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A NEW BEST PROXIMITY POINT RESULT WITH AN APPLICATION TO DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, we first introduce a new class of contractions via a new concept called a p^* -cyclic contraction mapping, combining the ideas of a cyclic contraction mapping and a p^* -contraction. Then we introduce the definition of a cyclically-complete pair within the context of partial metric spaces. Next, we establish several best proximity point results for p^* -cyclic contraction mappings on $D \cup E$, considering the two cases where (D, E) is a cyclically complete pair and a cyclically 0-complete pair in partial metric spaces. Finally we are investigating the sufficient conditions required to demonstrate the existence of a solution to the differential inclusions using the main result.

Keywords: best proximity point; p^* -cyclic contractions; differential Inclusions.

2020 AMS Subject Classification: 54H25, 47H10, 45B05.

1. INTRODUCTION AND PRELIMINARIES

The first result in fixed point theory is that of Banach [5] known as Banach contraction principle, in 1922. Then after many authors have generalized and refined it in different ways [10, 16, 18], Popescu [17] introduced the concept of p -contraction mapping and obtained a

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fixed point result for these mappings. According to this result, every p -contraction mapping $Y : \Lambda \rightarrow \Lambda$ on a complete metric space (Λ, ρ) , that is there exists $q \in [0, 1)$ such that:

$$\rho(Yx, Y\xi) \leq q[\rho(x, \xi) + |\rho(x, Yx) - \rho(\xi, Y\xi)|]$$

for all $x, \xi \in \Lambda$ has a unique fixed point.

Latterly, Kirk and al.[14] achieved a new generalisation of Banach's contraction principle by using a new concept of cyclic mapping. In their research, cyclic mapping may not be continuous, in contrast to Banach's result. This is the important characteristic of their result. Subsequently, many researchers have studied the possibility of obtaining fixed point results for cyclic mappings[12, 23].

However, one of the interesting generalisations of Banach's contraction principle has been attained by taking into account non-self mappings. Let $D, E \neq \emptyset$ be subsets of a metric space (Λ, ρ) and $Y : D \rightarrow E$ be nonself mappings. If $D \cap E = \emptyset$, Y cannot have a fixed point. Then, since $\rho(x, Yx) \leq \rho(D, E)$ for all $x \in D$, it is reasonable to search a point satisfying $\rho(x, Yx) = \rho(D, E)$. This point is said to be a best proximity point of Y . Note that a best proximity point of Y is an optimal solution for the problem $\min_{x \in D} \rho(x, Yx)$. Additionally, a best proximity point turns into a fixed point in the case of $D = E = \Lambda$. For this reason, many authors have studied on this topic [20, 22, 9].

Based on the ideas of best proximity and cyclic mapping, the well-known concept of cyclic contraction mapping was introduced by Eldred and Veeramani [8]. These ideas were thus unified. So far, some properties such as bounded compactness have been used to guarantee the existence of best proximity points for cyclic contraction mappings. Recently, introducing a nice notion called cyclically completeness, Karpagam and Agrawal [13] show the existence of best proximity point of a cyclic contraction mapping in the absence of the property of bounded compactness. Subsequently, many authors have attained some best proximity point results in virtue of this concept [2, 6].

In 1994, motivated by the experience of computer science, Matthews, [15], relaxed the condition $x = \xi$ implies $\rho(x, \xi) = 0$ in a metric space (Λ, ρ) by introducing the partial metric. Partial metric spaces are a subject of study in mathematics. Subsequent to this, numerous authors obtained a variety of fixed point results. The best proximity point is achieved in the context of

partial metric spaces [4, 11, 7]. In recent times, Romaguera [19] introduced the concept of 0-complete partial metric space. Consequently, a weaker form of completeness on partial metric spaces has been obtained.

In this paper, we introduce a new concept of p^* -cyclic contraction mapping by modifying the definitions of p -cyclic contraction in [21].

Then, with some additional conditions we can get a result of best proximity point results for p^* -cyclic contraction mappings on $D \cup E$ where (D, E) is a cyclically -complete pair in a partial metric space. Finally, We are investigating the sufficient conditions required to demonstrate the existence of a solution to the differential inclusions.

In this section, we give some definitions, lemmas and theorems which are important in our main result. We begin this section with the following result for cyclic mappings, which was obtained by Kirk et al.[14].

Theorem 1.1 ([14]). *Let $\emptyset \neq D, E$ be closed subsets of a complete metric space (Λ, ρ) and $Y : D \cup E \rightarrow D \cup E$ be a cyclic mapping, that is, $Y(D) \subseteq E$ and $Y(E) \subseteq D$. Then, Y has a fixed point in $D \cap E$ if there is $q \in [0, 1)$ such that*

$$\rho(Yx, Y\xi) \leq q\rho(x, \xi)$$

for all $x \in D$ and $\xi \in E$.

If $D \cap E = \emptyset$ in Theorem 1.1 the famous concept of cyclic contraction mapping was introduced by Eldred and Veeramani [8]. Then, they obtain a best proximity point result as follows:

Definition 1.2 ([8]). *Let $\emptyset \neq D, E$ be closed subsets of a complete metric space (Λ, ρ) and $Y : D \cup E \rightarrow D \cup E$ be a cyclic mapping. Then, Y is called cyclic contraction mapping if there exists $q \in [0, 1)$ such that*

$$\rho(Yx, Y\xi) \leq q\rho(x, \xi) + (1 - q)\rho(D, E)$$

for all $x \in D$ and $\xi \in E$.

Theorem 1.3 ([8]). *Let (Λ, ρ) be a metric space, $\emptyset \neq D, E \subseteq \Lambda$ where D, E are closed and $Y : D \cup E \rightarrow D \cup E$ be a cyclic contraction mapping. If either D or E is boundedly compact, then Y has a best proximity point in $D \cup E$.*

Next, we recall the concept of cyclically completeness.

Definition 1.4 ([13]). Let (Λ, ρ) be a metric space and $\emptyset \neq D, E \subseteq \Lambda$. A sequence $\{x_n\}$ in $D \cup E$ with $\{x_{2n}\} \subset D$ and $\{x_{2n+1}\} \subset E$ is called a cyclically Cauchy sequence if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ satisfying

$$\rho(x_n, x_m) < \rho(D, E) + \varepsilon$$

for all $n, m \geq n_0$ with m is odd, n is even.

Definition 1.5 ([13]). A pair (D, E) of subsets of a metric space is said to be cyclically complete if for every cyclically Cauchy sequence $\{x_n\} \subset D \cup E$ either $\{x_{2n}\}$ or $\{x_{2n+1}\}$ are convergent.

Now, we give the definition of the partial metric space and its topological properties.

Definition 1.6 ([15]). Let $\Lambda \neq \emptyset$ and $\theta : \Lambda \times \Lambda \rightarrow [0, \infty)$ be a mapping satisfying the following conditions:

- (1) $\theta(x, x) = \theta(x, \xi) = \theta(\xi, \xi)$ if and only if $x = \xi$,
- (2) $\theta(x, x) \leq \theta(x, \xi)$,
- (3) $\theta(\xi, x) = \theta(x, \xi)$,
- (4) $\theta(x, \eta) \leq \theta(x, \xi) + \theta(\xi, \eta) - \theta(\xi, \xi)$

for all $x, \eta, \xi \in \Lambda$. Then, θ is called a partial metric. Additionally, (Λ, θ) is called a partial metric space.

Every metric space is a partial metric space, but the converse may not be true ((see for more details [1, 3]). Assume that θ is a partial metric on Λ . There is an T_0 topology τ_θ on Λ . Additionally, the family open θ -balls $\{B_\theta(x, \varepsilon) : x \in \Lambda, \varepsilon > 0\}$ is a base for the topology τ_θ where $B_\theta(x, \varepsilon) = \{\xi \in \Lambda : \theta(x, \xi) < \theta(x, x) + \varepsilon\}$ for all $x \in \Lambda$ and $\varepsilon > 0$.

If we take a sequence $\{x_n\} \subset \Lambda$ and $x \in \Lambda$, then it is clear that $\{x_n\} \subset \Lambda$ converges to x w.r.t. τ_θ .

If we take a sequence $\{x_n\} \subseteq \Lambda$ and $x \in \Lambda$ then it is clear that:

- (1) $\{x_n\}$ converges to x w.r.t τ_θ if and only if $\lim_{n \rightarrow \infty} \theta(x_n, x) = \theta(x, x)$
- (2) The sequence x_n is said to be Cauchy sequence if $\lim_{n, m \rightarrow \infty} \theta(x_n, x_m)$ exists and is finite

Definition 1.7 ([19]). Let (Λ, θ) be a partial metric space and x_n be a sequence in Λ .

- i) $\{x_n\}$ is called 0-Cauchy sequence if $\lim_{n,m \rightarrow \infty} \theta(x_n, x_m) = 0$.
- ii) (Λ, θ) is called 0-complete partial metric space if every 0-Cauchy sequence converges to a point x in Λ w.r.t. τ_θ such that $\lim_{n,m \rightarrow \infty} \theta(x_n, x_m) = \theta(x, x)$.

Now we introduce the definition of p*-cyclic contraction mapping on partial metric spaces.

Definition 1.8. Let $\emptyset \neq D, E$ be subsets of a partial metric space (Λ, θ) and $Y: D \cup E \rightarrow D \cup E$ be a cyclic mapping. Then Y is said to be p*-cyclic contraction mapping if there exists $q \in [0, 1)$ such that

$$\theta(Yx, Y\xi) \leq q\{\theta(x, \xi) + |\theta(\xi, Yx) - \theta(x, Y\xi)|\} + (1 - q)\theta(D, E)$$

for all $x \in D$ and $\xi \in E$, where $\theta(D, E) = \inf\{\theta(x, \xi), x \in D, \xi \in E\}$.

Now, we recall the definition of cyclically Cauchy sequence and cyclically 0-complete pair in a partial metric spaces.

Definition 1.9 ([21]). Let (Λ, θ) a partial metric space, $\emptyset \neq D, E \subseteq \Lambda$. A sequence $\{x_n\}$ in $D \cup E$ with $\{x_{2n}\} \subseteq D$ and $\{x_{2n+1}\} \subseteq E$ is called a cyclically Cauchy sequence if for each ε there is $n_0 \in \mathbb{N}$ satisfying

$$\theta(x_n, x_m) \leq \theta(D, E) + \varepsilon$$

for all $m, n \geq n_0$ with m is odd, n is even.

Definition 1.10 ([21]). Let $\emptyset \neq D, E$ a subsets of a partial metric space (Λ, θ) . A pair (D, E) is said to be cyclically 0-complete pair if for every cyclically Cauchy sequence $\{x_n\}$ in $D \cup E$ either $\{x_{2n}\}$ has a convergent subsequence $\{x_{2n_k}\}$ to a point $x^* \in D$ w.r.t. τ_θ such that

$$\lim_{k,l \rightarrow \infty} \theta(x_{2n_k}, x_{2n_l}) = \theta(x^*, x^*) = 0$$

or $\{x_{2n+1}\}$ has a convergent subsequence $\{x_{2n_k+1}\}$ to a point $\zeta^* \in E$ w.r.t. τ_θ such that

$$\lim_{k,l \rightarrow \infty} \theta(x_{2n_k+1}, x_{2n_l+1}) = \theta(\zeta^*, \zeta^*) = 0.$$

Remark 1.11. If D or E is a closed subset of 0-complete partial metric space (Λ, θ) and $\theta(D, E) = 0$, then (D, E) is a cyclically 0-complete pair.

Then we introduce the definition of cyclically complete pair in a partial metric spaces.

Definition 1.12. Let $\emptyset \neq D, E$ a subsets of a partial metric space (Λ, θ) . A pair (D, E) is said to be cyclically -complete pair if for every cyclically Cauchy sequence $\{x_n\}$ in $D \cup E$ either $\{x_{2n}\}$ convergent to a point $x^* \in D$ w.r.t. τ_θ such that

$$\lim_{n,m \rightarrow \infty} \theta(x_{2n}, x_{2m}) = \theta(x^*, x^*)$$

or $\{x_{2n+1}\}$ convergent to a point $\zeta^* \in E$ w.r.t. τ_θ such that

$$\lim_{n,m \rightarrow \infty} \theta(x_{2n+1}, x_{2m+1}) = \theta(\zeta^*, \zeta^*);$$

Now, we give a new definition in partial metric spaces.

Definition 1.13. Let $\emptyset \neq D, E$ a subsets of a partial metric space (Λ, θ) . Then, D is called boundedly compact if every bounded sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ to a point $x^* \in D$ w.r.t. τ_θ such that

$$\lim_{k,l \rightarrow \infty} \theta(x_{2l+1}, x_{2k+1}) = \theta(\zeta^*, \zeta^*).$$

Remark 1.14. Note that if either D or E is a boundedly compact, then the pair (D, E) is a cyclically complete pair.

2. MAIN RESULTS

We begin with this properties.

Proposition 2.1. Let (Λ, θ) a partial metric space and $\emptyset \neq D, E \subseteq \Lambda$. Assume that $Y: D \cup E \rightarrow D \cup E$ is a p^* -cyclic contraction mapping. Then for every sequence $\{x_n\} \subseteq D \cup E$ defined by $x_{n+1} = Yx_n$ with the initial point $x_0 \in D$, such that

$$|\theta(x_{n+1}, x_{n-1}) - \theta(x_n, x_n)| \leq \theta(x_{n-1}, x_n) - \theta(x_n, x_{n+1}),$$

for all $n \in \mathbb{N}$, we have $\theta(x_n, x_{n+1}) \rightarrow \theta(D, E)$ as $n \rightarrow \infty$.

Proof. Let $\{x_n\} \subseteq D \cup E$ be a sequence constructed as in Proposition 2.1. Since Y is p^* -cyclic contraction, then by all condition of proposition 2.1, we have:

$$\begin{aligned}
 \theta(x_n, x_{n+1}) &= \theta(Yx_{n-1}, Yx_n) \\
 &\leq q\theta(x_n, x_{n-1}) + q|\theta(x_{n+1}, x_{n-1}) - \theta(x_n, x_n)| + (1-q)\theta(D, E) \\
 &\leq q\theta(x_n, x_{n-1}) + q\theta(x_n, x_{n-1}) - q\theta(x_n, x_{n+1}) + (1-q)\theta(D, E) \\
 &= 2q\theta(x_n, x_{n-1}) - q\theta(x_n, x_{n+1}) + (1-q)\theta(D, E)
 \end{aligned}$$

and so we get:

$$\theta(x_n, x_{n+1}) \leq \alpha\theta(x_n, x_{n-1}) + \beta\theta(D, E)$$

for all $n \in \mathbb{N}$, where $\alpha = \frac{2q}{1+q} < 1$ and $\beta = \frac{1-q}{1+q}$. By using the last inequality, we get:

$$\begin{aligned}
 \theta(D, E) &\leq \theta(x_n, x_{n+1}) \\
 &\leq \alpha\theta(x_n, x_{n-1}) + \beta\theta(D, E) \\
 &\leq \alpha(\alpha\theta(x_{n-2}, x_{n-1}) + \beta\theta(D, E)) + \beta\theta(D, E) \\
 &= \alpha^2\theta(x_{n-2}, x_{n-1}) + \beta\theta(D, E)(1 + \alpha) \\
 &\quad \vdots \\
 &\leq \alpha^n\theta(x_0, x_1) + \beta\theta(D, E)(1 + \alpha + \alpha^2 + \dots + \alpha^{n-1}) \\
 &= \alpha^n\theta(x_0, x_1) + \beta\theta(D, E)\left(\frac{1 - \alpha^n}{1 - \alpha}\right) \\
 &= \alpha^n\theta(x_0, x_1) + \beta\theta(D, E)(1 - \alpha^n)
 \end{aligned}$$

for all $n \in \mathbb{N}$. Hence, we get

$$\lim_{n \rightarrow \infty} \theta(x_n, x_{n+1}) = \theta(D, E)$$

□

The following proposition is crucial for our main result.

Proposition 2.2. *Let $\emptyset \neq D, E$ a subsets of a partial metric space (Λ, θ) . Assume that $Y : D \cup E \rightarrow D \cup E$ is a p^* -cyclic contraction mapping. Then for every sequence $\{x_n\} \subseteq D \cup E$ defined by $x_{n+1} = Yx_n$ with the initial point $x_0 \in D$, such that all conditions of proposition 2.1 are satisfied and*

$$(1) \quad |\theta(x_n, x_{m-1}) - \theta(x_m, x_{n-1})| \leq q\theta(x_m, x_{m-1})$$

for all $n, m \in \mathbb{N}$, is bounded.

Proof. Let $\{x_n\} \subseteq D \cup E$ be a sequence constructed as in Proposition 2.1. Hence from proposition 2.1 the sequence $\{\theta(x_{2n}, x_{2n+1})\}$ converges to $\theta(D, E)$, and so the sequence $\{\theta(x_{2n}, x_{2n+1})\}$ is bounded. Then, there exists $L > 0$ such that:

$$\theta(x_{2n}, x_{2n+1}) \leq L$$

for all $n \in \mathbb{N}$. Since Y is p^* -cyclic contraction, and from the conditions in proposition 2.1 we have:

$$\begin{aligned} \theta(x_{2n+1}, x_1) &\leq \theta(x_{2n+1}, x_{2n+2}) + \theta(x_{2n+2}, x_1) \\ &\leq L + \theta(x_{2n+2}, x_1) \\ &= L + \theta(Yx_{2n+1}, Yx_0) \\ &\leq L + q\theta(x_{2n+1}, x_0) + q|\theta(x_{2n+1}, Yx_0) - \theta(Yx_{2n+1}, x_0)| + (1-q)\theta(D, E) \\ &\leq L + q\theta(x_{2n+1}, x_0) + q\theta(x_{2n+2}, x_{2n+1}) + (1-q)\theta(D, E) \quad (\text{by inequality (1)}) \\ &\leq 2L + q\theta(x_{2n+1}, x_1) + 2q\theta(x_0, x_1) + (1-q)\theta(D, E) \end{aligned}$$

for each $n \in \mathbb{N}$, which implies that: $\theta(x_{2n+1}, x_1) \leq \frac{2L}{1-q} + \frac{2q}{1-q}\theta(x_0, x_1) + \theta(D, E)$

Let

$$M = \frac{2L}{1-q} + \frac{2q}{1-q}\theta(x_0, x_1) + \theta(D, E).$$

Hence, $\{x_{2n+1}\}$ is bounded. Additionally, we get

$$\theta(x_{2n}, x_1) \leq \theta(x_{2n}, x_{2n+1}) + \theta(x_{2n+1}, x_1) \leq L + M.$$

Hence, $\{x_{2n}\}$ is bounded, and so $\{x_n\}$ is bounded. □

The main result of this paper is the following theorem.

Theorem 2.3. *Let $\emptyset \neq D, E$ a subsets of a partial metric space (Λ, θ) where (D, E) is a cyclically - complete pair. Suppose that $Y: D \cup E \rightarrow D \cup E$ is a p^* -cyclic contraction mapping. Then, if for every sequence $\{x_n\} \subseteq D \cup E$ defined by $x_{n+1} = Yx_n$ with the initial point $x_0 \in D$, such that*

$$|\theta(x_{n+1}, x_{n-1}) - \theta(x_n, x_n)| \leq \theta(x_{n-1}, x_n) - \theta(x_n, x_{n+1}),$$

for all $n \in \mathbb{N}$,

$$(2) \quad |\theta(x_{2n-1}, Yx^*) - \theta(x_{2n}, x^*)| \leq \theta(x_{2n-2}, Yx^*) - \theta(D, E), \text{ for all } n \in \mathbb{N}, x^* \in D,$$

$$(3) \quad |\theta(x_n, x_{m-1}) - \theta(x_m, x_{n-1})| \leq q\theta(x_m, x_{m-1}) \quad \text{for all } n, m \in \mathbb{N}$$

then Y has a best proximity point.

Proof. We have from proposition 2.1:

$$(4) \quad \lim_{n \rightarrow \infty} \theta(x_n, x_{n+1}) = \theta(D, E)$$

Now, let us show that $\{x_n\}$ is a cyclically Cauchy sequence. Assume $n, m \in \mathbb{N}$ with $n \geq m$. Since Y is a p^* -cyclic contraction mapping and (3), we get

$$\begin{aligned} \theta(x_n, x_m) &= \theta(Yx_{n-1}, Yx_{m-1}) \\ &\leq q\theta(x_{n-1}, x_{m-1}) + q|\theta(x_n, x_{m-1}) - \theta(x_m, x_{n-1})| + (1-q)\theta(D, E) \\ &\leq q\theta(x_{n-1}, x_{m-1}) + q\theta(x_m, x_{m-1}) + (1-q)\theta(D, E) \\ &= q\theta(x_{n-1}, x_{m-1}) + q\theta(x_m, x_{m-1}) + (1-q)\theta(D, E) \end{aligned}$$

for all $n, m \in \mathbb{N}$ with $n \geq m$. Additionally from proposition 2.1 we have:

$$\theta(x_n, x_{n+1}) \leq \alpha\theta(x_n, x_{n+1}) + \beta\theta(D, E)$$

for all $n \in \mathbb{N}$, where $\alpha = \frac{2q}{1+q} < 1$ and $\beta = \frac{1-q}{1+q}$. Therefore, we have

$$\begin{aligned} \theta(D, E) &\leq \theta(x_n, x_m) \\ &\leq \theta(Yx_{n-1}, Yx_{m-1}) \\ &\leq q\theta(x_{n-1}, x_{m-1}) + q\theta(x_m, x_{m-1}) + (1-q)\theta(D, E) \\ &\leq q\{q\theta(x_{n-2}, x_{m-2}) + q\theta(x_{m-2}, x_{m-1}) + (1-q)\theta(D, E)\} \\ &\quad + q\{\alpha\theta(x_{m-2}, x_{m-1}) + \beta\theta(D, E)\} + (1-q)\theta(D, E) \\ &\leq q^2\theta(x_{n-2}, x_{m-2}) + (q^2 + \alpha q)\theta(x_{m-2}, x_{m-1}) + \beta q\theta(D, E) + (1+q)(1-q)\theta(D, E) \\ &\quad \vdots \end{aligned}$$

$$\begin{aligned}
&\leq q^m \theta(x_{n-m}, x_0) \\
&+ q^m \left\{ 1 + \frac{\alpha}{q} + \left(\frac{\alpha}{q}\right)^2 + \left(\frac{\alpha}{q}\right)^3 + \left(\frac{\alpha}{q}\right)^{m-1} \right\} \theta(x_0, x_1) \\
&+ \beta \left\{ \begin{array}{l} \{q + q^2 + q^3 + \dots + q^{m-2}\} \\ + \alpha \{q + q^2 + q^3 + \dots + q^{m-1}\} \\ + \alpha^2 \{q + q^2 + q^3 + \dots + q^{m-2}\} \\ \vdots \\ + \alpha^{m-1} q \end{array} \right\} \theta(D, E) \\
&+ (1 - 2q) \{1 + q + q^2 + q^3 + \dots + q^{m-1}\} \theta(D, E) \\
&\leq q^m \theta(x_{n-m}, x_0) + q^m \sum_{i=0}^{m-1} \left(\frac{\alpha}{q}\right)^i \theta(x_0, x_1) \\
&+ \beta \left\{ \sum_{i=1}^{\infty} q^i + \alpha \sum_{i=1}^{\infty} q^i + \alpha^2 \sum_{i=1}^{\infty} q^i + \dots + \alpha^{m-1} \sum_{i=1}^{\infty} q^i \right\} \theta(D, E) \\
&+ (1 - q) \sum_{i=1}^{\infty} q^i \theta(D, E) \\
&\leq q^m \theta(x_{n-m}, x_0) + q^m \frac{1 - \left(\frac{\alpha}{q}\right)^m}{1 - \frac{\alpha}{q}} \theta(x_0, x_1) \\
&+ \beta \frac{q}{1 - q} \{1 + \alpha + \alpha^2 + \alpha^3 + \dots + \alpha^{m-1}\} \theta(D, E) + (1 - q) \frac{1 - q^m}{1 - q} \theta(D, E)
\end{aligned}$$

Since $\{x_n\}$ is bounded sequence, then

$$\begin{aligned}
\theta(D, E) &\leq \theta(x_n, x_m) \\
&\leq q^m M + \frac{q^m - \alpha^m}{1 - \frac{\alpha}{q}} \theta(x_0, x_1) + \frac{q}{1 - q} \theta(D, E) \\
&+ (1 - q^m) \theta(D, E)
\end{aligned}$$

for all $m, n \in \mathbb{N}$ with $n \geq m$, and for some $M > 0$. Hence, we have

$$\lim_{n, m \rightarrow \infty} \theta(x_n, x_m) = \theta(D, E)$$

Since (D, E) is a cyclically -complete pair, without loss of the generality $\{x_n\}$ has a subsequence such that

$$\lim_{i,j \rightarrow \infty} \theta(x_{2n_i}, x_{2n_j}) = \lim_{i \rightarrow \infty} \theta(x_{2n_i}, x^*) = \theta(x^*, x^*)$$

Additionally, we have:

$$\begin{aligned} \theta(x_{2n_i}, Yx^*) &\leq \theta(Yx_{2n_i-1}, Yx^*) \\ &\leq q \{ \theta(x_{2n_i-1}, x^*) + |\theta(x_{2n_i-1}, Yx^*) - \theta(Yx_{2n_i-1}, x^*)| \} + (1-q)\theta(D, E) \\ &\leq q \{ \theta(x_{2n_i-1}, x_{2n_i}) + \theta(x_{2n_i}, x^*) - \theta(x_{2n_i}, x_{2n_i}) \} \\ &\quad + q \{ |\theta(x_{2n_i-1}, Yx^*) - \theta(Yx_{2n_i-1}, x^*)| \} + (1-q)\theta(D, E) \end{aligned}$$

Taking limit $i \rightarrow \infty$, in the last inequality, from (3) and (4) we get:

$$\begin{aligned} \theta(x^*, Yx^*) &= \lim_{n \rightarrow \infty} \theta(x_{2n_i}, Yx^*) \\ &\leq q\theta(D, E) + q \lim_{n \rightarrow \infty} \theta(x_{2n_i-2}, Yx^*) - q\theta(D, E) + (1-q)\theta(D, E) \\ &= q\theta(x^*, Yx^*) + (1-q)\theta(D, E) \end{aligned}$$

Therefore we have, $\theta(x^*, Yx^*) \leq \theta(D, E)$, and so $\theta(x^*, Yx^*) = \theta(D, E)$. Hence x^* is a best proximity point of Y in D . If $\{x_{2n+1}\}$ has a subsequence $\{x_{2n_i+1}\}$ such that

$$(5) \quad \lim_{i,j \rightarrow \infty} \theta(x_{2n_i+1}, x_{2n_j+1}) = \lim_{i \rightarrow \infty} \theta(x_{2n_i+1}, \xi^*) = \theta(\xi^*, \xi^*)$$

for some $\xi^* \in E$. Then, by the similar way, it can be shown that ξ^* is a best proximity point of Y in E . \square

Example 2.4. Let $\Lambda = [0, \infty) \times [0, \infty)$ and $\theta : \Lambda \times \Lambda \rightarrow \mathbb{R}$ be a function defined by

$$\theta(x, \xi) = \max\{x_1, \xi_1\} + |x_2 - \xi_2|$$

for $x = (x_1, x_2)$ and $\xi = (\xi_1, \xi_2) \in \Lambda$.

It is clear that (Λ, θ) is a partial metric space, Let

$$D = \{(a, 1) : a \in [0, \infty)\},$$

and

$$E = \{(a, 2) : a \in [0, \infty)\}$$

then $\theta(D, E) = 1$. The pair (D, E) is a cyclically 0-complete pair (see [20])

If we define a mapping $Y : D \cup E \rightarrow D \cup E$

$$Yx = \begin{cases} (\frac{a}{2}, 2) & \text{if } x = (a, 1) \in D \\ (\frac{a}{2}, 1) & \text{if } x = (a, 2) \in E \end{cases}$$

Y is a p^* -cyclic contraction mapping for $q = \frac{1}{2}$,

Let $x_0 = (a, 1) \in D$ then,

$$\theta(x_{n+1}, x_{n-1}) = \max\left\{\frac{a}{2^{n+1}}, \frac{a}{2^{n-1}}\right\} = \frac{a}{2^{n-1}}, \quad \theta(x_{n+1}, x_n) = \max\left\{\frac{a}{2^{n+1}}, \frac{a}{2^n}\right\} + 1 = \frac{a}{2^n} + 1,$$

$$\theta(x_n, x_{n-1}) = \max\left\{\frac{a}{2^n}, \frac{a}{2^{n-1}}\right\} + 1 = \frac{a}{2^{n-1}} + 1, \quad \theta(x_n, x_n) = \max\left\{\frac{a}{2^n}, \frac{a}{2^n}\right\} = \frac{a}{2^n}.$$

we have ,

$$\theta(x_{n+1}, x_{n-1}) - \theta(x_{n-1}, x_n) + \theta(x_n, x_{n+1}) + \theta(x_n, x_n) = \frac{a}{2^{n-1}} - \frac{a}{2^{n-1}} - 1 + \frac{a}{2^n} + 1 - \frac{a}{2^n} = 0 \leq 0$$

for all $n \in \mathbb{N}$. Therefore

$$|\theta(x_{n+1}, x_{n-1}) - \theta(x_n, x_n)| \leq \theta(x_{n-1}, x_n) - \theta(x_n, x_{n+1}),$$

for all $n \in \mathbb{N}$, Now we prove that:

$$|\theta(x_{2n-1}, Yx^*) - \theta(x_{2n}, x^*)| \leq \theta(x_{2n-2}, Yx^*) - \theta(D, E),$$

for all $n \in \mathbb{N}$ and $x^* \in D$.

Let $x^* = (b, 1) \in D$ then,

$$\theta(x_{2n-1}, Yx^*) = \max\left\{\frac{a}{2^{2n-1}}, \frac{b}{2}\right\}, \quad \theta(x_{2n}, x^*) = \max\left\{\frac{a}{2^{2n}}, b\right\}, \quad \theta(x_{2n-2}, Yx^*) = \max\left\{\frac{a}{2^{2n-2}}, \frac{b}{2}\right\} + 1$$

it suffices to show that:

$$(6) \quad \left| \max\left\{\frac{a}{2^{2n-1}}, \frac{b}{2}\right\} - \max\left\{\frac{a}{2^{2n}}, b\right\} \right| \leq \max\left\{\frac{a}{2^{2n-2}}, \frac{b}{2}\right\}$$

Case 1: $b = 0$ we have:

$$\max\left\{\frac{a}{2^{2n-1}}, \frac{b}{2}\right\} = \frac{a}{2^{2n-1}}, \quad \max\left\{\frac{a}{2^{2n}}, b\right\} = \frac{a}{2^{2n}}, \quad \max\left\{\frac{a}{2^{2n-2}}, \frac{b}{2}\right\} = \frac{a}{2^{2n-2}}$$

then:

$$\left| \max\left\{\frac{a}{2^{2n-1}}, \frac{b}{2}\right\} - \max\left\{\frac{a}{2^{2n}}, b\right\} \right| = \frac{a}{2^{2n-1}} - \frac{a}{2^{2n}} = \frac{a}{2^{2n}}$$

and

$$\max \left\{ \frac{a}{2^{2n-2}}, \frac{b}{2} \right\} = \frac{a}{2^{2n-2}}$$

hence the inequality (6) is verified.

Case 2: $\frac{b}{2} > \frac{a}{2^{2n-2}} > \frac{a}{2^{2n-1}} > \frac{a}{2^{2n}}$: for all $n \in \mathbb{N}$

Then

$$\left| \max \left\{ \frac{a}{2^{2n-1}}, \frac{b}{2} \right\} - \max \left\{ \frac{a}{2^{2n}}, b \right\} \right| = b - \frac{b}{2} = \frac{b}{2}$$

and

$$\max \left\{ \frac{a}{2^{2n-2}}, \frac{b}{2} \right\} = \frac{b}{2}$$

the inequality (6) is verified

Case 3: if for some $j \in \mathbb{N}$ we have $\frac{a}{2^{2j-2}} > \frac{a}{2^{2j-1}} > \frac{a}{2^{2j}} > \frac{b}{2}$:

Then

$$\left| \max \left\{ \frac{a}{2^{2j-1}}, \frac{b}{2} \right\} - \max \left\{ \frac{a}{2^{2j}}, b \right\} \right| = \frac{a}{2^{2j-1}} - \frac{a}{2^{2j}} = \frac{a}{2^{2j}}$$

and

$$\max \left\{ \frac{a}{2^{2j-2}}, \frac{b}{2} \right\} = \frac{a}{2^{2j-2}}$$

the inequality (6) is verified.

Case 4: If for some $j \in \mathbb{N}$ we have $\frac{a}{2^{2j-2}} > \frac{b}{2} > \frac{a}{2^{2j-1}} > \frac{a}{2^{2j}}$:

Then

$$\left| \max \left\{ \frac{a}{2^{2j-1}}, \frac{b}{2} \right\} - \max \left\{ \frac{a}{2^{2j}}, b \right\} \right| = b - \frac{b}{2} = \frac{b}{2}$$

and

$$\max \left\{ \frac{a}{2^{2j-2}}, \frac{b}{2} \right\} = \frac{a}{2^{2j-2}}$$

the inequality (6) is verified

Case 5: If for some $j \in \mathbb{N}$ we have $\frac{a}{2^{2j-2}} > \frac{a}{2^{2j-1}} > \frac{b}{2} > \frac{a}{2^{2j}}$: Then we have two cases
 if $\frac{a}{2^{2j-2}} > b > \frac{a}{2^{2j-1}} > \frac{b}{2} > \frac{a}{2^{2j}}$:

$$\left| \max \left\{ \frac{a}{2^{2j-1}}, \frac{b}{2} \right\} - \max \left\{ \frac{a}{2^{2j}}, b \right\} \right| = b - \frac{a}{2^{2j-1}}$$

and

$$\max \left\{ \frac{a}{2^{2j-2}}, \frac{b}{2} \right\} = \frac{a}{2^{2j-2}}$$

Moreover

$$\frac{a}{2^{2j-2}} + \frac{a}{2^{2j-1}} - b \geq 0$$

the inequality (6) is verified

if $b > \frac{a}{2^{2j-2}} > \frac{a}{2^{2j-1}} > \frac{b}{2} > \frac{a}{2^{2j}}$: Then:

$$\left| \max \left\{ \frac{a}{2^{2j-1}}, \frac{b}{2} \right\} - \max \left\{ \frac{a}{2^{2j}}, b \right\} \right| = b - \frac{a}{2^{2j-1}}$$

and

$$\max \left\{ \frac{a}{2^{2j-2}}, \frac{b}{2} \right\} = \frac{a}{2^{2j-2}}$$

Moreover

$$\frac{a}{2^{2j-2}} + \frac{a}{2^{2j-1}} - b = \frac{3a}{2^{2j-1}} - b \geq 0. \quad \left(\frac{a}{2^{2j-1}} > \frac{b}{2} \text{ implies } \frac{2a}{2^{2j-1}} > b \right)$$

the inequality (6) is verified

Let us prove that:

$$(7) \quad |\theta(x_n, x_{m-1}) - \theta(x_m, x_{n-1})| \leq \theta(x_m, x_{m-1})$$

for all $n, m \in \mathbb{N}$ with $n \geq m$, In fact, we have:

$$\theta(x_n, x_{m-1}) = \max \left\{ \frac{a}{2^n}, \frac{a}{2^{m-1}} \right\} \quad \text{and} \quad \theta(x_{n-1}, x_m) = \max \left\{ \frac{a}{2^{n-1}}, \frac{a}{2^m} \right\} + 1$$

,

$$\theta(x_m, x_{m-1}) = \max \left\{ \frac{a}{2^m}, \frac{a}{2^{m-1}} \right\} + 1$$

then

$$|\theta(x_n, x_{m-1}) - \theta(x_m, x_{n-1})| = \left| \frac{a}{2^{m-1}} - \frac{a}{2^m} - 1 \right| = \left| \frac{a}{2^{m-1}} - 1 \right|$$

and

$$\theta(x_m, x_{m-1}) = \frac{a}{2^{m-1}} + 1$$

which implies (7) is verified. Therefore, all conditions of Theorem are satisfied hence Y has a best proximity point x^* in $D \cup E$.

Corollary 2.5. *Let $\emptyset \neq D, E$ a subsets of 0-complete partial metric space (Λ, θ) . and $Y: \Lambda \rightarrow \Lambda$ a mapping. If there exist $q \in [0, 1)$ such that*

$$(8) \quad \theta(Yx, Y\xi) \leq q\{\theta(x, \xi) + |\theta(\xi, Yx) - \theta(x, Y\xi)|\}$$

and for every sequence $\{x_n\} \subseteq D \cup E$ defined by $x_{n+1} = Yx_n$ with the initial point $x_0 \in \Lambda$, such that

$$|\theta(x_{n+1}, x_{n-1}) - \theta(x_n, x_n)| \leq \theta(x_{n-1}, x_n) - \theta(x_n, x_{n+1}),$$

for all $n \in \mathbb{N}$, and,

$$(9) \quad |\theta(x_{2n-1}, Yx^*) - \theta(x_{2n}, x^*)| \leq \theta(x_{2n-2}, Yx^*),$$

for all $n \in \mathbb{N}$, $x^ \in D$ and*

$$(10) \quad |\theta(x_n, x_{m-1}) - \theta(x_m, x_{n-1})| \leq \theta(x_m, x_{m-1})$$

for all $n, m \in \mathbb{N}$, then Y has a fixed point.

Proof. Let (Λ, θ) be a 0-complete partial metric space. If we take $D = E = \Lambda$ and $\theta(D, E) = 0$, taking into Remark 1.11 we can say that (D, E) is a cyclically 0-complete pair. Additionally, from inequality (3), Y is p^* -cyclic contraction mapping. Since all hypotheses of Theorem 2.4 are satisfied, we conclude that there exists a point x^* such that

$$\theta(x^*, Yx^*) = \theta(D, E) = 0$$

which implies that $x^* = Yx^*$. □

Now, we give a theorem of existence of best proximity point in the case (D, E) is a cyclically 0-complete pair.

Proposition 2.6. *Let $\emptyset \neq D, E$ a subsets of a partial metric space (Λ, θ) . If $Y: D \cup E \rightarrow D \cup E$ verify there exists $q \in [0, 1)$ such that:*

$$\theta(Yx, Y\xi) \leq \frac{q}{3}\{\theta(x, \xi) + \theta(x, Yx) + \theta(\xi, Y\xi)\} + (1 - q)\theta(D, E) \quad \text{for all } x \in D, \xi \in E$$

, where $\theta(D, E) = \inf\{\theta(x, \xi), x \in D, \xi \in E\}$. If for every sequence $\{x_n\}$ defined by $x_{n+1} = Yx_n$ with the initial point $x_0 \in D$, is bounded.

Proof. We proceed as in the article [21].

We show in the theoreme below that $\theta(x_{n+2}, x_{n+1}) \rightarrow \theta(D, E)$. Hence the sequence $\{\theta(x_{2n}, x_{2n+1})\}$ converges to $\theta(D, E)$, and so the sequence $\{\theta(x_{2n}, x_{2n+1})\}$ is bounded. Then, there exists $L > 0$ such that:

$$\theta(x_{2n}, x_{2n+1}) \leq L$$

for all $n \in \mathbb{N}$. we have:

$$\begin{aligned} \theta(x_{2n+1}, x_1) &\leq \theta(x_{2n+1}, x_{2n+2}) + \theta(x_{2n+2}, x_1) \\ &\leq L + \theta(x_{2n+2}, x_1) \\ &= L + \theta(Yx_{2n+1}, Yx_0) \\ &\leq L + q\theta(x_{2n+1}, x_0) + q\theta(x_{2n+1}, x_{2n+2}) + q\theta(x_1, x_0) + (1-q)\theta(D, E); \quad (\text{Since } \frac{q}{3} \leq q) \\ &\leq 2L + q\theta(x_{2n+1}, x_1) + 2q\theta(x_0, x_1) + (1-q)\theta(D, E) \end{aligned}$$

for each $n \in \mathbb{N}$, wich implies that: $\theta(x_{2n+1}, x_1) \leq \frac{2L}{1-q} + \frac{2q}{1-q}\theta(x_0, x_1) + \theta(D, E)$

Let

$$M = \frac{2L}{1-q} + \frac{2q}{1-q}\theta(x_0, x_1) + \theta(D, E).$$

Hence, $\{x_{2n+1}\}$ is bounded. Additionally, we get

$$\theta(x_{2n}, x_1) \leq \theta(x_{2n}, x_{2n+1}) + \theta(x_{2n+1}, x_1) \leq L + M.$$

Hence, $\{x_{2n}\}$ is bounded, and so $\{x_n\}$ is bounded. □

Theorem 2.7. Let $\emptyset \neq D, E$ a subsets of a partial metric space (Λ, θ) where (D, E) is a cyclically 0- complete pair. If $Y : D \cup E \rightarrow D \cup E$ verify there exists $q \in [0, 1)$ such that:

$$\theta(Yx, Y\xi) \leq \frac{q}{3}\{\theta(x, \xi) + |\theta(\xi, Yx) - \theta(x, Y\xi)|\} + (1-q)\theta(D, E) \quad \text{for all } x \in D, \xi \in E,$$

where $\theta(D, E) = \inf\{\theta(x, \xi), x \in D, \xi \in E\}$. Then Y has a best proximity.

Proof. Y verify:

$$\theta(Yx, Y\xi) \leq \frac{q}{3}\{\theta(x, \xi) + \theta(\xi, Y\xi) + \theta(x, Yx)\} + (1-q)\theta(D, E)$$

for all $x \in D$ and $\xi \in E$, indeed:

If: $\theta(x, Y\xi) \geq \theta(\xi, Yx)$ then:

$$\theta(Yx, Y\xi) \leq \frac{q}{3}\{\theta(x, \xi) + \theta(x, Y\xi) - \theta(\xi, Yx)\} + (1-q)\theta(D, E)$$

By the triangle inequality, we get:

$$\theta(x, Y\xi) \leq \theta(x, \xi) + \theta(\xi, Y\xi) \text{ and } \theta(\xi, Yx) \geq \theta(\xi, x) - \theta(x, Yx)$$

then:

$$\begin{aligned} \theta(x, Y\xi) - \theta(\xi, Yx) &\leq \theta(x, \xi) + \theta(\xi, Y\xi) - \theta(\xi, x) + \theta(x, Yx) \\ &\leq \theta(\xi, Y\xi) + \theta(x, Yx) \end{aligned}$$

Hence:

$$\begin{aligned} \theta(Yx, Y\xi) &\leq \frac{q}{3}\{\theta(x, \xi) + \theta(x, Y\xi) - \theta(\xi, Yx)\} + (1-q)\theta(D, E) \\ &\leq \frac{q}{3}\{\theta(x, \xi) + \theta(\xi, Y\xi) + \theta(x, Yx) + (1-q)\theta(D, E)\} \end{aligned}$$

If: $\theta(x, Y\xi) \leq \theta(\xi, Yx)$: we deduce by the similar way.

Now let us show that $\theta(x_{n+2}, x_{n+1}) \rightarrow \theta(D, E)$

We have

$$\theta(x_{n+2}, x_{n+1}) \leq \frac{q}{3}\{\theta(x_{n+1}, x_n) + \theta(x_{n+2}, x_{n+1}) + \theta(x_{n+1}, x_n)\} + (1-q)\theta(D, E)$$

which is equivalent to

$$\left(1 - \frac{q}{3}\right)\theta(x_{n+2}, x_{n+1}) \leq \frac{2q}{3}\theta(x_{n+1}, x_n) + (1-q)\theta(D, E)$$

and hence

$$\theta(x_{n+2}, x_{n+1}) \leq \frac{2q}{3-q}\theta(x_{n+1}, x_n) + (1-q)\frac{3}{3-q}\theta(D, E) \quad \text{for } n = 0, 1, 2, \dots$$

inductively, we have:

$$\begin{aligned} &\theta(x_{n+2}, x_{n+1}) \\ &\leq \frac{2q}{3-q}\theta(x_{n+1}, x_n) + (1-q)\frac{3}{3-q}\theta(D, E) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2q}{3-q} \left[\frac{2q}{3-q} \theta(x_n, x_{n-1}) + (1-q) \frac{3}{3-q} \theta(D, E) \right] + (1-q) \frac{3}{3-q} \theta(D, E) \\
&\leq \left(\frac{2q}{3-q} \right)^2 \theta(x_n, x_{n-1}) + \left(\frac{2q}{3-q} \right) \left((1-q) \frac{3}{3-q} \right) \theta(D, E) + (1-q) \frac{3}{3-q} \theta(D, E) \\
&\leq \left(\frac{2q}{3-q} \right)^2 \left[\frac{2q}{3-q} \theta(x_{n-1}, x_{n-2}) + (1-q) \frac{3}{3-q} \theta(D, E) \right] \\
&\quad + \left(\frac{2q}{3-q} \right) \left((1-q) \frac{3}{3-q} \right) \theta(D, E) + (1-q) \frac{3}{3-q} \theta(D, E) \\
&\leq \left(\frac{2q}{3-q} \right)^3 \theta(x_{n-1}, x_{n-2}) + \left(\frac{2q}{3-q} \right)^2 \left((1-q) \frac{3}{3-q} \right) \theta(D, E) \\
&\quad + \left(\frac{2q}{3-q} \right) \left((1-q) \frac{3}{3-q} \right) \theta(D, E) + (1-q) \frac{3}{3-q} \theta(D, E) \\
&\leq \left(\frac{2q}{3-q} \right)^{n+1} \theta(x_1, x_0) \\
&\quad + \left[\left(\frac{2q}{3-q} \right)^n + \dots + \left(\frac{2q}{3-q} \right)^2 + \left(\frac{2q}{3-q} \right) + 1 \right] \left((1-q) \frac{3}{3-q} \right) \theta(D, E)
\end{aligned}$$

As $n \rightarrow \infty$, the expression $\left(\frac{2q}{3-q} \right)^{n+1} \theta(x_1, x_0) \rightarrow 0$ and $\sum \left(\frac{2q}{3-q} \right)^n \rightarrow \frac{3-3q}{3-q}$. Thus

$$\theta(Yx_{n+1}, x_{n+1}) = \theta(x_{n+2}, x_{n+1}) \rightarrow \theta(D, E)$$

Now, let us show that $\{x_n\}$ a cyclically Cauchy sequence. Assume $n, m \in \mathbb{N}$ with $m \geq n$.

Since Y verify the contraction condition, we get

$$\begin{aligned}
&\theta(x_m, x_n) \\
&\leq \frac{q}{3} \{ \theta(x_{m-1}, x_{n-1}) + \theta(x_{m-1}, x_m) + \theta(x_{n-1}, x_n) \} + (1-q) \theta(D, E) \\
&\leq \frac{q}{3} \theta(x_{m-1}, x_{n-1}) + \frac{q}{3} \theta(x_{m-1}, x_m) + \frac{q}{3} \theta(x_{n-1}, x_n) + (1-q) \theta(D, E) \\
&\leq \frac{q}{3} \left\{ \frac{q}{3} \theta(x_{m-2}, x_{n-2}) + \frac{q}{3} \theta(x_{m-2}, x_{m-1}) + \frac{q}{3} \theta(x_{n-2}, x_{n-1}) + (1-q) \theta(D, E) \right\} \\
&\quad + \frac{q}{3} \theta(x_{m-1}, x_m) + \frac{q}{3} \theta(x_{n-1}, x_n) + (1-q) \theta(D, E) \\
&\leq \left(\frac{q}{3} \right)^2 \theta(x_{m-2}, x_{n-2}) + \left(\frac{q}{3} \right)^2 \theta(x_{m-2}, x_{m-1}) + \left(\frac{q}{3} \right)^2 \theta(x_{n-2}, x_{n-1}) \\
&\quad + \frac{q}{3} \theta(x_{m-1}, x_m) + \frac{q}{3} \theta(x_{n-1}, x_n) + \frac{q}{3} (1-q) \theta(D, E) + (1-q) \theta(D, E) \\
&\leq \left(\frac{q}{3} \right)^3 \theta(x_{m-3}, x_{n-3}) + \left(\frac{q}{3} \right)^3 \theta(x_{m-3}, x_{m-2}) + \left(\frac{q}{3} \right)^2 \theta(x_{m-2}, x_{m-1}) + \frac{q}{3} \theta(x_{m-1}, x_m)
\end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{q}{3}\right)^3 \theta(x_{n-3}, x_{n-2}) + \left(\frac{q}{3}\right)^2 \theta(x_{n-2}, x_{n-1}) + \frac{q}{3} \theta(x_{n-1}, x_n) + \frac{q}{3} (1-q) \theta(D, E) \\
 & + (1-q) \theta(D, E) + \left(\frac{q}{3}\right)^2 (1-q) \theta(D, E) \\
 & \leq \left(\frac{q}{3}\right)^n \theta(x_{m-n}, x_0) + \sum_{i=1}^n \left(\frac{q}{3}\right)^i \theta(x_{m-i}, x_{m-i+1}) + \sum_{i=1}^n \left(\frac{q}{3}\right)^i \theta(x_{n-i}, x_{n-i+1}) \\
 & + \sum_{i=1}^{n-1} \left(\frac{q}{3}\right)^i (1-q) \theta(D, E)
 \end{aligned}$$

Since $\{x_n\}$ is bounded, then $\left(\frac{q}{3}\right)^n \theta(x_{m-n}, x_0) \rightarrow 0$, Since $\theta(x_n, x_{n+1}) \rightarrow \theta(D, E)$, then for all $\varepsilon > 0$ there exist a n_0 such that for all $n \geq n_0$

$$(11) \quad \theta(x_n, x_{n+1}) < \theta(D, E) + \varepsilon,$$

then

$$\sum_{i=1}^n \left(\frac{q}{3}\right)^i \theta(x_{n-i}, x_{n-i+1}) + \sum_{i=1}^n \left(\frac{q}{3}\right)^i \theta(x_{m-i}, x_{m-i+1}) \leq 2 \sum_{i=1}^n \left(\frac{q}{3}\right)^i \theta(D, E) + 2 \sum_{i=1}^n \left(\frac{q}{3}\right)^i \varepsilon$$

for all $n \geq n_0$, moreover we have:

$$2 \sum_{i=1}^n \left(\frac{q}{3}\right)^i \theta(D, E) \rightarrow \frac{2q}{3-q} \theta(D, E)$$

and,

$$\sum_{i=1}^{n-1} \left(\frac{q}{3}\right)^i (1-q) \theta(D, E) \rightarrow \frac{3(1-q)}{3-q} \theta(D, E)$$

yeilds

$$\theta(x_m, x_n) \leq \frac{3(1-q)}{3-q} \theta(D, E) + \frac{2q}{3-q} \theta(D, E) + \varepsilon \frac{2q}{3-q}$$

for all $n \geq n_0$, which implies that

$$\begin{aligned}
 \theta(x_m, x_n) & \leq \theta(D, E) + \frac{2q}{3-q} \varepsilon \\
 & < \theta(D, E) + \varepsilon
 \end{aligned}$$

for all $n, m \geq n_0$. Hence $\{x_n\}$ a cyclically Cauchy sequence.

Since (D, E) is a cyclically 0-complete pair, without loss of the generality we can assume that $\{x_{2n}\}$ converges to x^* such that

$$(12) \quad \lim_{n,m \rightarrow \infty} \theta(x_{2n}, x_{2m}) = \lim_{n \rightarrow \infty} \theta(x_{2n}, x^*) = \theta(x^*, x^*) = 0$$

Additionally, we have:

$$\begin{aligned} \theta(x_{2n}, Yx^*) &\leq \theta(Yx_{2n-1}, Yx^*) \\ &\leq \frac{q}{3} \{ \theta(x_{2n-1}, x^*) + \theta(x_{2n-1}, x_{2n}) + \theta(Yx^*, x^*) \} + (1-q)\theta(D, E) \end{aligned}$$

Taking limit $n \rightarrow \infty$, in the last inequality we get:

$$\begin{aligned} \theta(x^*, Yx^*) &= \lim_{n \rightarrow \infty} \theta(x_{2n}, Yx^*) \\ &\leq \frac{q}{3} \theta(D, E) + \frac{q}{3} \theta(D, E) + \frac{q}{3} \theta(Yx^*, x^*) + (1-q)\theta(D, E) \\ &= \frac{q}{3} \theta(Yx^*, x^*) + (1 - \frac{q}{3}) \theta(D, E) \end{aligned}$$

Therefore we have, $(1 - \frac{q}{3})\theta(x^*, Yx^*) \leq (1 - \frac{q}{3})\theta(D, E)$, and so $\theta(x^*, Yx^*) \leq \theta(D, E)$, then $\theta(x^*, Yx^*) = \theta(D, E)$. Hence x^* is a best proximity point of Y in D . If $\{x_{2n+1}\}$ converges to ξ^* such that

$$(13) \quad \lim_{n,m \rightarrow \infty} \theta(x_{2n+1}, x_{2m+1}) = \lim_{n \rightarrow \infty} \theta(x_{2n+1}, \xi^*) = \theta(\xi^*, \xi^*) = 0$$

for some $\xi^* \in E$. Then, by the similar way, it can be shown that ξ^* is a best proximity point of Y in E . \square

3. APPLICATION: BEST PROXIMITY POINT THEOREM FOR DIFFERENTIAL INCLUSIONS IN A PARTIAL METRIC SPACE

Let

$$\Lambda = \{u \in C[0, 1] : u(t) \geq 0 \text{ for all } t \in [0, 1]\}$$

and define the partial metric $\theta : \Lambda \times \Lambda \rightarrow [0, \infty)$ by

$$\theta(u, v) = \begin{cases} \sup_{t \in [0, 1]} u(t), & \text{if } u = v, \\ \sup_{t \in [0, 1]} \{u(t) + v(t)\}, & \text{if } u \neq v. \end{cases}$$

Fix $\beta > 0$, and set $\alpha := \frac{3}{2}\beta$ and $c := \beta$. Define

$$D = \left\{ u \in \Lambda : \beta < \sup_{t \in [0,1]} u(t) \leq \alpha \right\}, \quad E = \left\{ u \in \Lambda : 0 \leq \sup_{t \in [0,1]} u(t) \leq \beta \right\}.$$

Then $D \cap E = \emptyset$ and $\theta(D, E) = \beta$ (as an infimum over $D \times E$).

Consider the differential inclusion

$$(14) \quad \dot{x}(t) \in F(x(t)), \quad t \in [0, 1],$$

where $F : [0, \alpha] \rightrightarrows [0, \infty)$ is a set-valued map with nonempty values. Assume the following (standard) regularity and growth conditions hold:

Assume the following (standard) regularity and growth conditions hold:

(Sel) For each $u \in \Lambda$ there exists a (Bochner) measurable selection $f_u : [0, 1] \rightarrow [0, \infty)$ with $f_u(t) \in F(u(t))$ a.e.

(Gr) There exists $\lambda \in (0, 1)$ (in what follows we take $\lambda = \frac{1}{2}$) such that for every $u \in \Lambda$ and the selection f_u in **(Sel)**,

$$0 \leq f_u(t) \leq \lambda \sup_{r \in [0,1]} u(r) \quad \text{for a.e. } t \in [0, 1].$$

Define the operator $Y : \Lambda \rightarrow \Lambda$ by $(Yu)(t) := \int_0^t f_u(s) ds$.

Define the *cyclic extension* $\tilde{Y} : \Lambda \rightarrow \Lambda$ by

$$\tilde{Y}(u)(t) = \begin{cases} (Yu)(t), & u \in D, \\ (Yu)(t) + c, & u \in E. \end{cases}$$

Theorem 3.1 (Best proximity for the inclusion-induced cyclic map). *Assume **(Sel)** and **(Gr)** with $\lambda = \frac{1}{2}$. Suppose moreover:*

(i) (Shrinkage) *For all $u \in \Lambda$,*

$$\sup_{t \in [0,1]} (Yu)(t) \leq \lambda \sup_{t \in [0,1]} u(t) \quad (\lambda = \frac{1}{2}).$$

(ii) (p^* -cyclic contractivity) *There exists $q \in [0, 1)$ such that for all $u \in D, v \in E$ and all $t, s \in [0, 1]$,*

$$(Yu)(t) + (Yv)(t) \leq q \left(u(s) + v(s) + \left| \sup_r \{u(r) + (Yv)(r)\} - \sup_r \{(Yu)(r) + v(r)\} \right| \right) - 2c + (1 - q)\beta.$$

Equivalently (by taking suprema and using the definition of \tilde{Y}),

$$(15) \quad \theta(\tilde{Y}u, \tilde{Y}v) \leq q \left(\theta(u, v) + |\theta(\tilde{Y}u, v) - \theta(u, \tilde{Y}v)| \right) + (1 - q)\beta, \quad u \in D, v \in E.$$

(iii) (Iterative reduction) For every sequence $U_{n+1} = \tilde{Y}U_n$ with $U_0 \in \Lambda$,

$$U_{n+1}(t) \leq \frac{1}{2}U_n(t) \quad \forall n \in \mathbb{N}.$$

(iv) E is boundedly compact in (Λ, θ) (hence the cyclic pair (D, E) is cyclically complete).

(v) for all $n \in \mathbb{N}$, $x^* \in D$,

$$\beta + \theta(U_{2n}, x^*) \leq \theta(U_{2n-2}, Yx^*) + \theta(U_{2n-1}, Yx^*),$$

Then \tilde{Y} admits a best proximity point between D and E : there exists $x^* \in D$ such that

$$\theta(x^*, \tilde{Y}x^*) = \theta(D, E) = \beta.$$

Proof. By (i) and the choice $\alpha = \frac{3}{2}\beta$, if $u \in D$ then

$$\sup_t \tilde{Y}(u)(t) = \sup_t (Yu)(t) \leq \lambda \sup_t u(t) \leq \frac{1}{2} \cdot \frac{3}{2}\beta = \frac{3}{4}\beta < \beta,$$

so $\tilde{Y}(D) \subset E$. If $v \in E$ then

$$\sup_t \tilde{Y}(v)(t) = \sup_t ((Yv)(t) + c) \leq \lambda\beta + \beta = \frac{3}{2}\beta = \alpha,$$

and since $\lambda > 0$ we also have $\sup_t \tilde{Y}(v)(t) > \beta$, hence $\tilde{Y}(E) \subset D$. Thus \tilde{Y} is cyclic on (D, E) .

Assumption (ii) is precisely the p^* -cyclic contractivity in (15) with respect to the given partial metric θ . Assumption (iii) yields a uniform decay of sup-norms along the orbit of \tilde{Y} , which implies boundedness of the orbit. By (iv), the odd (resp. even) subsequences of the Picard iteration admit convergent subsequences in E (resp. in D). Assumption (i), (v) and (iii) ensure all conditions of theorem 2.3. The p^* -cyclic contractivity ensures that any two such limits (x^*, y^*) satisfy $\theta(x^*, y^*) = \beta$ with $y^* = \tilde{Y}x^*$, which is the best proximity property. Hence there exists $x^* \in D$ with $\theta(x^*, \tilde{Y}x^*) = \beta$. \square

Remark 3.2. (i) The theorem does not assert that x^* solves (14). It asserts that x^* is a best proximity point for the inclusion-induced cyclic map \tilde{Y} .

(ii) If one seeks an actual solution of (14) from a best proximity point, an additional alignment

hypothesis relating θ and the algebra of Y is needed; see Corollary 3.3 below.

(iii) The selection and growth hypotheses **(Sel)**–**(Gr)** are classical (Carathéodory-type) conditions ensuring that $Y : \Lambda \rightarrow \Lambda$ is well-defined and that **(i)** holds with the stated λ .

Additional alignment hypothesis and solution extraction. To turn a best proximity point into a solution of (14) one may impose:

(A) (Alignment) If $u \in D$ and $\theta(u, Yu) = \beta$, then

$$u(t) = c + (Yu)(t) \quad \text{for all } t \in [0, 1].$$

(This is an explicit structural link between the geometry of θ and the additive relation.)

Corollary 3.3 (From best proximity to an inclusion solution). *Assume the hypotheses of Theorem 3.1 and, in addition, (A). Then the best proximity point $x^* \in D$ produced by Theorem 3.1 satisfies*

$$x^*(t) = c + (Yx^*)(t) = c + \int_0^t f_{x^*}(s) ds$$

for the selection f_{x^*} of **(Sel)**. In particular, x^* is absolutely continuous with $\dot{x}^*(t) = f_{x^*}(t) \in F(x^*(t))$ a.e. on $[0, 1]$, so x^* solves the differential inclusion (14) with $x^*(0) = c$.

Proof. From Theorem 3.1 there exists $x^* \in D$ with $\theta(x^*, \tilde{Y}x^*) = \beta$. Since $x^* \in D$, $\tilde{Y}x^* = Yx^*$. Thus $\theta(x^*, Yx^*) = \beta$, and **(A)** yields $x^* = c + Yx^*$ pointwise. The remainder is immediate. \square

Remark 3.4. Hypothesis **(A)** is not automatic for the partial metric θ used here; it must be postulated (or one may adopt a different partial metric that encodes the additive shift by c). Keeping your original θ while adding **(A)** is the most conservative fix.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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