Available online at http://scik.org
Advances in Fixed Point Theory, 3 (2013), No. 2, 355-382
ISSN: 1927-6303

A REVIEW OF THE KKM THEORY ON ϕ_A-SPACES OR GFC-SPACES

SEHIE PARK[1,2]

[1]The National Academy of Sciences, Republic of Korea, Seoul 137–044

[2]Department of Mathematical Sciences, Seoul National University, Seoul 151–747, Korea

Abstract. In the KKM theory, G-convex spaces are extended to KKM spaces or abstract convex spaces in 2006. Various types of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ appeared until 2007 can be made into G-convex spaces in several ways. Moreover, various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on G-convex spaces. Therefore, our G-convex space theory can be applied to various types of ϕ_A-spaces. However, Khanh et al. in 2009 introduced a disguised form of ϕ_A-spaces called GFC-spaces. In the present paper, we review their works on GFC-spaces and clarify that their basic results are consequences of known ones. Finally, further comments on each of seven papers on GFC-spaces are given.

Keywords: abstract convex space; KKM space; G-convex space; FC-space; GFC-space; KKM theorem; maximal element; minimax inequality; variational inequality.

2010 AMS Subject Classification: 47H04; 47H10; 49J27; 49J35; 52A99; 54H25; 58E35; 90C47; 91B50

1. Introduction

The Knaster–Kuratowski–Mazurkiewicz (simply, KKM) theorem in 1929 provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. The KKM theory, first called by the author in 1992, is the study on applications of numerous equivalent formulations or generalizations of the theorem; see [22-26].

Received March 30, 2013

Since 1993, the author has initiated the study of the KKM theory on generalized convex spaces (or G-convex spaces) $(X, D; \Gamma)$ as a common generalization of various general convexities without linear structures due to many other authors. We have established within such a frame the foundations of the KKM theory, as well as fixed point theorems and many other equilibrium results for multimaps. This direction of study has been followed by a number of other authors.

In the last decade, there have appeared authors who introduced spaces of the form $(X, \{\varphi_A\})$ having a family $\{\varphi_A\}$ of continuous functions defined on simplexes and claimed that such spaces generalize G-convex spaces without giving any justifications or proper examples. In fact, a number of modifications or imitations of G-convex spaces have followed; see [1-6,18,21,27-30,44,52] and other literature. Some authors also introduced various types of generalized KKM maps and tried to generalize the KKM theorem for their own settings; see [2-6,18,21,44,52] and the references therein. Most of such generalizations are disguised forms of known results.

In order to destroy some inadequate concepts and to upgrade the KKM theory, in 2006-09, we proposed new concepts of abstract convex spaces and the KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [27-30,32,33,36-44]. Moreover, we noticed that all spaces of the form $(X, \{\varphi_A\})$ can be unified to ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ or spaces having a family $\{\phi_A\}_{A\in\langle D\rangle}$ of singular simplexes.

In fact, in [28-31,34-36], we showed that various types of ϕ_A-spaces can be made into G-convex spaces in several ways, and various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on such G-convex spaces. Therefore, our G-convex space theory can be applied to various types of ϕ_A-spaces. As such examples, we obtained KKM type theorems and very general fixed point theorems on ϕ_A-spaces.

However, later in 2009, Khanh et al. [12] introduced another disguised form of ϕ_A-spaces called GFC-spaces, and published several papers on their spaces [7,12-17]. Our aim in the present paper is to review their works on GFC-spaces and clarify that their basic results are consequences of previously known ones obtained by the present author. In fact, we discuss on the definition of GFC-spaces [7,12-17] (Section 3), artificial terminology adopted by Khanh et al. [7,12-15] (Section 4), convex subsets of GFC-spaces [7,12-17] (Section 5), KKM theorems on GFC-spaces [12,13] (Section 6), an example of GFC-spaces in [17] (Section 7), continuous selection theorems in [17] (Section 8), and weakly KKM theorems in [17] (Section 9). These basic results are all consequences of previously known ones. Finally, Section 10 devotes to introduce abstracts and to give further comments on each of seven papers on GFC-spaces [7,12-17].

2. Abstract convex spaces

We follow our recent works [42,43,47] and the references therein. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$*-map* [resp., a $\mathfrak{KO}$*-map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

Typical examples of KKM spaces can be seen in [42,43] and the references therein. Recently, Kulpa and Szymanski [19] found some partial KKM spaces which are not KKM spaces.

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\begin{aligned} &\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Convex space} \Longrightarrow \text{H-space} \\ &\Longrightarrow \text{G-convex space} \Longrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space} \\ &\Longrightarrow \text{Partial KKM space} \Longrightarrow \text{Abstract convex space.} \end{aligned}$$

Recall that, in 2010 [43], we derived generalized forms of the Ky Fan minimax inequality, the von Neumann–Sion minimax theorem, the von Neumann–Fan intersection theorem, the Fan type analytic alternative, and the Nash equilibrium theorem for partial KKM spaces. Consequently, our results in [43] unify and generalize most of previously known particular cases of the same nature.

Moreover, in [42], we clearly derived a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently,

[42] unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces. Note that [42] contains some incorrectly stated statements such as (VI), Theorem 4, (XVI), and (XVII). These can be corrected easily.

In a previous work [41] of the present author, we obtained three general KKM type theorems for abstract convex spaces. In a recent works [47,48], we showed that two of them can be stated for intersectionally closed-valued KKM maps in the sense of Luc et al. [20]. Each of such KKM type theorems contains a large number of previously known particular forms. In [48], we recalled some of them due to the present author in order to give comments or minor corrections of the proofs of them.

Consider the following related four conditions for a multimap $G : D \multimap Z$:

(a) $\bigcap_{a\in D} \overline{G(a)} \neq \emptyset$ implies $\bigcap_{a\in D} G(a) \neq \emptyset$.

(b) $\bigcap_{a\in D} \overline{G(a)} = \overline{\bigcap_{a\in D} G(a)}$ (G is *intersectionally closed-valued* [20]).

(c) $\bigcap_{a\in D} \overline{G(a)} = \bigcap_{a\in D} G(a)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

The following is essentially given in [41,48]:

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $G : D \multimap Z$ a map such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*

(2) *there exists a nonempty compact subset K of Z such that either*

(i) *$K \supset \bigcap\{\overline{G(a)} \mid a \in M\}$ for some $M \in \langle D\rangle$; or*

(ii) *for each $N \in \langle D\rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{F(L_N)}$ is compact, and*

$$K \supset \overline{F(L_N)} \cap \bigcap_{a\in D'} \overline{G(a)}.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{a\in D} \overline{G(a)} \neq \emptyset.$$

Furthermore,

(α) *if G is transfer closed-valued, then* $\overline{F(E)} \cap K \cap \bigcap\{G(a) \mid a \in D\} \neq \emptyset$; *and*

(β) *if G is intersectionally closed-valued, then* $\bigcap\{G(a) \mid a \in D\} \neq \emptyset$.

3. G-convex spaces, ϕ_A-spaces and GFC-spaces

Let Δ^n be the standard n-simplex with vertices $\{e_i\}_{i=0}^n$. For any $A \in \langle D \rangle$, let $|A|$ be its cardinality, $\Delta^A := \Delta^{|A|-1}$ and Δ_J^A be the face of Δ^A corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$, then $\Delta_J^A = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$.

The following is well-known:

Definition 3.1. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$, there exists a continuous function $\phi_A : \Delta^A \to \Gamma_A$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J^A) \subset \Gamma_J$.

For details on G-convex spaces, see [23-26,49] and the references therein, where basic theory was indicated and lots of examples of G-convex spaces were given.

Recently, there have appeared authors of [1,3-6] and many others who introduced spaces of the form $(X, \{\varphi_A\})$; see [40,50] and the references therein. Some of them tried to rewrite certain results on G-convex spaces by simply replacing Γ_A by $\varphi_A(\Delta^A)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples.

Motivated by such situation, we introduced the following in 2007:

Definition 3.2. A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta^A \to X$ (that is, singular $(|A| - 1)$-simplexes) for each $A \in \langle D \rangle$.

It is clear that any G-convex space is a ϕ_A-space. Conversely, a ϕ_A-space can be made into a G-convex space [28,31,34-36,39,40] in several ways. The following is one of them:

Proposition 3.1. *Any ϕ_A-space $(X, D; \{\phi_A\})$ can be made into a G-convex space $(X, D; \Gamma)$ with*

$$\Gamma_N := \bigcup_{M \supset N} \phi_M(\Delta_N^M) \text{ for each } N \in \langle D \rangle$$

where M is taken in $\langle D \rangle$.

In order to apply our theory of abstract convex spaces to ϕ_A-spaces, we observed the following in [50]:

Proposition 3.2. *A ϕ_A-space $(X, D; \{\phi_A\})$ becomes an abstract convex space $(X, D; \Gamma)$ with $\Gamma_A = \Gamma(A) = \phi_A(\Delta^A)$ for $A \in \langle D \rangle$*

The resulting abstract convex space is not a G-convex space in general, but we have the following in [50]:

Proposition 3.3. *Any ϕ_A-space $(X, D; \{\phi_A\})$ is a KKM space.*

Therefore, any ϕ_A-space possesses a large number of properties given in [42,43].

In 2005, Ding [4] introduced the following notion of "a finitely continuous" topological space (in short, FC-space):

Definition 3.3. ([4], Def. 1.1) $(Y, \{\varphi_N\})$ is said to be a FC-space if Y is a topological space and for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. A subset D of $(Y, \{\varphi_N\})$ is said to be a FC-subspace of Y if for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ and for each $\{y_{i_0}, \dots, y_{i_k}\} \subset N \cap D$, $\varphi_N(\Delta_k) \subset D$ where $\Delta_k = \mathrm{co}\{e_{i_j} : j = 0, \dots, k\}$.

In our previous comments on FC-spaces [28,29,31-36,44], we criticized this definition in several points.

Recall that most of modifications or variants of G-convex spaces $(X, D; \Gamma)$ are based on the replacement of $\Gamma(N)$ by the corresponding $\phi_N(\Delta^N)$; see [44]. The following form of ϕ_A-spaces $(X, D; \{\phi_A\})$ gives one of such examples:

Definition 3.4. ([7,12-17]) Let X be a topological space, A be a nonempty set and Φ be a family of continuous mappings $\varphi : \Delta_n \to X,\ n \in \mathbb{N}$. Then a triple (X, A, Φ) is said to be a generalized finitely continuous topological space (GFC-space in short) iff, for each finite subset $N = \{a_0, a_1, \dots, a_n\}$ of A, there is $\varphi_N : \Delta_n \to X$ of the family Φ. (Later, we also use $(X, A, \{\varphi_N\})$ to denote (X, A, Φ).)

In fact, in 2009, Khanh et al. [12] introduced this definition and that of certain generalized KKM mappings. They noted incorrectly that G-convex spaces are special case of GFC-spaces and that the G-convex space and the FC-space are incomparable. They also missed to give any proper examples to support their claims. Actually, their GFC-spaces are simply our ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ and, hence, are essentially same to G-convex spaces.

4. Artificial Terminology in GFC-spaces

Since 1995, there have appeared concepts of the *compact closure (ccl), compact interior (cint), transfer compactly closed-valued multimap, transfer compactly l.s.c. multimap, transfer compactly local intersection property*, respectively, instead of the basic concepts of the *closure, interior, closed-valued multimap, l.s.c. multimap, and possession of a finite open cover property.* The aim of introducing such unfamiliar terms is to claim theorems of utmost generality, but, unfortunately, such terms are very difficult to check in practice and, in many cases, have no proper examples. Therefore they are artificial, impractical, and useless.

In our previous work [26] in 2000, we showed that we can invalidate many of such terms on G-convex spaces by replacing the original topology of the underlying space by a finer topology. However, in the last decade, a number of authors continued to use such inappropriate terms. Even in 2010, there were authors who were still using such obsolete terminology.

In a recent paper in 2011 [46], we showed that such adoption of terms is inappropriate and artificial in the KKM theory of abstract convex spaces. In fact, any theorem with a "transfer" attached term is equivalent to the corresponding one without "transfer". Moreover, we can invalidate terms with "compactly" attached by giving a finer topology on the underlying space. In such ways, we obtained simpler formulations of KKM type theorems, Fan-Browder type fixed point theorems, and other results in the KKM theory on abstract convex spaces. Consequently, we can upgrade the theory by eliminating the above-mentioned inappropriate terms and by stating theorems in more elegant forms.

Since 2009, in the works of Khanh et al. [7,12-15] and others, they adopted the above mentioned artificial terminology. However, later papers of Khanh et al. [16,17] in 2011 did not use such artificial terminology.

5. Definition of convex subsets of GFC-spaces

We have used the following definition:

Definition 5.1. Let $(E, D; \Gamma)$ be a G-convex space and $X \subset E,\ D' \subset D$. Then X is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $J \in \langle D' \rangle$, we have $\Gamma_J \subset X$.

This can be stated for ϕ_A-spaces as follows:

Definition 5.2. Let $(E, D; \{\phi_A\}_{A \in \langle D \rangle})$ be a ϕ_A-space and $X \subset E,\ D' \subset D$. Then X is called a Γ-*convex subset* of the ϕ_A-space relative to D' if for any $N \in \langle D \rangle$ and any $J \in \langle N \cap D' \rangle$, we have $\phi_N(\Delta_J^N) \subset X$.

Proposition 5.1. *The preceding two definitions are equivalent.*

Proof. Assume Definition 5.1. We already showed that every ϕ_A-space can be made into a G-convex space $(E, D; \Gamma)$ such that for each $N \in \langle D \rangle$, there exists a continuous function $\phi_N : \Delta^N \to \Gamma_N$ such that $J \in \langle N \rangle$ implies $\phi_N(\Delta_J^N) \subset \Gamma_J$. For any $J \in \langle N \cap D' \rangle$, we assumed $\Gamma_J \subset X$. Hence $\phi_N(\Delta_J^N) \subset X$. Therefore, Definition 5.2 holds.

Conversely, as in Proposition 3.1, for each $N \in \langle D \rangle$, we defined $\Gamma_N := \bigcup_{M \supset N} \phi_M(\Delta_N^M)$ such that $(E, D; \Gamma)$ is a G-convex space. Suppose that X is a Γ-convex subset of the ϕ_A-space relative to D'. Then for any $J \in \langle D' \rangle$ and $M \in \langle D' \rangle$ such that $M \supset J$, we have $\phi_M(\Delta_J^M) \subset X$ by Definition 5.2. Hence $\Gamma_J := \bigcup_{M \supset J} \phi_M(\Delta_J^M) \subset X$. This means Definition 5.1. $\square$

In 2006, Ding added the following to [4, Def. 1.1]:

Definition 5.3. ([5, Def. 2.1]) Let $(Y, \{\varphi_N\})$ be an FC-space. If A and B are two subsets of Y, B is said to be an FC-subspace of Y relative to A if for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and for any $\{y_{i_0}, \ldots, y_{i_k}\} \subset A \cap N$, $\varphi_N(\Delta_k) \subset B$ where $\Delta_k = \mathrm{co}\{e_{i_j} : j = 0, \ldots, k\}$. If $A = B$, then B is called an FC-subspace of Y.

Note that, replacing (E, D, X, D', ϕ) by (Y, Y, B, A, φ), Definition 5.2 reduces to Ding's above definition.

Therefore, instead of using the concept of an FC-subspace of $(Y, \{\varphi_N\})$ relative to A as in Definition 5.3 [5, Def. 2.1], we may use a Γ-convex subset of the G-convex space $(Y, D; \Gamma)$ relative to $A \subset D$. Any interested reader can check this matter in all of the papers on FC-spaces.

Now we consider the following due to Khanh et al.:

Definition 5.4. ([7]) Let $(X, A, \{\varphi_N\})$ be a GFC-space. Let $P, Q \subset A$ and $S : A \to 2^X$ be given. P is called an S-subset of A (an S-subset of A w.r.t. Q) if $\forall N = \{y_0, y_1, \ldots, y_n\} \in \langle D \rangle$, $\forall \{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\} \subset N \cap P$ ($N \cap Q$, resp.), $\phi_N(\Delta_k) \subset S(P)$, where Δ_k is the face of Δ_n corresponding to $\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\}$.

We show that this definition is particular to our Definition 5.2. In fact, by putting $E := X,\ X := S(P),\ D := A$ and $D' := P$, $S(P)$ is a Γ-convex subset relative to P. Moreover, by putting $E := X,\ X := S(P),\ D := A$ and $D' := P$, $S(P)$ is a Γ-convex subset relative to Q.

Remark. 1. In Definition 5.4, Khanh et al. followed Definition 5.3 of FC-subspace due to Ding ["If, in addition, S is the identity map then an S-subset of X coincides with an FC-subspace of X" [13]. This statement seems to be an evidence of the artificial generalization.].

2. Khanh et al. [17] stated that their $(S(P), P, \Phi)$ is a GFC-space. However, in Definition 5.4, the map S is hard to construct even in simplest cases. For example, a G-convex space $(E, D; \Gamma) := (\Delta^n, \{e_i\}_{i=0}^n; \mathrm{co})$ has a Γ-convex subset $(X, D'; \Gamma') := (\Delta^{n-1}, \{e_i\}_{i=0}^{n-1}; \mathrm{co})$. How could they construct $S : D' \multimap X$ such that $S(D') = X$ in this case? They could adopt $S(e_i) := X = \Delta^{n-1}$ trivially for each i, $0 \le i \le n-1$, but this is no use as they said in [17].

6. KKM theorems of GFC-spaces in [12]

In our previous work [28] and others, we obtained the following:

Proposition 6.1. *For a ϕ_A-space $(X, D; \{\phi_A\})$, any map $T : D \multimap X$ satisfying*

$$\phi_A(\Delta_J) \subset T(J) \quad \textit{for each} \quad A \in \langle D \rangle \textit{ and } J \in \langle A \rangle$$

is a KKM map on a G-convex space $(X, D; \Gamma)$.

The following has a quite long history in the KKM theory developed by ourselves:

Definition 6.1. Let $(E, D; \Gamma)$ be a G-convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{a \in A} G(a) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

As in Proposition 5.1, this definition is easily seen to be equivalent to the following generalization of Proposition 6.1:

Definition 6.2. Let $(E, D; \{\phi_A\})$ be a ϕ_A-space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\phi_N(\Delta_J^N)) \subset G(J) := \bigcup_{a \in J} G(a) \qquad \text{for all} \quad N \in \langle D \rangle \text{ and } J \in \langle N \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

Khanh et al. gave the following:

Definition 6.3. ([13]) (i) Let $(X, Y; \Phi)$ be a GFC-space and Z be a topological space. Let $F : Y \to 2^Z$ and $T : X \to 2^Z$ be set-valued mappings. F is called a generalized KKM mapping w.r.t. T (T-KKM mapping in short) if, for each $N = \{y_0, y_1, \ldots, y_n\} \in \langle Y \rangle$ and each $\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\} \subset N$, one has $T(\phi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} F(y_{i_j})$, where Δ_k is as above.

(ii) We say that a set-valued mapping $F : X \to 2^Z$ has the generalized KKM property if for each T-KKM mapping $F : Y \to 2^Z$, the family $\{\overline{F(y)} : y \in D$ has the finite intersection property. By $\mathrm{KKM}(X, Y, Z)$ we denote the class of all the mappings $F : X \to 2^Z$ which enjoy the generalized KKM property.

Note that (i) is the same to Definition 6.2 and that $\mathrm{KKM}(X, Y, Z)$ is our $\mathfrak{KC}(X, Y, Z)$.

The following KKM theorem of Khanh et al. follows from our Theorem C:

Theorem 6.2. ([12]) *Let (X, Y, Φ) be a GFC-space, Z be a topological space, $S : Y \to 2^X$, $T : X \to 2^Z$ and $F : Y \to 2^Z$ be multifunctions, where $T \in \mathrm{KKM}(X, Y, Z)$. assume that*

(i_1) *for each compact subset $D \subset X$, $\overline{T(D)}$ is compact;*

(i_2) *there is a compact subset K of Z such that for each $N \in \langle Y \rangle$, there is an S-subset L_N of Y, containing N with $S(L_N)$ or $\overline{S(L_N)}$ being compact and*

$$\overline{T(S(L_N))} \cap \bigcap_{y \in L_N} \mathrm{ccl} F(y) \subset K;$$

(ii) *F is T-KKM and transfer compactly closed-valued.*

Then

$$\overline{T(S(Y))} \cap \bigcap_{y \in Y} F(y) \neq \emptyset.$$

For a long time, the present author and many followers used to denote that L_N is a Γ-convex subset of the whole space X. But, in Theorem 6.2, $S(L_N)$ is a Γ-convex subset of X in our sense. Note that S seems to be an artificial extension of the identity map on $X = Y$ in Ding's FC-spaces.

Proof using Theorem C. We can replace the topology of X by its compactly generated extension; see [26,46]. In order to apply Theorem C(α), consider the 6-tuple

$$(E, F, G, D, D', L_N))$$

instead of

$$(S(Y), T, F, Y, L_N, S(L_N)).$$

From $T \in \mathrm{KKM}(X, Y, Z) = \mathfrak{KC}(X, Y, Z)$, it is known that $T|_{S(Y)} \in \mathfrak{KC}(S(Y), Y, Z)$ and hence $F \in \mathfrak{KC}(E, D, Z)$. Note that (ii) implies (1) in Theorem C. We show that (i_1) and (i_2) imply condition (ii) of Theorem C. Since $S(L_N)$ or $\overline{S(L_N)}$ is compact, $\overline{T(S(L_N))} = \overline{F(L_N)}$ is compact by (i_1). [The proof of Theorem C works for compact $\overline{F(L_N)}$!] Then by (i_2), we have

$$\overline{T(S(L_N))} \cap \bigcap_{y \in L_N} \mathrm{ccl} F(y) \subset K;$$

which implies

$$K \supset \overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)}.$$

Then by our Theorem C(α), we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} G(y) \neq \emptyset.$$

or

$$\overline{T(S(Y))} \cap K \cap \bigcap_{y \in Y} F(y) \neq \emptyset.$$

This conclusion is slightly better than that of Theorem 6.2. □

From the seminal works of Ky Fan, this kind of whole intersection property can be reformulated to many equivalent forms like coincidence theorems or geometric forms. This was done several earlier works of the present author et al.

7. An example of GFC-spaces in [17]

The authors of [17] stated as follows:

"The class of GFC-spaces contains a large number of spaces with various kinds of generalized convexity structures such as FC-spaces, G-convex spaces. Recall that a G-convex space of Park is a triple (X, A, Γ), where X and A are as Definition 3.4 and $\Gamma : \langle A \rangle \multimap X$ is such that, for each $N \in \langle A \rangle$ with cardinality $|N| = n+1$, there exists a continuous map $\varphi_N : \Delta_n \to \Gamma(A)$ such that, for each $M \in \langle N \rangle$, $\varphi_N(\Delta_M) \subset \Gamma(M)$. A G-convex space (X, A, Γ) is called trivial iff, for all $N \in \langle A \rangle$, $\Gamma(N) = X$. Of course, any above-mentioned space can be made into a trivial G-convex space, but a trivial G-convex space has no use. In [29] it is asserted that any GFC-space is a (nontrivial) G-convex space, i.e., the latter is more general. However, in fact, the notion of a GFC-space is properly more general than that of a G-convex space as shown by an example."

Comments to the example in [17]. 1. This is an example of a GFC-space which is a 'trivial' G-convex space.

2. We showed that any ϕ_A spaces can be made into G-convex spaces in several ways; see [28,29,31,34-36].

3. We did not say that any GFC-space is a 'nontrivial' G-convex space. Actually, we did not exclude 'trivial' G-convex spaces.

Contrary to the claim in [17], we have the following:

Proposition 7.1. *A ϕ_A-space $(X, D; \{\phi_A\})$ is a G-convex space $(X, D; \Gamma)$ iff it has a KKM map $G : D \multimap X$.*

Proof. Suppose that $(X, D; \{\phi_A\})$ has a KKM map $G : D \multimap X$. Then we have

$$\phi_A(\Delta_J^A) \subset G(J) \quad \text{for each} \quad A \in \langle D \rangle \text{ and } J \in \langle A \rangle.$$

Define a map $\Gamma : \langle D \rangle \multimap X$ by $\Gamma(J) := G(J) = \bigcup\{G(a) \mid a \in J\}$ for each and each $J \in \langle A \rangle$. Then we have

$$\phi_A(\Delta_J^A) \subset \Gamma(J) \quad \text{for each} \quad A \in \langle D \rangle \text{ and } J \in \langle A \rangle.$$

Therefore $(X, D; \Gamma)$ is a G-convex space.

Conversely, suppose that $(X, D; \{\phi_A\})$ is a G-convex space $(X, D; \Gamma)$. By putting $G(z) := \Gamma(\{z\})$ for $z \in D$, we have a KKM map $G : D \multimap X$ as above. In fact, for each $A \in \langle D \rangle$, we have a continuous function $\phi_A : \Delta^A \to G(A) =: \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta^A_J) \subset G(J) =: \Gamma(J)$. Hence $\phi_A(\Delta^A_J) \subset \Gamma(J) \subset G(J)$ for each $A \in \langle D \rangle$ and $J \in \langle A \rangle$. Therefore $G : D \multimap X$ is a KKM map on $(X, D; \{\phi_A\})$. □

Let us say that a KKM map $G : D \multimap X$ is 'trivial' if $G(z) = X$ for all $z \in D$. Then the above proposition shows that

Proposition 7.2. *A ϕ_A-space $(X, D; \{\phi_A\})$ is a trivial G-convex space $(X, D; \Gamma)$ iff it has a trivial KKM map $G : D \multimap X$.*

Proposition 7.3. *A ϕ_A-space $(X, D; \{\phi_A\})$ is a nontrivial G-convex space $(X, D; \Gamma)$ iff it has a nontrivial KKM map $G : D \multimap X$.*

As expressed in [17], of course, any GFC-space without nontrivial KKM map has no use in the KKM theory. Consequently, in order to show that GFC-spaces properly generalize G-convex spaces, the authors of [17] should give an example of a GFC-space with a nontrivial KKM map that is not G-convex. This is impossible by Proposition 7.3.

8. Continuous selection theorems in [17]

Let X be a topological space. The following originates from Horvath [8,9]:

Definition 8.1. For a G-convex space $(Y, D; \Gamma)$, a multimap $T : X \multimap Y$ is called a Φ-*map* provided that there exists a companion multimap $S : X \multimap D$ satisfying

(a) for each $x \in X$, $J \in \langle S(x) \rangle$ implies $\Gamma_J \subset T(x)$; and

(b) $X = \bigcup\{\operatorname{Int} S^-(a) \mid a \in D\}$.

We can define the following equivalent:

Definition 8.2. For a ϕ_A-space $(Y, D; \phi_A)$, a multimap $T : X \multimap Y$ is called a Φ-*map* provided that there exists a companion multimap $S : X \multimap D$ satisfying

(a) for each $x \in X$, each $N \in \langle D \rangle$ and each $J \in \langle N \cap S(x) \rangle$, we have $\phi_N(\Delta_J^A) \subset T(x)$; and

(b) $X = \bigcup\{\operatorname{Int} S^-(a) \mid a \in D\}$.

We begin with the following modification of [49, Lemma 3.1]:

Lemma 8.1. *Let X be a normal space, $(Y, D; \{\phi_A)\}$ a ϕ_A-space, and $S : X \multimap D$ a multimap such that*

$$X = \bigcup\{\operatorname{Int} S^-(a) \mid a \in A\}.$$

for some $A \in \langle S(X) \rangle$. Then there exist a continuous map $s : X \to \phi_A(\Delta^A)$ such that $s(x) \in \phi_A(\Delta^A_{A \cap S(x)})$ for all $x \in X$.

Proof. Let $A := \{a_0, a_1, \dots, a_n\} \in \langle D \rangle$ such that $X = \bigcup_{i=0}^n \operatorname{Int} S^-(a_i)$. Note that $a_i \in S(X)$ for all i. Since $(Y, D; \{\phi_A\})$ is a ϕ_A-space, there exists a continuous map $\phi_A : \Delta^A \to Y$. Let $\{\alpha_i\}_{i=0}^n$ be a partition of unity subordinated to the cover $\{\operatorname{Int} S^-(a_i)\}_{i=0}^n$ of the normal space X; that is,

(1) for each i, $\alpha_i : X \to [0, 1]$ is continuous;

(2) $\operatorname{Supp} \alpha_i \subset \operatorname{Int} S^-(a_i)$ for each i; and

(3) for each $x \in X$, $\sum_{i=0}^n \alpha_i(x) = 1$.

Define a continuous map $p : X \to \Delta^A$ by

$$p(x) = \sum_{i=0}^n \alpha_i(x) e_i = \sum_{a_i \in A_x} \alpha_i(x) e_i \quad \text{for } x \in X,$$

where $\{e_i\}_{i=0}^n$ are vertices of Δ_n,

$$a_i \in A_x \subset A \Longleftrightarrow \alpha_i(x) \neq 0 \Longrightarrow x \in \operatorname{Int} S^-(a_i) \Longrightarrow a_i \in S(x)$$

and hence $A_x \in \langle A \cap S(x) \rangle$ and $p(x) \in \Delta^A_{A_x}$. Therefore,

$$(\phi_A \circ p)(x) \in \phi_A(\Delta^A_{A_x}) \subset \phi_A(\Delta^A_{A \cap S(x)})$$

and $s := \phi_A \circ p : X \to Y$ is a continuous map such that $s(x) \in \phi_A(\Delta^A_{A \cap S(x)})$. $\square$

Corollary 8.2. ([49, Lemma 3.1]) *Let X be a normal space, $(Y, D; \Gamma)$ a G-convex space, and $S : X \multimap D$ a multimap such that*

$$X = \bigcup \{\operatorname{Int} S^{-}(a) \mid a \in A\}.$$

for some $A \in \langle S(X) \rangle$. Then there exist a continuous map $s : X \to \Gamma_A$ such that $s(x) \in \Gamma(A \cap S(x))$ for all $x \in X$.

Proof. Since $(Y, D; \Gamma)$ is G-convex, there exists a continuous map $\phi_A : \Delta^A \to Y$ such that $\phi_A(\Delta^A) \subset \Gamma_A$ and $\phi_A(\Delta^A_J) \subset \Gamma_J$ for each $J \in \langle A \rangle$. Therefore, by Lemma 8.1, there exists $s : X \to \phi_A(\Delta^A) \subset \Gamma_A$ such that $s(x) \in \phi_A(\Delta^A_{A \cap S(x)}) \subset \Gamma(A \cap S(x))$ for all $x \in X$. □

Some history of Corollary 8.2 was given in [49].

The following extends a part of [49, Theorem 1]:

Theorem 8.3. *Let X be a normal space, $(Y, D; \{\phi_A\})$ a ϕ_A-space, $T : X \multimap Y$ a Φ-map with a companion map $S : X \multimap D$ such that*

$$X = \bigcup \{\operatorname{Int} S^{-}(a) \mid a \in A\}.$$

for some $A \in \langle S(X) \rangle$.

Then we have the following:

(i) *T has a continuous selection $f : X \to Y$ such that $f(X) \subset \phi_A(\Delta^A)$. More precisely, there exist two continuous maps $p : X \to \Delta^A$ and $\phi_A : \Delta^A \to \phi_A(\Delta^A) \subset Y$ such that $f = \phi_A \circ p$.*

(ii) *If $g : Y \to X$ is a continuous map, then there exists a $y_0 \in Y$ such that $y_0 \in T(g(y_0))$.*

(iii) *If $R : X \multimap Y$ is a map such that R^{-} has a continuous selection, then R and T have a coincidence point $x_0 \in X$; that is, $R(x_0) \cap T(x_0) \neq \emptyset$.*

Proof. (i) By Lemma 8.1, there exist a continuous map $f = s : X \to Y$ such that $f = \phi_A \circ p$ and

$$f(x) \in \phi_A(\Delta^A_{A \cap S(x)}) \subset \phi_A(\Delta^A) \quad \text{for all} \quad x \in X.$$

Since T is a Φ-map, $\phi_A(\Delta^A_{A\cap S(x)}) \subset T(x)$ and hence f is the required continuous selection of T.

(ii), (iii) just follow of the proofs of corresponding parts of [48, Theorem 3.2]. $\square$

Note that if $(Y, D; \{\phi_A)\}$ is a G-convex space, Theorem 8.2 reduces to [49, Theorem 3.2.]. Moreover, we have the following from Theorem 8.3(i):

Corollary 8.4. ([17, Theorem 2.1]) *Let Z be a normal (topological) space, $(X, A; \{\varphi_N\})$ a GFC-space and $G : Z \multimap X$. Assume that there be $F : Z \multimap A$ such that the following conditions hold*

(i) *for each $z \in Z$, each $N = \{a_0, \ldots, a_n\} \subset A$ and each $\{a_{i_0}, \ldots, a_{i_k}\} \subset N \cap F(z)$ one has $\varphi_N(\Delta_k) \subset G(z)$, where Δ_k is the simplex formed by $\{e_{i_0}, \ldots, e_{i_k}\}$;*

(ii) *$Z = \bigcup_{i=0}^m \operatorname{Int} F^-(a_i)$ for some $\{a_0, \ldots, a_m\} \subset A$.*

Then G has a continuous selection g of the form $g = \varphi \circ \psi$ for some continuous maps $\varphi : \Delta_m \to X$ and $\psi : Z \to \Delta_m$.

Theorem 8.5. *Let X be a normal space, $(Y, D; \{\phi_A\})$ a ϕ_A-space, $T : X \multimap Y$ a Φ-map with a companion map $S : X \multimap D$ such that*

(1) *the cover $\{\operatorname{Int} S^-(a) \mid a \in D\}$ of X has a locally finite refinement; and*

(2) *for each $A, B \in \langle D\rangle$ with $B \subset A$ implies $\phi_B = \phi_A|_{\Delta^B}$.*

Then T has a continuous selection $f : X \to Y$.

Proof. The proof of [49, Theorem 3.5] works with a slight modification of the last part as follows: Since S is a companion of the Φ-map T, we have $\phi_N(\Delta^N) \subset T(x)$. Since $f(x) \in \phi_N(\Delta^N)$, we have $f(x) \in T(x)$. This completes our proof. $\square$

Note that [49, Theorem 3.5] is a G-convex space version of Theorem 8.5. From Theorem 8.5, we immediately have the following:

Corollary 8.6. ([17, Theorem 2.3]) *Suppose Z be a paracompact space, $(X, A, \{\varphi_N\})$ be a GFC-space and $G : Z \multimap X$. Assume that there be $F : Z \to A$ such that*

(i) *for all $z \in Z$, all $N = \{a_0, \ldots, a_n\} \subset A$ and all $\{a_{i_0}, \ldots, a_{i_k}\} \subset N \cap F(z)$, one has $\varphi_N(\Delta_k) \subset G(z)$;*

(ii) $Z = \bigcup_{a \in A} \operatorname{Int} F^-(a)$.

Then G is locally continuously selectionable.

For the history of selection theorems in the KKM theory, see our recent work [49], where we showed that our previous selection theorems with a few generalized forms of them contain certain selection theorems in more than a dozen papers of other authors.

9. Weakly KKM maps

The following is given in 2008 [37]:

Definition 9.1. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(x) \cap G(A) \neq \emptyset \qquad \text{for all} \ \ A \in \langle D \rangle \ \ \text{and all} \ \ \ x \in \Gamma(A),$$

then G is called a *weakly KKM map* with respect to F.

Clearly each KKM map with respect to F is weakly KKM, and a weakly KKM map $G : D \multimap E$ with respect to the identity map 1_E is simply a KKM map.

In [37], particular forms of Definition 9.1 for vector spaces, G-convex spaces, and FC-spaces were given. Later, the following particular form appeared in 2011:

Definition 9.2. ([17, Definition 1.3]) Let (X, A, Φ) be a GFC-space and Y be a nonempty set. Let $T : X \multimap Y,\ F : A \multimap Y$ be two set-valued mappings. F is called a weak KKM mapping (KKM mapping, resp.) with respect to (wrt, for short) T , shortly, weak T-KKM mapping [T-KKM mapping, resp.] iff, for each $N = \{a_0, \ldots, a_n\} \subset A,\ \{a_{i_0}, \ldots, a_{i_k}\} \subset N$ and $x \in \varphi_N(\Delta^k),\ T(x) \cap \bigcup_{j=0}^k F(a_{i_j}) \neq \emptyset$ [$T(\varphi_N(\Delta^k)) \subset \bigcup_{j=0}^k F(a_{i_j})$, resp.].

In 2008, we obtained the following:

Theorem 9.1. ([37, Theorems 4.3 and 4.7]) *Let $(X, D; \Gamma)$ be a KKM space, Y a nonempty set, and $F : X \multimap Y$ and $G : D \multimap Y$ maps such that*

(1) *G is weakly KKM map with respect to F; and*

(2) *for each $a \in D$, the set $\{x \in X \mid F(x) \cap G(a) \neq \emptyset\}$ is closed.*

Then the following statements hold:

(i) *if X is compact, there exists an $x_0 \in X$ such that $F(x_0) \cap G(a) \neq \emptyset$ for each $a \in D$.*

(ii) *for each $A \in \langle D \rangle$ there exists an $x_0 \in \Gamma(A)$ such that $F(x_0) \cap G(a) \neq \emptyset$ for all $a \in A$.*

For the case (ii), closedness in (2) can be replaced by openness also.

In [17], its authors obtained the following weak KKM theorem and then applied it to minimax inequalities. Note that by Propositions 3.2 and 3.3, a GFC-space is a KKM space with $\Gamma_N = \varphi_N(\Delta^N)$. Hence, from Theorem 9.1, we have the following:

Corollary 9.2. ([17, Theorem 4.1]) *Let X be a Hausdorff space, $(X, A, \{\varphi_N\})$ a GFC-space, Y a nonempty set, $T : X \multimap Y$ and $H : A \multimap Y$. Assume that*

(i) *H be a weak T-KKM mapping;*

(ii) *for each $a \in A$, the set $\{x \in X : T(x) \cap H(a) \neq \emptyset\}$ be closed.*

Then the following statements hold:

(a) *if, additionally, X is compact, then there exists a point $\bar{x} \in X$ such that $T(\bar{x}) \cap H(a) \neq \emptyset$ for each $a \in A$;*

(b) *for each finite subset $N = \{a_0, \ldots, a_n\}$ of A, there exists a point $\bar{x} \in \varphi_N(\Delta^n)$ such that $T(\bar{x}) \cap H(a_i) \neq \emptyset$ for each $i \in \{0, 1, \ldots, n\}$.*

In Corollary 9.2, the Hausdorffness is redundant. In [17], its authors needed it because they are based on certain selection method.

As an example of applications of Theorem 9.1(1), we give the following:

Theorem 9.3. ([37, Theorem 4.11]) *Let $(X, D; \Gamma)$ be a compact KKM space, Y a topological space. Let $T : X \multimap Y$ be a u.s.c. map, $\psi : D \times Y \to \mathbb{R}$, $\varphi : X \times Y \to \mathbb{R}$ two functions and $\beta = \inf_{x \in X} \sup_{y \in T(x)} \varphi(x, y)$. Suppose that*

(1) *for each $a \in D$, $\psi(a, \cdot)$ is u.s.c. on Y;*

(2) *for any $\lambda < \beta$ and $y \in T(x)$, we have $\mathrm{co}_\Gamma \{a \in D \mid \psi(a, y) < \lambda\} \subset \{x \in X \mid \varphi(x, y) < \lambda\}$.*

(a) *Then the following holds:*

$$\inf_{x\in X}\sup_{y\in T(x)}\varphi(x,y)\le \sup_{x\in X}\inf_{a\in D}\sup_{y\in T(x)}\psi(a,y).$$

(b) *Further, if T is compact-valued, then there exists an $x_0\in X$ such that*

$$\inf_{x\in X}\sup_{y\in T(x)}\varphi(x,y)\le \inf_{a\in D}\sup_{y\in T(x_0)}\psi(a,y).$$

As in [37], particular forms or variants of Theorem 9.3 on G-convex spaces, FC-spaces, or GFC-spaces are possible. There is a variant in [17].

10. Comments on works on GFC-spaces

In a sequence of papers [7,12-17], Khanh et al. introduced GFC-spaces which were claimed to include properly both the incomparable G-convex spaces and FC-spaces without any justification nor any example. Their works do not reflect or they ignored recent studies on the KKM theory of abstract convex spaces or ϕ_A-spaces.

In the previous sections, we showed that actually their GFC-spaces are our ϕ_A-spaces given more early and can be made into G-convex spaces. We noted that they adopted artificial terminology, inadequate concept of subspaces or convex subset, KKM theorems with artificial coercivity condition, and an inadequate example. Their basic results including KKM theorems, continuous selection theorems and weakly KKM theorems are shown to be all consequences of known ones by the present author.

In the present section, abstracts and further comments on each of seven papers on GFC-spaces are given. Recently, basic results in the KKM theory are extended to our class of abstract convex spaces. Many results on GFC-spaces can be extended to this new class.

10.1 Khanh–Quan–Yao [12] NA 71 (2009)

Abstract: “We define a generalized KKM mapping, called T-KKM mapping, and the corresponding generalized KKM property, which include many counterparts existing in the literature. KKM-type theorems, coincidence theorems and geometric section theorems are established to generalize recent known results."

Comments: In this paper, the authors adopted inadequate definition of S-subsets or subspaces of GFC-spaces, artificial terminology, and inadequate coercivity conditions in their KKM theorems. Moreover, they ignored earlier concepts of abstract convex spaces and ϕ_A-spaces. Most of their results are modifications of earlier works in the literature and can be easily deduced or generalized by applying the material on abstract convex spaces in [39] in 2008.

10.2 Hai–Khanh–Quan [7] NA 71 (2009)

Abstract: "We establish a maximal element theorem, an intersection theorem and a coincidence-point theorem in product GFC-spaces. As examples of wide ranges of applications, we first deduce sufficient conditions for the solution existence of a mixed system of inclusions. Then using this we obtain existence results for systems of vector quasi-optimization problems and for multiobjective mathematical programs constrained by systems of inclusions. Our results are shown to improve and include recent ones in the literature."

Comments: This paper also adopts artificial terminology and S-subsets. Moreover, a modified form of our early class $\mathfrak{B}$ of better admissible multimaps and a trivial map in the class $\mathfrak{KC}$ that is not in their $\mathfrak{B}$ is given. Furthermore, the adoption of S-subsets lets all the results be complicated.

10.3 Khanh–Quan [13] Optim. 59 (2010)

Abstract: "We propose a definition of GFC-spaces to encompass G-convex spaces, FC-spaces and many recent existing spaces with generalized convexity structures. Intersection, coincidence and maximal-element theorems are then established under relaxed assumptions in GFC-spaces. These results contain, as true particular cases, a number of counterparts which were recently developed in the literature."

Comments: In this paper, its authors also adopted artificial terminology, inadequate S-sets, and inconvenient coercivity conditions. The so-called T-KKM map is originated by ourselves not by Chang and Yen. Since our definition of the class $\mathfrak{B}$ of better admissible multimaps appeared first in 1997, it is extended to G-convex spaces and absolute convex spaces. A number of authors imitated the concept and, here, Khanh and Quan defined a particular type of the original $\mathfrak{B}$. All of the results are generalizations of known results for G-convex spaces. Note that the basic Theorem 2.2 is a very particular form of our KKM theorem C. Finally, it is routine to reformulate the KKM theorem to coincidence theorems, maximal element theorems, and many others.

10.4 Khanh–Quan [14] NA 72 (2010)

Abstract: "We establish general theorems on maximal elements, coincidence points and nonempty intersections for set-valued mappings on GFC-spaces and show their equivalence. Applying them we derive equivalent forms of alternative theorems. As applications, we develop in detail general types of minimax theorems. The results obtained improve or include as special cases several recent ones in the literature."

Comments: This paper begins with the routine introduction of artificial terminology. The authors took the wrong way to generalize FC-spaces not G-convex spaces. In fact, in order to generalize FC-subspaces, the author defined S-subsets, S_{GFC}-subsets, "pre" GFC-spaces, "full" GFC-spaces, and GFC-hull w.r.t. S of a GFC-space (X, Y, Φ) without giving any practical examples. Note that their GFC-hull is a subset of Y. Moreover, the authors claimed that a GFC-hull w.r.t. S of a subset extends the notion of an FC-hull of a set in an FC-space and a G-convex hull of a set in a G-convex space. Recall that a Γ-convex hull of a set $D' \subset D$ in a G-convex space (or abstract convex space) $(E, D; \Gamma)$ is a subset of E not of D, and hence their GFC-hull can not extends Γ-convex hulls. The authors seem to confuse with the particular case $E = D$ as in FC-spaces. This is why we said that they took wrong way, and makes all of the arguments complicated and difficult to understand. If the authors followed simply Definition 2.1, then all of the above artificial definitions would not necessary.

In [14], Theorem 3.3 (Nonempty intersection theorem) is a simple consequence of our Theorem C, and Theorems 3.1 (Maximal elements) and Theorem 3.2 (Coincidence points) are routine equivalents of Theorem 3.1. Furthermore, all other results can be stated for abstract convex spaces by eliminating artificial concepts related to S-sets.

10.5 Khanh–Quan [15] JOTA 146 (2010)

Abstract: "Applying generalized KKM-type theorems established in our previous paper [7], we prove the existence of solutions to a general variational inclusion problem, which contains most of the existing results of this type. As applications, we obtain minimax theorems in various settings and saddle-point theorems in particular. Examples are given to explain advantages of our results."

Comments: After repeating artificial terminology with "compactly", the authors defined GFC-spaces and S-subsets with the routine claim that their definitions encompasses both of FC-spaces and G-convex spaces. They did not quote any of abstract convex spaces or ϕ_A-spaces. Theorem 3.1 is the KKM theorem in [12], which is a consequence of our Theorem C as mentioned already. Then they defined artificial $(k, T, r\alpha)$-quasiconcavity related to four multimaps $H,\ T,\ h,\ k$ and $\alpha \in \{\alpha_1, \alpha_2\},\ r \in \{r_1, r_2, r_3, r_4\}$, where it is unclear what α and r are.

Section 2 devotes to give some examples of such $(k, T, r\alpha)$-quasiconcave maps. In Section 3, besides $(X, Y, \Phi),\ Z,\ \Omega,\ D,\ H,\ T,\ h,\ k,\ h_{r\alpha}$, let multimaps $S_1,\ S_2,\ f,\ g$ be given additionally, and the authors claimed to prove the existence of a generalized variational inclusion problem and some applications. The possibility of existence of a problem related so many things is quite doubtful. Moreover, the proofs in Section 3 are beyond of our understanding.

10.6 Khanh–Quan [16] JOTA 148 (2011)

Abstract: "We propose the definition of T-KKM points and consider generic stability of T-KKM mappings and essential components of sets of T-KKM points. As applications, using a unified approach, we derive from these results the existence of essential components of solution sets to various optimization-related problems. We do this in two steps. First, we deduce the corresponding results for variational inclusions, which are new. Then we obtain, as consequences, the existence of essential components of solutions to other problems, which are new or include recent ones in the literature."

Comments: This paper does not reflect recent studies on the KKM theory. The authors claim that their GFC-spaces include properly both the incomparable G-convex spaces and FC-spaces without any justification nor any example. Actually their GFC-spaces are ϕ_A-spaces due to Park more early in 2007-2009. The better admissible class of multimaps due to Park is cited to other author (Ding 2005) who used to imitate the concept sometimes incorrectly. More early in 2006, basic results in the KKM theory are extended to the class of abstract convex spaces by Park. Many results in this paper can be extended to this new class.

10.7 Khanh–Long–Quan [17] JOTA 151 (2011)

Abstract: "We prove theorems on continuous selections, collectively fixed points, collectively coincidence points, weak Knaster-Kuratowski-Mazurkiewicz mappings and provide their applications in various optimization-related problems. Each of our theorems is demonstrated by using its preceding assertions. The results contain and improve a number of existing ones in the recent literature. They are shown to be also more advantageous in applications in optimization."

Comments: The authors repeated to define S-subsets of GFC-spaces and gave an inappropriate example of their spaces; see Sections 6 and 7 in the present paper. Then continuous selection theorems were given; see Section 8 of the present paper. Some solutions of the problems (CPP), (SVR), (MPIC), and (SVOP) are given. Finally, a weak KKM theorem and minimax inequalities are given; see Section 9 in the present paper.

References

[1] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano, and J.-V. Llinares, Abstract convexity and fixed points, J. Math. Anal. Appl. 222 (1998), 138–150.

[2] W. Briec and C. Horvath, Nash points, Ky Fan inequality and equilibria of abstract economies in Max-Plus and $\mathbb{B}$-convexity, J. Math. Anal. Appl. 341(1) (2008), 188–100.

[3] G.L. Cain Jr. and L. González, The Knaster-Kuratowski-Mazurkiewicz theorem and abstract convexities, J. Math. Anal. Appl. 338 (2008), 563–571.

[4] X.P. Ding, Maximal element theorems in product FC-spaces and generalized games, J. Math. Anal. Appl. 305 (2005), 29–42.

[5] X.P. Ding, Generalized KKM type theorems in FC-spaces with applications (I), J. Glob. Optim. 36 (2006), 581–596.

[6] L. Gonzáles, S. Kilmer, and J. Rebaza, From a KKM theorem to Nash equilibria in L-spaces, Topology Appl. 255 (2007), 165–170.

[7] N.X. Hai, P.Q. Khanh, and N.H. Quan, Some existence theorems in nonlinear analysis for mappings on GFC-spaces and applications, Nonlinear Anal. 71 (2009), 6170–6181.

[8] C.D. Horvath, Contractibility and generalized convexity, J. Math. Anal. Appl. 156 (1991), 341–357.

[9] C.D. Horvath, Extension and selection theorems in topological spaces with a generalized convexity structure, Ann. Fac. Sci. Toulouse 2 (1993), 253–269.

[10] C.D. Horvath, Topological convexities, selections and fixed points, Topology Appl. 155 (2008), 830–850.

[11] J.-C. Hou, Characterization of the existence of a pure-strategy Nash equilibrium, Appl. Math. Letters 22 (2009), 689–692.

[12] P.Q. Khanh, N.H. Quan, and J.C. Yao, Generalized KKM type theorems in GFC-spaces and applications, Nonlinear Anal. 71 (2009), 1227–1234.

[13] P.Q. Khanh and N.H. Quan, Intersection theorems, coincidence theorems and maximal-element theorems in GFC-spaces, Optimization 59 (2010), 115–124.

[14] P.Q. Khanh and N.H. Quan, General existence theorems, alternative theorems and applications to minimax problems, Nonlinear Anal. 72 (2010), 2706–2715.

[15] P.Q. Khanh and N.H. Quan, Existence results for general inclusions using generalized KKM theorems with applications to minimax problems, J. Optim. Theory Appl. 146 (2010), 640–653.

[16] P.Q. Khanh and N.H. Quan, Generic stability and essential components of generalized KKM points and applications, J. Optim. Theory Appl. 148 (2011), 488–504.

[17] P.Q. Khanh, V.S.T. Long, and N.H. Quan, Continuous selections, collectively fixed points and weak Knaster-Kuratowski-Mazurkiewicz mappings in optimization, J. Optim. Theory Appl. 151 (2011), 552–572.

[18] W. Kulpa and A. Szymanski, Applications of general infimum principles to fixed-point theory and game theory, Set-valued Anal. 16 (2008), 375–398.

[19] W. Kulpa and A. Szymanski, it Some remarks on Park's abstract convex spaces, Preprint.

[20] D. T. Luc, E. Sarabi, and A. Soubeyran, Existence of solutions in variational relation problems without convexity, J. Math. Anal. Appl. 364 (2010), 544–555.

[21] Q. Luo, KKM and Nash equilibria type theorems in topological ordered spaces, J. Math. Anal. Appl. 264 (2001), 262–269.

[22] S. Park, Some coincidence theorems on acyclic multifunctions and applications to KKM theory, Fixed Point Theory and Applications (K.-K. Tan, ed.), 248–277, World Scientific Publ., River Edge, NJ, 1992.

[23] S. Park, Ninety years of the Brouwer fixed point theorem, Vietnam J. Math. 27 (1999), 193–232.

[24] S. Park, Continuous selection theorems in generalized convex spaces, Numer. Funct. Anal. Optimiz. 20 (1999), 567–583.

[25] S. Park, Elements of the KKM theory for generalized convex spaces, Korean J. Comput. & Appl. Math. 7 (2000), 1–28.

[26] S. Park, Remarks on topologies of generalized convex spaces, Nonlinear Func. Anal. Appl. 5 (2000), 67–79.

[27] S. Park, On generalizations of the KKM principle on abstract convex spaces, Nonlinear Anal. Forum 11 (2006), 67–77.

[28] S. Park, Various subclasses of abstract convex spaces for the KKM theory, Proc. Nat. Inst. Math. Sci. 2(4) (2007), 35–47.

[29] S. Park, Comments on some abstract convex spaces and the KKM maps, Nonlinear Anal. Forum 12(2) (2007), 125–139.

[30] S. Park, Comments on recent studies on abstract convex spaces, Nonlinear Anal. Forum 13(1) (2008), 1–17.

[31] S. Park, Generalized convex spaces, L-spaces, and FC-spaces, J. Glob. Optim. 45(2) (2009), 203–210.

[32] S. Park, Elements of the KKM theory on abstract convex spaces, J. Korean Math. Soc. 45(1) (2008), 1–27.

[33] S. Park, Equilibrium existence theorems in KKM spaces, Nonlinear Analysis 69 (2008), 4352–4364.

[34] S. Park, Comments on the KKM theory on ϕ_A-spaces, PanAmerican Math. J. 18(2) (2008), 61–71.

[35] S. Park, Remarks on KKM maps and fixed point theorems in generalized convex spaces, CUBO, Math. J. 10(4), 1–13.

[36] S. Park, Remarks on fixed points, maximal elements, and equilibria of economies in abstract convex spaces, Taiwan. J. Math. 12(6) (2008), 1365–1383.

[37] S. Park, Remarks on weakly KKM maps in abstract convex spaces, Inter. J. Math. Math. Sci. vol. 2008, Article ID 423596, 10pp.

[38] S. Park, Applications of the KKM principle on abstract convex minimal spaces, Nonlinear Funct. Anal. Appl. 13(2) (2008), 173–191.

[39] S. Park, New foundations of the KKM theory, J. Nonlinear Convex Anal. 9(3) (2008), 331–350.

[40] S. Park, Remarks on the partial KKM principle, Nonlinear Anal. Forum 14 (2009), 51–62.

[41] S. Park, General KKM theorems for abstract convex spaces, J. Inform. Math. Sci. 1(1) (2009), 1–13.

[42] S. Park, The KKM principle in abstract convex spaces — Equivalent formulations and applications, Nonlinear Analysis 73 (2010), 1028–1042.

[43] S. Park, Generalizations of the Nash equilibrium theorem in the KKM theory, Takahashi Legacy, Fixed Point Theory and Appl., vol. 2010, Article ID 234706, 23pp, doi:10.1155/2010/ 234706.

[44] S. Park, The rise and decline of generalized convex spaces, Nonlinear Anal. Forum 15 (2010), 1–12.

[45] S. Park, Several episodes in recent studies on the KKM theory, Nonlinear Anal. Forum 15 (2010), 13–26.

[46] S. Park, Remarks on some basic concepts in the KKM theory, Nonlinear Anal. 74 (2011), 2439–2447.

[47] S. Park, New generalizations of basic theorems in the KKM theory, Nonlinear Anal. 74 (2011), 3000–3010.

[48] S. Park, Remarks on certain coercivity in general KKM theorems, Nonlinear Anal. Forum 16 (2011), 1–10.

[49] S. Park, Continuous selection theorems in generalized convex spaces: Revisited, Nonlinear Anal. Forum 16 (2011), 21–33.

[50] S. Park, Abstract convex spaces, KKM spaces and ϕ_A-spaces, Nonlinear Anal. Forum 17 (2012), 1–10.

[51] S. Park and W. Lee, A unified approach to generalized KKM maps in generalized convex spaces, J. Nonlinear Convex Anal. 2 (2001), 157–166.

[52] M.C. Sánchez and J.-V. Llinares, and B. Subiza, A KKM-result and an application for binary and non-binary choice function, Econom. Th. 21 (2003), 185–193.