1})$, $i=1,2,\cdots\mathfrak{r}$ (with $\overline{\Delta}_{i+1}=\overline{\Delta}_1$) and $a\in\mathscr{A}$ with $||a||<1$ and $\iota\in[0,1]$ such that

$$\xi(\rho_{c^*}(\mathscr{H}(\mathfrak{s},\iota),\mathscr{H}(\mathfrak{x},\iota),\mathscr{H}(\mathfrak{x},\iota)))\preceq\xi(a\rho_{c^*}(\mathfrak{s},\mathfrak{x},\mathfrak{x})a^*)-\varpi(a\rho_{c^*}(\mathfrak{s},\mathfrak{x},\mathfrak{x})a^*)$$

where $\xi:\mathscr{A}_+\to\mathscr{A}_+$ be an altering distance function and $\varpi:\mathscr{A}_+\to\mathscr{A}_+$ is lower semi-continuous, such that $\varpi(\mathfrak{s})\succ 0_\mathscr{A}$ for all $\mathfrak{s}\succ 0_\mathscr{A}$ and discontinuous at $\mathfrak{s}=0_\mathscr{A}$ with $\varpi(0_\mathscr{A})=0_\mathscr{A}$.

iii) $\exists\ \mathbb{M}\in\mathscr{A}_+\ni\xi(\rho_{c^*}(\mathscr{H}(\mathfrak{s},\iota),\mathscr{H}(\mathfrak{s},\ell),\mathscr{H}(\mathfrak{s},\ell)))\preceq||\mathbb{M}|||\iota-\ell|$ for every $\mathfrak{s}\in\cup_{i=1}^{\mathfrak{r}}\overline{\Delta}_i$, $\iota,\ell\in[0,1]$

Then $\mathscr{H}(.,0)$ has a fixed point $\iff\mathscr{H}(.,1)$ has a fixed point.

Proof Let the set $\mathbb{B}=\left\{\ \iota\in[0,1]:\mathscr{H}(\mathfrak{s},\iota)=\mathfrak{s}\text{ for some }\mathfrak{s}\in\cup_{i=1}^{\mathfrak{r}}\Delta_i\ \right\}$.

Suppose that $\mathscr{H}(.,0)$ has a fixed point in $\cup_{i=1}^{\mathfrak{r}}\Delta_i$, we have that $0_\mathscr{A}\in\mathbb{B}$. So that $\mathbb{B}\neq\emptyset$. Now we show that $\mathbb{B}$ is both closed and open in $[0,1]$ and hence by the connectedness $\mathbb{B}=[0,1]$. As a result, $\mathscr{H}(.,1)$ has a fixed point in $\cup_{i=1}^{\mathfrak{r}}\Delta_i$. First we show that $\mathbb{B}$ closed in $[0,1]$. To see this, Let $\{\mathfrak{i}_\mathfrak{v}\}_{\mathfrak{v}=1}^{\infty}\subseteq\mathbb{B}$ with $\mathfrak{i}_\mathfrak{v}\to\mathfrak{i}\in[0,1]$ as $\mathfrak{v}\to\infty$. We must show that $\mathfrak{i}\in\mathbb{B}$. Since $\mathfrak{i}_\mathfrak{v}\in\mathbb{B}$ for $\mathfrak{v}=0,1,2,3,\cdots$, there exists sequences $\{\mathfrak{s}_\mathfrak{v}\}\subseteq\cup_{i=1}^{\mathfrak{r}}\Delta_i$ with $\mathfrak{s}_\mathfrak{v}=\mathscr{H}(\mathfrak{s}_\mathfrak{v},\mathfrak{i}_\mathfrak{v})$.

Consider

$$\begin{aligned}\rho_{c^*}(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{s}_{\mathfrak{v}+1}) &= \rho_{c^*}(\mathscr{H}(\mathfrak{s}_\mathfrak{v},\mathfrak{i}_\mathfrak{v}),\mathscr{H}(\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}+1}),\mathscr{H}(\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}+1}))\\ &\preceq \begin{array}{c}\rho_{c^*}(\mathscr{H}(\mathfrak{s}_\mathfrak{v},\mathfrak{i}_\mathfrak{v}),\mathscr{H}(\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{i}_\mathfrak{v}),\mathscr{H}(\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{i}_\mathfrak{v}))\\ +\rho_{c^*}(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{i}_\mathfrak{v}),\mathscr{H}(\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}+1}),\mathscr{H}(\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{i}_{\mathfrak{v}+1}))\end{array}\\ &\preceq \rho_{c^*}(\mathscr{H}(\mathfrak{s}_\mathfrak{v},\mathfrak{i}_\mathfrak{v}),\mathscr{H}(\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{i}_\mathfrak{v}),\mathscr{H}(\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{i}_\mathfrak{v}))+||\mathbb{M}|||\mathfrak{i}_\mathfrak{v}-\mathfrak{i}_{\mathfrak{v}+1}|.\end{aligned}$$

Letting $\mathfrak{v} \to \infty$, and applying ξ properties, we get

$$\begin{aligned}\lim_{\mathfrak{v}\to\infty} \xi(\rho_{c^*}(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{s}_{\mathfrak{v}+1})) &\preceq \lim_{\mathfrak{v}\to\infty} \xi(\rho_{c^*}(\mathscr{H}(\mathfrak{s}_\mathfrak{v},\mathfrak{i}_\mathfrak{v}),\mathscr{H}(\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{i}_\mathfrak{v}),\mathscr{H}(\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{i}_\mathfrak{v})))\\ &\preceq \lim_{\mathfrak{v}\to\infty}\left(\xi(a\rho_{c^*}(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{s}_{\mathfrak{v}+1})a^*) - \varpi(a\rho_{c^*}(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{s}_{\mathfrak{v}+1})a^*) \right).\end{aligned}$$

By the definitions of ξ, ϖ and $||a|| < 1$, we conclude that

$$\lim_{\mathfrak{v}\to\infty} ||\rho_{c^*}(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{s}_{\mathfrak{v}+1})|| \le \lim_{\mathfrak{v}\to\infty} ||a||^2||\rho_{c^*}(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{s}_{\mathfrak{v}+1})||.$$

which implies that $\lim_{\mathfrak{v}\to\infty}\left(1-||a||^2\right)||\rho_{c^*}(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{s}_{\mathfrak{v}+1})|| \le 0$.
So that

$$\lim_{\mathfrak{v}\to\infty} \rho_{c^*}(\mathfrak{s}_\mathfrak{v},\mathfrak{s}_{\mathfrak{v}+1},\mathfrak{s}_{\mathfrak{v}+1}) = \tilde{0}_{\tilde{\mathscr{A}}}.$$

Next,we will show that $\{\mathfrak{s}_\mathfrak{v}\}$ is a Cauchy sequence in $\mathbb{V}$ with regard to $\mathscr{A}$. On the contrary, suppose that $\{\mathfrak{s}_\mathfrak{v}\}$ not be a Cauchy sequence,then for some $\varepsilon \succ 0_{\mathscr{A}}$ and the sequence of natural numbers $\{\mathfrak{v}(j)\}$ and $\{\mathfrak{u}(j)\}$ such that $\mathfrak{u}(j) > \mathfrak{v}(j) \ge j$ for $j \in \mathbb{N}$ and

$$\rho_{c^*}\left(\mathfrak{s}_{\mathfrak{v}(j)},\mathfrak{s}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}\right) \succeq \varepsilon \tag{17}$$

corresponding to $\mathfrak{v}(j)$. We can choose $\mathfrak{v}(j)$ to be the smallest such that(17) is satisfied. Then we have

$$\rho_{c^*}\left(\mathfrak{s}_{\mathfrak{v}(j)},\mathfrak{s}_{\mathfrak{u}(j)-1},\mathfrak{s}_{\mathfrak{u}(j)-1}\right) \prec \varepsilon. \tag{18}$$

From (17) and (18), we have

$$\varepsilon \preceq \rho_{c^*}\left(\mathfrak{s}_{\mathfrak{v}(j)},\mathfrak{s}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}\right) \preceq \rho_{c^*}\left(\mathfrak{s}_{\mathfrak{v}(j)},\mathfrak{s}_{\mathfrak{v}(j)+1},\mathfrak{s}_{\mathfrak{v}(j)+1}\right) + \rho_{c^*}\left(\mathfrak{s}_{\mathfrak{v}(j)+1},\mathfrak{s}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}\right).$$

Letting $j \to \infty$ and applying ξ on both sides, we get

$$\xi(\varepsilon) \preceq \lim_{j\to\infty} \xi\left(\rho_{c^*}\left(\mathfrak{s}_{\mathfrak{v}(j)+1},\mathfrak{s}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}\right)\right). \tag{19}$$

But

$$\begin{aligned}&\lim_{j\to\infty} \xi\left(\rho_{c^*}\left(\mathfrak{s}_{\mathfrak{v}(j)+1},\mathfrak{s}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}\right)\right)\\ =\ &\lim_{j\to\infty} \xi\left(\rho_{c^*}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}(j)+1},\mathfrak{i}_{\mathfrak{v}(j)+1}),\mathscr{H}(\mathfrak{s}_{\mathfrak{u}(j)},\mathfrak{i}_{\mathfrak{u}(j)}),\mathscr{H}(\mathfrak{s}_{\mathfrak{u}(j)},\mathfrak{i}_{\mathfrak{u}(j)})\right)\right)\\ \preceq\ &\lim_{\mathfrak{v}\to\infty}\left(\xi\left(a\rho_{c^*}\left(\mathfrak{s}_{\mathfrak{v}(j)+1},\mathfrak{s}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}\right)a^*\right) - \varpi\left(a\rho_{c^*}\left(\mathfrak{s}_{\mathfrak{v}(j)+1},\mathfrak{s}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}\right)a^*\right) \right).\end{aligned}$$

It follows that $\lim_{j\to\infty}(1-||a||^2)||\rho_{c^*}\left(\mathfrak{s}_{\mathfrak{v}(j)+1},\mathfrak{s}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}\right)||\leq 0$.
Thus, $\lim_{j\to\infty}\rho_{c^*}\left(\mathfrak{s}_{\mathfrak{v}(j)+1},\mathfrak{s}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}\right)=0_{\mathscr{A}}$. Hence, from Eq.(19) and properties of ξ, we have $\varepsilon\leq 0_{\mathscr{A}}$, which is a contradiction. Hence $\{\mathfrak{s}_{\mathfrak{v}}\}$ is Cauchy sequences in $\mathbb{V}$ with regard to $\mathscr{A}$. However, $(\mathbb{V},\mathscr{A},\rho_{c^*})$ is complete, so there exists $\mathfrak{z}\in\mathbb{V}$ such that $\lim_{\mathfrak{v}\to\infty}\mathfrak{s}_{\mathfrak{v}}=\mathfrak{z}$. Now we show that $\mathfrak{z}\in\cap_{i=1}^{\mathfrak{r}}\Delta_i$. From the cyclic representation of Ξ concerning $\mathscr{H}$, and since $\mathfrak{s}_0\in\Delta_1$, we have $\{\mathfrak{s}_{\mathfrak{v}\mathfrak{r}}\}_{\mathfrak{v}\geq 0}\subseteq\Delta_1$. Since $\overline{\Delta}_1$ is closed, so that $\mathfrak{z}\in\Delta_1$. Again using a cyclic representation of Ξ concerning $\mathscr{H}$, we get $\{\mathfrak{s}_{\mathfrak{v}\mathfrak{r}+1}\}_{\mathfrak{v}\geq 0}\subseteq\Delta_2$. Since Δ_2 is closed, so we obtain $\mathfrak{z}\in\Delta_2$. Proceeding this way, we get $\mathfrak{z}\in\cap_{i=1}^{\mathfrak{r}}\Delta_i$ and hence, $\cap_{i=1}^{\mathfrak{r}}\Delta_i\neq\emptyset$.
Now, we have

$$\begin{aligned}\xi(\rho_{c^*}(\mathfrak{z},\mathscr{H}(\mathfrak{z},\mathfrak{i}),\mathscr{H}(\mathfrak{z},\mathfrak{i}))) &= \lim_{\mathfrak{v}\to\infty}\xi(\rho_{c^*}(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{i}),\mathscr{H}(\mathfrak{z},\mathfrak{i}),\mathscr{H}(\mathfrak{z},\mathfrak{i})))\\ &\preceq \lim_{\mathfrak{v}\to\infty}\Big(\ \xi(a\rho_{c^*}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{z},\mathfrak{z})a^*)-\varpi(a\rho_{c^*}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{z},\mathfrak{z})a^*)\ \Big)=0_{\mathscr{A}}.\end{aligned}$$

It follows that $\mathscr{H}(\mathfrak{z},\mathfrak{i})=\mathfrak{z}$. Thus, $\mathfrak{i}\in\mathbb{B}$. Hence $\mathbb{B}$ is closed in $[0,1]$. Let $\mathfrak{i}_0\in\mathbb{B}$, then there exist $\mathfrak{s}_0\in\cup_{i=1}^{\mathfrak{r}}\Delta_i$ with $\mathfrak{s}_0=\mathscr{H}(\mathfrak{s}_0,\mathfrak{i}_0)$. Since $\Delta_1,\Delta_2,\cdots,\Delta_{\mathfrak{r}}$ are open, then there exist $r>0$ such that $B_{\rho_{c^*}}(\mathfrak{s}_0,r)\subseteq\cup_{i=1}^{\mathfrak{r}}\Delta_i$. Choose $\mathfrak{i}\in(\mathfrak{i}_0-\varepsilon,\mathfrak{i}_0+\varepsilon)$ such that $|\mathfrak{i}-\mathfrak{i}_0|\leq\frac{1}{||\mathbb{M}^{\mathfrak{v}}||}<\frac{\varepsilon}{2}$, then for
$\mathfrak{s}\in\overline{B_{\rho_{c^*}}(\mathfrak{s}_0,r)}=\left\{\mathfrak{s}\in\cup_{i=1}^{\mathfrak{r}}\Delta_i/\rho_{c^*}(\mathfrak{s},\mathfrak{s}_0,\mathfrak{s}_0)\preceq r+\rho_{c^*}(\mathfrak{s}_0,\mathfrak{s}_0,\mathfrak{s}_0)\right\}$. Now we have

$$\begin{aligned}\rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{i}),\mathfrak{s}_0,\mathfrak{s}_0) &= \rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{i}_0),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{i}_0))\\ &\preceq \rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{i}))+\rho_{c^*}(\mathscr{H}(\mathfrak{s}_0,\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{i}_0),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{i}_0))\\ &\preceq \rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{i}))+\frac{1}{||\mathbb{M}^{\mathfrak{v}-1}||}.\end{aligned}$$

Letting $\mathfrak{v}\to\infty$ and applying ξ properties, we obtain

$$\begin{aligned}\xi(\rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{i}),\mathfrak{s}_0,\mathfrak{s}_0)) &\preceq \xi(\rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{i})))\\ &\preceq \xi(a\rho_{c^*}(\mathfrak{s},\mathfrak{s}_0,\mathfrak{s}_0)a^*)-\varpi(a\rho_{c^*}(\mathfrak{s},\mathfrak{s}_0,\mathfrak{s}_0)a^*)\\ &\preceq \xi(a\rho_{c^*}(\mathfrak{s},\mathfrak{s}_0,\mathfrak{s}_0)a^*).\end{aligned}$$

Since ξ is continuous and non-decreasing, we obtain

$$\rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{i}),\mathfrak{s}_0,\mathfrak{s}_0)\preceq a\rho_{c^*}(\mathfrak{s},\mathfrak{s}_0,\mathfrak{s}_0)a^*$$

which implies that

$$\begin{aligned} ||\rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{i}),\mathfrak{s}_0,\mathfrak{s}_0)|| &\leq ||a||^2||\rho_{c^*}(\mathfrak{s},\mathfrak{s}_0,\mathfrak{s}_0)|| < ||\rho_{c^*}(\mathfrak{s},\mathfrak{s}_0,\mathfrak{s}_0)|| \\ &\leq r + ||\rho_{c^*}(\mathfrak{s}_0,\mathfrak{s}_0,\mathfrak{s}_0)||. \end{aligned}$$

Thus for each fixed $\mathfrak{i} \in (\mathfrak{i}_0 - \varepsilon, \mathfrak{i}_0 + \varepsilon)$, $\mathscr{H}(.,\mathfrak{i}) : \overline{B_{\rho_{c^*}}(\mathfrak{s}_0, r)} \to \overline{B_{\rho_{c^*}}(\mathfrak{s}_0, r)}$. Since also (ii) holds and ξ is continuous and non-decreasing and ϖ is continuous with $\varpi(t) \succ 0_{\mathscr{A}}$ fort $t \succ 0_{\mathscr{A}}$, then all the conditions of Theorem 4.2 are satisfied. Thus, we conclude that $\mathscr{H}(.,\mathfrak{i})$ has a fixed point in $\cap_{i=1}^{\mathfrak{r}} \overline{\Delta}_i$. But this must be in $\cap_{i=1}^{\mathfrak{r}} \Delta_i$ since (i) holds. Thus, $\mathfrak{i} \in \mathbb{B}$ for any $\mathfrak{i} \in (\mathfrak{i}_0 - \varepsilon, \mathfrak{i}_0 + \varepsilon)$. Hence $(\mathfrak{i}_0 - \varepsilon, \mathfrak{i}_0 + \varepsilon) \subseteq \mathbb{B}$. Clearly $\mathbb{B}$ is open in [0, 1]. For the reverse implication, we use the same strategy.

5. Conclusions

This study culminates in the successful formulation of common fixed point theorems (CFPT) for generalized (α, β)-(ξ, ϖ)-weakly cyclic contraction mappings, situated within the abstract framework of $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$. The findings contribute significantly to the advancement of fixed point theory and underscore its applicability to solving functional equations and exploring aspects of homotopy theory. These results not only deepen theoretical understanding but also open avenues for further mathematical inquiry and interdisciplinary applications.

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