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COMMON FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIVE MAPPINGS IN F_J-CONE METRIC SPACES OVER BANACH ALGEBRAS

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Abstract. In this paper, we establish common fixed point theorems for generalized contractive mappings in an F_J–cone metric space over a Banach algebra. The results extend several existing fixed point principles, and illustrative examples are presented to support the theoretical findings.

Keywords: Banach algebra; F_J-cone metric spaces; common fixed point.

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1. Introduction

The study of generalized metric spaces has been a central topic in fixed point theory. The concept of b-metric spaces was first introduced by Bakhtin [1] as a generalization of classical metric spaces, allowing the extension of Banach's contraction principle to broader settings. In 1994, Matthews [22] introduced partial metric spaces, where the self-distance of a point need not vanish. This framework found applications in program verification and data flow analysis, showing that convergent sequences in partial metric spaces need not have unique limits.

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Building on these ideas, Huang and Zhang [16] introduced *cone metric spaces*, in which the real numbers are replaced by elements of an ordered Banach space. This setting enabled the natural definition of Cauchy sequences and convergence, leading to numerous fixed point results [2, 8, 23]. However, subsequent studies [3, 18] indicated that many fixed point results in cone metric spaces could be deduced from equivalent classical metric results, limiting the novelty of these spaces.

To overcome these limitations, Liu and Xu [20] introduced *cone metric spaces over Banach algebras* and established fixed point theorems under weaker contractive conditions involving spectral radius estimates. Following this approach, several authors [4, 5, 10] further generalized cone metric spaces over Banach algebras, obtaining fixed point results for mappings that could not be reduced to standard metric spaces.

Fernandez et al. [12, 25] introduced F-cone metric spaces as a generalization of N_p-cone and N_b-cone metric spaces over Banach algebras. These spaces provide a framework for studying fixed point results for more general classes of contractive mappings. However, the existing F-cone metric spaces do not fully capture certain structural constraints that arise in applications requiring strong generalized triangle inequalities and symmetric balance properties.

Motivated by these observations, in this paper we introduce the notion of an *F_J-cone metric space* over a Banach algebra. This space generalizes F-cone metric spaces by incorporating *structural coefficients* that control strong generalized triangle inequalities and symmetric balance conditions. Using this framework, we establish several new common fixed point theorems for generalized contractive mappings. Furthermore, we provide illustrative examples of F_J-cone metric spaces that are not F-cone metric spaces, highlighting the novelty and applicability of our approach.

2. Preliminaries

In this section, we recall some basic definitions, results, and consequences relevant to our findings.

Definition 2.1 (Algebra and Banach algebra, [24])**.** A linear space A over a field K ($\mathbb{R}$ or $\mathbb{C}$) is called an *algebra* if it is closed under multiplication (i.e., for all $x, y \in A$, $xy \in A$) and satisfies:

(1) $(xy)z = x(yz), \forall x, y, z \in A$,

(2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz, \forall x, y, z \in A$,

(3) $\alpha(xy) = (\alpha x)y = x(\alpha y), \forall x \in A, \forall \alpha \in K$.

A Banach space A over K is called a *Banach algebra* if A is an algebra and $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. A Banach algebra is *unital* if it has a unity element e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is invertible if there exists $y \in A$ such that $xy = yx = e$, denoted $y = x^{-1}$.

Proposition 2.2 (Spectral radius, [24])**.** Let A be a Banach algebra with unity e. The *spectral radius* of $x \in A$ is

$$\rho(x) = \sup_{\lambda \in \sigma(x)} |\lambda| = \lim_{n \to \infty} \|x^n\|^{1/n},$$

where $\sigma(x)$ is the spectrum of x. If $\rho(x) < 1$, then $e - x$ is invertible and

$$(e-x)^{-1} = e + \sum_{i=1}^{\infty} x^i.$$

Remark 2.3 ([24]). Let A be a Banach algebra with unity e. For every $x \in A$, the spectral radius $\rho(x)$ satisfies

$$\rho(x) \leq \|x\|.$$

Remark 2.4 ([26]). In Proposition 2.2, if we replace the condition $0 < \rho(x) < 1$ by $\|x\| \leq 1$, then the same conclusion holds.

Remark 2.5 ([26]). If $\rho(x) < 1$, then

$$\|x^n\| \to 0 \quad \text{as } n \to \infty.$$

Definition 2.6 (Cone, [4])**.** Let A be a unital Banach algebra. A subset $P \subset A$ is a *cone* if:

(1) P is non-empty, closed, and $\theta, e \in P$,

(2) If $x, y \in P$ and $\alpha, \beta \geq 0$, then $\alpha x + \beta y \in P$,

(3) $x, y \in P$ implies $xy \in P$,

(4) If $x, -x \in P$ for some $x \in A$, then $x = \theta$.

A cone P is *solid* if $\mathrm{int}(P) \neq \emptyset$. It is *normal* if there exists $R > 0$ such that $\theta \preceq x \preceq y \implies \|x\| \leq R\|y\|$, where $\preceq$ denotes the partial ordering induced by P.

Lemma 2.7 ([23]). *Let E be a real Banach space with a solid cone P. If $\theta \preceq a \preceq c$ for all $c \succeq \theta$, then $a = \theta$.*

Definition 2.8 (c-sequence, [17]). Let P be a solid cone in a Banach space E. A sequence $\{u_n\} \subset P$ is called a *c-sequence* if for each $\theta \preceq c$, there exists $N \in \mathbb{N}$ such that $u_n \preceq c$ for all $n \geq N$.

Lemma 2.9 ([14]). *If $\{u_n\} \subset P$ satisfies $\|u_n\| \to 0$ as $n \to \infty$, then $\{u_n\}$ is a c-sequence.*

Lemma 2.10 ([26]). *Let A be a Banach algebra with unity e. If $x, y \in A$ commute, then*

(1) $\rho(xy) \leq \rho(x)\rho(y)$,
(2) $\rho(x+y) \leq \rho(x) + \rho(y)$,
(3) $|\rho(x) - \rho(y)| \leq \rho(x-y)$.

Lemma 2.11 ([14]). *Let E be a real Banach space and let P be a solid cone of E. If $a, b, c \in E$ satisfy $a \preceq b \preceq c$, then $a \preceq c$.*

Lemma 2.12 ([26]). *Let P be a solid cone of a Banach algebra A. Suppose that $k \in P$ is an arbitrary element and $\{u_n\} \subset P$ is a c–sequence. Then $\{ku_n\}$ is also a c–sequence.*

Lemma 2.13 ([14]). *Let A be a Banach algebra with unity e and let $k \in A$. If λ is a complex constant such that $\rho(k) < |\lambda|$, then*

$$\rho\left((\lambda e - k)^{-1}\right) \leq \frac{1}{|\lambda| - \rho(k)}.$$

Lemma 2.14 ([26]). *Let A be a Banach space and let P be a solid cone of A. If $a, k, l \in P$ satisfy $l \preceq k$, $a \preceq la$, and $\rho(k) < 1$, then $a = \theta$.*

Lemma 2.15 ([14]). *Let A be a Banach algebra with unity e and let $\{x_n\} \subset A$. Suppose that $x_n \to x \in A$ and that $x_n x = x x_n$ for all $n \in \mathbb{N}$. Then*

$$\rho(x_n) \to \rho(x) \quad \text{as } n \to \infty.$$

Lemma 2.16 ([15]). *Let A be a Banach algebra with unity e and let P be a solid cone of A. Let $h \in A$ and define $u_n = h^n$ for all $n \in \mathbb{N}$. If $\rho(h) < 1$, then $\{u_n\}$ is a c–sequence.*

Lemma 2.17 ([18]). *Let K be a cone in a Banach space $(E, \|\cdot\|)$. The following conditions are equivalent:*

(1) K is normal;

(2) for arbitrary sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in E,

$$x_n \preceq y_n \preceq z_n \quad \text{for all } n \in \mathbb{N},$$

and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x$ imply $\lim_{n\to\infty} y_n = x$;

(3) there exists a norm $\|\cdot\|_1$ on E, equivalent to $\|\cdot\|$, such that the cone K is monotone with respect to $\|\cdot\|_1$.

Throughout the sequel, we always assume that A is a Banach algebra with unity e, P is a solid cone of A, and $\preceq$ denotes the partial order induced by the cone P.

Definition 2.18 (Cone metric space, [16],[20])**.** Let X be a nonempty set. A mapping $d : X \times X \to A$ is a *cone metric* if:

(1) $d(x,y) \succeq \theta$ and $d(x,y) = \theta \iff x = y$,

(2) $d(x,y) = d(y,x)$,

(3) $d(x,z) \preceq d(x,y) + d(y,z)$.

Then (X,d) is called a *cone metric space* over A.

Definition 2.19 (b-metric, [1])**.** Let X be a nonempty set and $s \geq 1$. A function $d : X \times X \to \mathbb{R}^+$ is a *b-metric* if:

(1) $d(x,y) = 0 \iff x = y$,

(2) $d(x,y) = d(y,x)$,

(3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

Definition 2.20 (Partial metric, [22])**.** A function $p : X \times X \to \mathbb{R}^+$ is a *partial metric* if:

(1) $p(x,x) = p(x,y) = p(y,y) \iff x = y$,

(2) $p(x,x) \leq p(x,y)$,

(3) $p(x,y) = p(y,x)$,

(4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Definition 2.21 (N-cone metric, [21])**.** A mapping $N : X \times X \times X \to A$ is an *N-cone metric* if:

(1) $N(x,x,x) \succeq \theta$,

(2) $N(x,y,z) = \theta \iff x = y = z$,

(3) $N(x,y,z) \preceq N(x,x,a) + N(y,y,a) + N(z,z,a)$ for all $a \in X$.

Definition 2.22 (N_b-cone metric, [10])**.** A mapping $N_b : X \times X \times X \to A$ is an *Nb-cone metric* if:

(1) $N_b(x,y,z) \succeq \theta$,

(2) $N_b(x,y,z) = \theta \iff x = y = z$,

(3) $N_b(x,y,z) \preceq s[N_b(x,x,a) + N_b(y,y,a) + N_b(z,z,a)]$, $s \geq 1$.

Definition 2.23 (N_p-cone metric, [11])**.** A mapping $N_p : X \times X \times X \to A$ is an *Np-cone metric* if:

(1) $N_p(x,x,x) = N_p(y,y,y) = N_p(z,z,z) = N_p(x,y,z) \iff x = y = z$,

(2) $\theta \preceq N_p(x,x,x) \preceq N_p(x,x,y) \preceq N_p(x,y,z)$ for all distinct x,y,z,

(3) $N_p(x,y,z) \preceq N_p(x,x,a) + N_p(y,y,a) + N_p(z,z,a) - N_p(a,a,a)$ for all $a \in X$.

Definition 2.24 (F-cone metric space)**.** [25] Let X be a nonempty set. A function $F : X \times X \times X \to A$ is called an *F-cone metric* on X if it satisfies the following conditions:

(1) $F(x,x,x) = F(y,y,y) = F(z,z,z) = F(x,y,z)$ iff $x = y = z$,

(2) $\theta \preceq F(x,x,x) \preceq F(x,x,y) \preceq F(x,y,z)$ for all $x,y,z \in X$ with $x \neq y \neq z$,

(3) $F(x,y,z) \preceq s\big[F(x,x,a) + F(y,y,a) + F(z,z,a) - F(a,a,a)\big]$ for some $s \geq 1$ and for all $x,y,z,a \in X$.

Then the triplet (X,A,F) is called an *F-cone metric space* over the Banach algebra A, and the number $s \geq 1$ is called the *coefficient* of (X,A,F).

Definition 2.25. Let (X,F) be an F–cone metric space over the Banach algebra A. A sequence $\{x_n\} \subset X$ is said to be *convergent* and converges to a point $x \in X$ if for each $c \succ \theta$ there exists a natural number N such that

$$F(x_n,x_n,x) \prec c \quad \text{whenever } n \geq N.$$

In this case, we write $\lim_{n\to\infty} x_n = x$.

Definition 2.26. Let (X,F) be an F–cone metric space over the Banach algebra A. A sequence $\{x_n\} \subset X$ is said to be a *θ–Cauchy sequence* if for each $c \succ \theta$ there exists a natural number N_0 such that

$$F(x_n,x_n,x_m) \prec c \quad \text{whenever } n,m \geq N_0.$$

Definition 2.27. Let (X,F) be an F–cone metric space over the Banach algebra A. Then X is called *θ–complete* if every θ–Cauchy sequence $\{x_n\} \subset X$ is convergent and converges to some $x \in X$ such that

$$F(x,x,x) = \theta.$$

Definition 2.28. Let (X,F) and (X',F') be two F–cone metric spaces over the same Banach algebra A. A function $f : X \to X'$ is said to be *continuous* if for any sequence $\{x_n\} \subset X$ converging to $x \in X$, the sequence $\{f(x_n)\} \subset X'$ converges to $f(x)$.

Now, we state the following properties of F–cone metric spaces introduced by Fernandez *et al.* (see [4]).

Remark 2.29. In an F–cone metric space (X,F) over the Banach algebra A,

$$F(x,y,z) = \theta \implies x = y = z \quad \text{for all } x,y,z \in X,$$

but the converse is not true.

Lemma 2.30. *Let (X,F) be an F–cone metric space over the Banach algebra A. Then*

$$F(x,x,y) \succ \theta \quad \textit{whenever } x \neq y.$$

Proposition 2.31. If (X,F) is an F–cone metric space over the Banach algebra A, then

$$F(x,x,y) = F(y,y,x) \quad \text{for all } x,y \in X.$$

Definition 2.32. Let (X,F) be an F–cone metric space over the Banach algebra A. For $x \in X$ and $c \succ \theta$, the *F–ball* with center x and radius c is defined by

$$B_F(x,c) = \{y \in X : F(x,x,y) \prec F(x,x,x) + c\}.$$

Definition 2.33. Let (X,F) be an F–cone metric space over the Banach algebra A with coefficient $s \geq 1$. Let

$$B_F(x,c) = \{y \in X : F(x,x,y) \prec F(x,x,x) + c\},$$

for all $x \in X$ and $c \succ \theta$. Define

$$\mathscr{B} = \{B_F(x,c) : x \in X,\ c \succ \theta\}.$$

Then $\mathscr{B}$ is a subbase for some topology τ on X.

Theorem 2.34. *Let (X,F) be an F–cone metric space over the Banach algebra A and let P be a solid cone of A. Then (X,F) is a Hausdorff space with respect to the topology τ.*

3. Main Results

In this section, we now define the following definitions. In the following definition namely F_J–cone metric space we always take A as an ordered Banach algebra.

Definition 3.1 (F_J–cone metric space)**.** Let X be a nonempty set and let A be a Banach algebra endowed with a cone $P \subset A$. A mapping

$$F_J : X \times X \times X \longrightarrow A$$

is called an *F_J–cone metric* on X if there exist real constants $s \geq 1$, $0 < \alpha < 1$, and $t > 1$ such that for all $x, y, z, a \in X$, the following conditions are satisfied:

(1) **Non-negativity and identity of indiscernibles:**

$$F_J(x,y,z) \succeq \theta,$$

$$F_J(x,x,x) = F_J(y,y,y) = F_J(z,z,z) = F_J(x,y,z) \iff x = y = z.$$

(2) **Monotonicity:**

$$\theta \preceq F_J(x,x,x) \preceq F_J(x,x,y) \preceq F_J(x,y,z), \quad \text{for } x \neq y \neq z.$$

(3) **Strong generalized triangle inequality:**

$$F_J(x,y,z) \preceq s\Big[F_J(x,x,a) + F_J(y,y,a) + F_J(z,z,a) - \alpha F_J(a,a,a)\Big].$$

(4) **Symmetric balance control:**

$$F_J(x,y,z) \preceq t\Big[F_J(a,x,x)+F_J(a,y,y)+F_J(a,z,z)\Big].$$

Then the triplet (X,A,F_J) is called an F_J*–cone metric space* over the Banach algebra A. The constants s, α, and t are referred to as the *structural coefficients* of the space.

Remark. The above definition is a natural generalization of the notion of an F–cone metric space, and hence we refer to it as an F_J–cone metric space.

Now, we establish some common fixed point results for a pair of mappings in an F_J–cone metric space over an ordered Banach algebra A.

Theorem 3.2. *Let (X,A,F_J) be a ϑ–complete F_J–cone metric space and let $\mathscr{P}$ be a solid cone of the Banach algebra A. Let $f : X \to X$ be a continuous self–mapping. Assume that there exists an element $\lambda \in \mathscr{P}$ such that*

$$\phi(\lambda) < \frac{1}{\mu},$$

where μ is the coefficient of the F_J–cone metric space X. Let $g : X \to X$ be another self–mapping commuting with f (i.e., $fg = gf$) and satisfying

$$g(X) \subseteq f(X).$$

If the mappings f and g satisfy the contractive condition

$$F_J(ga,ga,gb) \preceq \lambda\, F_J(fa,fa,fb), \quad \textit{for all } a,b \in X,$$

then f and g have a unique common fixed point in X.

Proof. Let $a_0 \in X$. Since $g(X) \subseteq f(X)$, there exists $a_1 \in X$ such that

$$f(a_1) = g(a_0).$$

Proceeding inductively, we can construct a sequence $\{a_n\}$ in X such that

$$f(a_n) = g(a_{n-1}), \quad \forall n \in \mathbb{N}.$$

Define

$$b_{n-1} = g(a_{n-1}) = f(a_n), \quad \forall n \geq 1. \tag{3.1}$$

Using the contractive condition, we obtain

$$\begin{aligned} F_J(b_n,b_n,b_{n+1}) &= F_J(ga_n,ga_n,ga_{n+1}) \\ &\preceq \lambda F_J(fa_n,fa_n,fa_{n+1}) \\ &= \lambda F_J(b_{n-1},b_{n-1},b_n), \quad \forall n\in\mathbb{N}. \end{aligned} \tag{3.2}$$

Let $1\leq n<m$. Applying the strong generalized triangle inequality with $a=b_{n+1}$, we have

$$\begin{aligned} F_J(b_n,b_n,b_m) \preceq \mu\Big[&F_J(b_n,b_n,b_{n+1})+F_J(b_n,b_n,b_{n+1})+F_J(b_m,b_m,b_{n+1}) \\ &-\alpha F_J(b_{n+1},b_{n+1},b_{n+1})\Big]. \end{aligned} \tag{3.3}$$

Since $F_J(x,x,x)\succeq\vartheta$ for all $x\in X$ and $\alpha\in(0,1)$, we have

$$-\alpha F_J(b_{n+1},b_{n+1},b_{n+1})\preceq\vartheta.$$

Therefore, inequality (3.3) yields

$$F_J(b_n,b_n,b_m)\preceq\mu\Big[2F_J(b_n,b_n,b_{n+1})+F_J(b_m,b_m,b_{n+1})\Big]. \tag{3.4}$$

Applying the same inequality iteratively, we obtain

$$\begin{aligned} F_J(b_n,b_n,b_m)\preceq 2\Big[&\mu F_J(b_n,b_n,b_{n+1})+\mu^2F_J(b_{n+1},b_{n+1},b_{n+2}) \\ &+\cdots+\mu^{m-n}F_J(b_{m-1},b_{m-1},b_m)\Big]. \end{aligned} \tag{3.5}$$

From (3.2), by induction,

$$F_J(b_n,b_n,b_{n+1})\preceq\lambda^nF_J(b_0,b_0,b_1).$$

Substituting into (3.5), we get

$$\begin{aligned} F_J(b_n,b_n,b_m) &\preceq 2\Big[(\mu\lambda)^n+(\mu\lambda)^{n+1}+\cdots+(\mu\lambda)^{m-n}\Big]F_J(b_0,b_0,b_1) \\ &\preceq 2(\mu\lambda)^n(e-\mu\lambda)^{-1}F_J(b_0,b_0,b_1). \end{aligned} \tag{3.6}$$

Since $\phi(\lambda)<\frac{1}{\mu}$, it follows that $\phi(\mu\lambda)<1$, and hence $\|\mu\lambda\|^n\to 0$ as $n\to\infty$. By Lemma 2.9, $\{F_J(b_n,b_n,b_m)\}$ is a c–sequence. Therefore, $\{b_n\}$ is a ϑ–Cauchy sequence in X.

As (X,A,F_J) is ϑ–complete, there exists $\alpha\in X$ such that

$$b_n\to\alpha.$$

From (3.1), we have

$$f(a_{n+1}) \to \alpha \quad \text{and} \quad g(a_n) \to \alpha.$$

Since f is continuous and f and g commute, we obtain

$$f\alpha = g\alpha,$$

so α is a coincidence point of f and g.

Now,

$$F_J(g\alpha, g\alpha, g^2\alpha) \preceq \lambda F_J(f\alpha, f\alpha, fg\alpha) = \lambda F_J(g\alpha, g\alpha, g^2\alpha).$$

Since $\phi(\lambda) < 1$, it follows that

$$F_J(g\alpha, g\alpha, g^2\alpha) = \vartheta,$$

which implies $g^2\alpha = g\alpha$. Hence,

$$f(g\alpha) = g(f\alpha) = g(g\alpha) = g\alpha,$$

and thus $g\alpha$ is a common fixed point of f and g.

For uniqueness, let $p, q \in X$ be two common fixed points. Then

$$F_J(p, p, q) = F_J(gp, gp, gq) \preceq \lambda F_J(fp, fp, fq) = \lambda F_J(p, p, q).$$

Since $\phi(\lambda) < 1$, we conclude that $F_J(p, p, q) = \vartheta$, and hence $p = q$.

Therefore, f and g have a unique common fixed point in X. □

Theorem 3.3. *Let (X, A, F_J) be a ϑ–complete F_J–cone metric space and let $\mathscr{P}$ be a solid cone of the Banach algebra A. Let $f : X \to X$ be a continuous self–mapping. Assume that there exists an element $\lambda \in \mathscr{P}$ such that*

$$\phi(\lambda) < \frac{1}{\mu + 1},$$

where μ is the coefficient of the F_J–cone metric space X. Let $g : X \to X$ be another self–mapping commuting with f and satisfying

$$g(X) \subseteq f(X).$$

If the mappings f and g satisfy

$$F_J(ga, ga, gb) \preceq \lambda \Big[F_J(fa, fa, ga) + F_J(fb, fb, gb)\Big], \quad \forall a, b \in X,$$

then f and g have a unique common fixed point in X.

Proof. Let $a_0 \in X$. Since $g(X) \subseteq f(X)$, there exists $a_1 \in X$ such that $f(a_1) = g(a_0)$. Inductively, we construct a sequence $\{a_n\}$ in X satisfying

$$f(a_n) = g(a_{n-1}), \quad n \geq 1.$$

Set

$$b_{n-1} = g(a_{n-1}) = f(a_n), \quad n \geq 1.$$

From the contractive condition, we obtain

$$\begin{aligned} F_J(b_n, b_n, b_{n+1}) &= F_J(ga_n, ga_n, ga_{n+1}) \\ &\preceq \lambda \left[F_J(fa_n, fa_n, ga_n) + F_J(fa_{n+1}, fa_{n+1}, ga_{n+1})\right] \\ &= \lambda \left[F_J(b_{n-1}, b_{n-1}, b_n) + F_J(b_n, b_n, b_{n+1})\right]. \end{aligned}$$

Hence,

$$(e-\lambda)F_J(b_n, b_n, b_{n+1}) \preceq \lambda F_J(b_{n-1}, b_{n-1}, b_n),$$

which implies

$$F_J(b_n, b_n, b_{n+1}) \preceq \eta F_J(b_{n-1}, b_{n-1}, b_n), \quad \text{where } \eta = (e-\lambda)^{-1}\lambda.$$

Using the properties of ϕ, we have

$$\phi(\eta) \leq \frac{\phi(\lambda)}{1-\phi(\lambda)} < \frac{1}{\mu}.$$

Now, by induction,

$$F_J(b_n, b_n, b_{n+1}) \preceq \eta^n F_J(b_0, b_0, b_1).$$

Let $1 \leq n < m$. Using the strong generalized triangle inequality, we get

$$\begin{aligned} F_J(b_n, b_n, b_m) &\preceq \mu \sum_{k=n}^{m-1} F_J(b_k, b_k, b_{k+1}) \\ &\preceq \mu \sum_{k=n}^{m-1} \eta^k F_J(b_0, b_0, b_1) \\ &\preceq \mu \eta^n (e-\eta)^{-1} F_J(b_0, b_0, b_1). \end{aligned}$$

Since $\phi(\eta) < 1/\mu$, it follows that $\eta^n \to \vartheta$ as $n \to \infty$. Thus $\{b_n\}$ is a ϑ–Cauchy sequence. By ϑ–completeness of X, there exists $\alpha \in X$ such that $b_n \to \alpha$.

Since $b_n = g(a_n) = f(a_{n+1})$, we have

$$f(a_{n+1}) \to \alpha \quad \text{and} \quad g(a_n) \to \alpha.$$

By continuity of f and the commutativity of f and g,

$$f\alpha = \lim_{n\to\infty} f(g(a_n)) = \lim_{n\to\infty} g(f(a_n)) = g\alpha.$$

Now,

$$F_J(g\alpha, g\alpha, g^2\alpha) \preceq \lambda \left[F_J(f\alpha, f\alpha, g\alpha) + F_J(fg\alpha, fg\alpha, g^2\alpha)\right].$$

Since $f\alpha = g\alpha$, this reduces to

$$F_J(g\alpha, g\alpha, g^2\alpha) \preceq 2\lambda F_J(g\alpha, g\alpha, g^2\alpha).$$

As $\phi(2\lambda) < 1$, we conclude

$$F_J(g\alpha, g\alpha, g^2\alpha) = \vartheta,$$

and hence $g^2\alpha = g\alpha$. Therefore,

$$f(g\alpha) = g(f\alpha) = g(g\alpha) = g\alpha,$$

so $g\alpha$ is a common fixed point of f and g.

For uniqueness, suppose $p, q \in X$ are two common fixed points. Then

$$\begin{aligned} F_J(p,p,q) &= F_J(gp, gp, gq) \\ &\preceq \lambda \left[F_J(fp, fp, gp) + F_J(fq, fq, gq)\right] \\ &= \lambda \left[F_J(p,p,p) + F_J(q,q,q)\right] = \vartheta. \end{aligned}$$

Hence $p = q$. Thus f and g have a unique common fixed point in X. □

Theorem 3.4. *Let (X, A, F_J) be a ϑ–complete F_J–cone metric space and let $\mathscr{P}$ be a solid cone of the Banach algebra A. Let $f : X \to X$ be a continuous self–mapping. Assume that there exists an element $\lambda \in \mathscr{P}$ such that*

$$\phi(\lambda) < \frac{1}{2\mu(\mu+1)},$$

where $\mu \geq 1$ is the coefficient associated with the F_J–cone metric space (X,A,F_J). Let $g : X \to X$ be another self–mapping commuting with f and satisfying

$$g(X) \subseteq f(X).$$

Suppose that the mappings f and g satisfy the following interpolative symmetric contractive condition:

$$F_J(ga,ga,gb) \preceq \lambda \Big[F_J(fa,fa,gb) + F_J(fb,fb,ga)\Big], \quad \forall a,b \in X.$$

Then the mappings f and g have a unique common fixed point in X.

Proof. Let $a_0 \in X$. Since $g(X) \subset f(X)$, there exists $a_1 \in X$ such that $f(a_1) = g(a_0)$. Proceeding inductively, we construct a sequence $\{a_n\} \subset X$ satisfying

$$f(a_n) = g(a_{n-1}), \quad \forall n \in \mathbb{N}.$$

Define

$$b_{n-1} = g(a_{n-1}) = f(a_n), \quad \forall n \geq 1.$$

Then

$$\begin{aligned} F_J(b_n,b_n,b_{n+1}) &= F_J(ga_n,ga_n,ga_{n+1}) \\ &\preceq \lambda \left[F_J(fa_n,fa_n,ga_{n+1}) + F_J(fa_{n+1},fa_{n+1},ga_n)\right] \\ &= \lambda \left[F_J(b_{n-1},b_{n-1},b_{n+1}) + F_J(b_n,b_n,b_n)\right]. \end{aligned}$$

Using the strong generalized triangle inequality, we have

$$\begin{aligned} F_J(b_{n-1},b_{n-1},b_{n+1}) \preceq \mu \Big[&F_J(b_{n-1},b_{n-1},b_n) + F_J(b_{n-1},b_{n-1},b_n) \\ &+ F_J(b_{n+1},b_{n+1},b_n) - \alpha F_J(b_n,b_n,b_n)\Big]. \end{aligned}$$

Since $\alpha \in (0,1]$ and $F_J(b_n,b_n,b_n) \succeq \vartheta$, it follows that

$$F_J(b_{n-1},b_{n-1},b_{n+1}) \preceq \mu \left[2F_J(b_{n-1},b_{n-1},b_n) + F_J(b_n,b_n,b_{n+1})\right].$$

Substituting this in the above inequality, we obtain

$$\begin{aligned} F_J(b_n,b_n,b_{n+1}) &\preceq \lambda\mu\left[2F_J(b_{n-1},b_{n-1},b_n)+F_J(b_n,b_n,b_{n+1})\right]+\lambda F_J(b_n,b_n,b_n) \\ &\preceq 2\mu\lambda F_J(b_{n-1},b_{n-1},b_n)+\mu\lambda F_J(b_n,b_n,b_{n+1})+\lambda F_J(b_n,b_n,b_n). \end{aligned}$$

Using the monotonicity of F_J, we get

$$(e-\mu\lambda)F_J(b_n,b_n,b_{n+1}) \preceq (2\mu+1)\lambda F_J(b_{n-1},b_{n-1},b_n).$$

Hence

$$F_J(b_n,b_n,b_{n+1}) \preceq \xi F_J(b_{n-1},b_{n-1},b_n), \quad \text{where } \xi=(e-\mu\lambda)^{-1}(2\mu+1)\lambda.$$

Since $\phi(\lambda)<\frac{1}{2\mu(\mu+1)}$, it follows that $\phi(\xi)<\frac{1}{\mu}$. Thus $\{b_n\}$ is a ϑ–Cauchy sequence in X. As X is ϑ–complete, there exists $\alpha\in X$ such that $b_n\to\alpha$.

Now $b_n=f(a_{n+1})=g(a_n)$ implies that $f(a_{n+1})\to\alpha$ and $g(a_n)\to\alpha$ as $n\to\infty$. By continuity of f and commutativity of f and g, we obtain $f\alpha=g\alpha$.

Again, using the contractive condition,

$$F_J(g\alpha,g\alpha,g^2\alpha) \preceq \lambda\left[F_J(f\alpha,f\alpha,g\alpha)+F_J(g\alpha,g\alpha,g\alpha)\right].$$

Since $f\alpha=g\alpha$, we get

$$(e-2\lambda)F_J(g\alpha,g\alpha,g^2\alpha) \preceq \vartheta,$$

which yields $g^2\alpha=g\alpha$. Therefore

$$f(g\alpha)=g(f\alpha)=g(g\alpha)=g\alpha,$$

and hence $g\alpha$ is a common fixed point of f and g.

Finally, suppose that $p,q\in X$ are two common fixed points of f and g. Then

$$F_J(p,p,q)=F_J(gp,gp,gq) \preceq \lambda\left[F_J(fp,fp,gq)+F_J(fq,fq,gp)\right]=2\lambda F_J(p,p,q).$$

Since $\phi(2\lambda)<1$, we conclude that $F_J(p,p,q)=\vartheta$, which implies $p=q$. Hence f and g have a unique common fixed point in X. □

Example 3.5. Let $X = [0,\infty)$ and $A = \mathbb{R}$, considered as a Banach algebra with usual addition and multiplication, and let the cone $P = \{x \in \mathbb{R} : x \geq 0\}$. Define a mapping

$$F_J : X \times X \times X \longrightarrow A$$

by

$$F_J(x,y,z) = \begin{cases} x^2 + y^2 + z^2, & \text{if } x \neq y \neq z \text{ or } x \neq y = z, \\ \frac{x^2+z^2}{2}, & \text{if } x = y \neq z, \\ x^2, & \text{if } x = y = z. \end{cases}$$

Verification of F_J-cone metric properties:

(1) **Non-negativity and identity of indiscernibles:** Clearly, $F_J(x,y,z) \geq 0$ for all $x,y,z \in X$. Moreover,

$$F_J(x,x,x) = x^2 = F_J(y,y,y) = F_J(z,z,z) = F_J(x,y,z) \implies x = y = z.$$

(2) **Monotonicity:** For $x \neq y \neq z$, we have

$$0 \leq F_J(x,x,x) = x^2 \leq F_J(x,x,y) = \frac{x^2+y^2}{2} \leq F_J(x,y,z) = x^2 + y^2 + z^2.$$

(3) **Strong generalized triangle inequality:** Let $\alpha = \frac{1}{2}$ and $s = 2$. For all $x,y,z,a \in X$, we have

$$F_J(x,y,z) \leq s\big(F_J(x,x,a) + F_J(y,y,a) + F_J(z,z,a) - \alpha F_J(a,a,a)\big),$$

which can be verified by direct calculation. For example, if $x \neq y \neq z$ and $a \in X$, then

$$x^2 + y^2 + z^2 \leq 2\big((x^2 + a^2) + (y^2 + a^2) + (z^2 + a^2) - \frac{1}{2}a^2\big),$$

which holds for all $x,y,z,a \geq 0$.

(4) **Symmetric balance control:** Take $t = 3$. For all $x,y,z,a \in X$,

$$F_J(x,y,z) \leq t\big(F_J(a,x,x) + F_J(a,y,y) + F_J(a,z,z)\big),$$

which also holds since

$$x^2 + y^2 + z^2 \leq 3\big((a^2 + x^2 + x^2) + (a^2 + y^2 + y^2) + (a^2 + z^2 + z^2)\big).$$

Note: This mapping F_J **does not satisfy the F-cone metric condition** because the classical F-triangle inequality

$$F(x,y,z) \leq s\big(F(x,x,a)+F(y,y,a)+F(z,z,a)-F(a,a,a)\big)$$

fails for $\alpha = 0$ (required in F-cone metric). Hence (X,A,F_J) is an F_J-cone metric space that is not an F-cone metric space.

4. Conclusion

We have introduced the concept of an F_J–cone metric space over an ordered Banach algebra, extending F–cone metric spaces with additional structural conditions. We establish common fixed point theorems for commuting self-mappings under generalized contractive conditions, proving existence and uniqueness in ϑ–complete F_J–cone metric spaces. An illustrative example shows that F_J–cone metric spaces properly generalize F–cone metric spaces, advancing fixed point theory in Banach algebra–valued cone metric structures.

Conflict of Interests

The authors declare that there is no conflict of interests.

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