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NONCOMPATIBLE MAPPINGS IN INTUITIONISTIC FUZZY METRIC SPACE

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Abstract. The purpose of this paper is to obtain a common fixed point theorem for non-compatible mappings in Intuitionistic Fuzzy metric space using the concept of R-weak commutativity without assuming the completeness of the space or continuity of the mappings involved. We also find an affirmative answer in Intuitionistic Fuzzy metric space to the problem of Rhoades [7].

Keywords: Intuitionistic fuzzy metric space, Common fixed point, noncompatible maps, R-weak commutativity of the mappings.

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1. Introduction

The foundation of fuzzy Mathematics is laid by Lofti A. Zadeh [10] with the introduction of fuzzy sets in 1965, as a way to represent vagueness in everyday life. Atanassov [3] introduced and studied the concept of intuitionistic fuzzy set as a generalization of fuzzy sets. In 2004, Park [6] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms. Recently, in 2006, Alaca et. al.[2] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [5]. In 2006, Turkoglu [9] proved Jungck's [4] common fixed point theorem in the setting of intuitionistic fuzzy metric space for commuting mappings. Afterwards, many author proved common fixed point theorem using different variants in such spaces. In this paper, our objective is to prove a common fixed point theorem for Noncompatible self-maps satisfying the property (E.A) using the notion of R-weakly commuting maps of type (A_g).

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2. Preliminaries

Definition 2.1. A binary operation $*$: $[0,1] \times [0,1] \to [0,1]$ is continuous t – norm if $*$ is satisfying the following conditions :

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all $a \in [0, 1]$;

(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.2. A binary operation $\Diamond$: $[0, 1] \times [0, 1] \to [0, 1]$ is continuous t – conorm if $\Diamond$ is satisfying the following conditions :

(i) $\Diamond$ is commutative and associative;

(ii) $\Diamond$ is continuous;

(iii) $a \Diamond 0 = a$ for all $a \in [0, 1]$;

(iv) $a \Diamond b \le c \Diamond d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3. [2] A 5 – tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t – norm, $\Diamond$ is a continuous t – conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions :

(i) $M(x, y, t) + N(x, y, t) \le 1$ for all $x, y \in X$ and $t > 0$;

(ii) $M(x, y, 0) = 0$ for all $x, y \in X$;

(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(v) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(vi) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;

(vii) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;

(viii) $N(x, y, 0) = 1$ for all $x, y \in X$;

(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(xi) $N(x, y, t) \Diamond N(y, z, s) \ge N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(xii) for all $x, y \in X$, $N(x, y, .) : [0, \infty) \to [0, 1]$ is right continuous;

(xiii) $\lim_{t \to \infty} N(x, y, t) = 0$ for all x, y in X :

The functions M(x, y, t) and N(x, y, t) denote the degree of nearness and the degree of non – nearness between x and y with respect to t, respectively.

Example 2.1. Let $X = \{\frac{1}{n} : n \in N\} \cup \{0\}$ with $*$ continuous t – norm and $\Diamond$ continuous t – conorm defined by $a * b = ab$ and $a \Diamond b = \min\{1, a+b\}$ respectively, for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ and $x, y \in X$, define M and N by

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0, \\ 0 & t = 0 \end{cases} \quad \text{and} \quad N(x,y,t) = \begin{cases} \dfrac{|x-y|}{t+|x-y|}, & t > 0, \\ 1 & t = 0. \end{cases}$$

Then, $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space, (for k = 1).

Definition 2.4. Let A and B be two self mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$. Then the maps A and B are said to be compatible if for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 2.5. Self maps A and S of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ are said to be R-weaklycommuting if there exists some real number R such that
$M(ASx, SAx, t) \geq M(Ax, Sx, t/R)$ and $N(ASx, SAx, t) \leq N(Ax, Sx, t/R)$ for each $x \in X$ and $t > 0$.

Definition 2.6. Self maps A and S of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ are said to be R-weakly commuting of type (A_g) if there exists some real number R such that
$M(AAx, SAx, t) \geq M(Ax, Sx, t/R)$ and $N(AAx, SAx, t) \leq N(Ax, Sx, t/R)$ for each $x \in X$ and $t > 0$.

Definition 2.7. Let A and B be two self mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$. Then the maps A and B satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Remark 2.1. From definition 2.4, it is inferred that two maps A and B on an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ are non-compatible if and only if there exists atleast one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$, but for some $t > 0$ either $\lim_{n\to\infty} M(ABx_n, BAx_n, t) \neq 1$ and $\lim_{n\to\infty} N(ABx_n, BAx_n, t) \neq 0$ or the limit does not exist.

3. Main Results

Theorem 3.1. Let f and g be pointwise R-weakly commuting self mappings of an Intuitionistic Fuzzy metric space $(X, M, N, *, \Diamond)$ satisfying the property (E.A) and :

(3.1) $f(X) \subset g(X)$

(3.2) $M(fx, fy, kt) \geq M(gx, gy, t)$ and $N(fx, fy, kt) \leq N(gx, gy, t)$, $k \geq 0$

(3.3) $$M(fx, f^2x, t) > \max\{M(gx, gfx, t), M(fx, gx,t), M(f^2x, gfx, t), M(fx, gfx, t), M(gx, f^2x, t)\}$$ and

$$N(fx, f^2x, t) < \min\{N(gx, gfx, t), N(fx, gx,t), N(f^2x, gfx, t), N(fx, gfx, t), N(gx, f^2x, t)\},$$ whenever $fx \neq f^2x$.

If the range of f or g is a complete subspace of X, then f and g have a common fixed point.

Proof. Since f and g satisfy the property (E.A.), then there exists a sequence $\{x_n\}$ such that $fx_n \to p$ and $gx_n \to p$, for some $p \in X$ as $n \to \infty$. Since $p \in f(X)$ and $f(X) \subset g(X)$, there exists some point u in X such that $p = gu$, where $p = \lim_n gx_n$. If $fu \neq gu$, then

$$M(fx_n, fu, kt) \geq M(gx_n, gu, t) \text{ and } N(fx_n, fu, kt) \leq N(gx_n, gu, t)$$

Letting $n \to \infty$,

$$M(gu, fu, kt) \geq (M(gu, gu, t) \text{ and } N(gu, fu, kt) \leq N(gu, gu, t)$$

Hence $fu = gu$.

Since f and g are R-weak commuting, there exists $R > 0$ such that

$$M(fgu, gfu, t) \geq (M(fu, gu, t/R) = 1 \text{ and } N(fgu, gfu, t) \leq N(fu, gu, t/R) = 0,$$

that is, fgu = gfu and ffu = gfu = ggu. If fu $\neq$ ffu, using (3.3), we get

$$M(fu, ffu, t) > \max\{M(gu, gfu, t), M(fu, gu, t), M(ffu, gfu, t),$$

$$M(ffu, gfu, t), M(gu, ffu, t)\}= M(fu, ffu, t)$$

and

$$N(fu, ffu, t) < \min\{N(gu, gfu, t), N(fu, gu, t), N(ffu, gfu, t), N(ffu, gfu, t), N(gu, ffu, t)\}$$

$$= N(fu, ffu, t),$$

a contradiction.

Hence, fu = ffu and fu = ffu = fgu = gfu = ggu.

Hence fu is a common fixed point of f and g. The case when f(X) is a complete subspace of X is similar to the above case since f(X) $\subset$ g(X). Hence we have the theorem.

We now give an example to illustrate the above theorem.

Example 3.1. Let X = [2, 20] and and d be the usual metric on X. For each t $\in$ [0, ∞), define (M, N) for x, y $\in$ X by

$$M(x, y, t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0, \\ 0 & t = 0 \end{cases} \quad \text{and} \quad N(x, y, t) = \begin{cases} \dfrac{|x-y|}{t+|x-y|}, & t > 0, \\ 1 & t = 0. \end{cases}$$

Clearly (X, M, N, $*$, $\Diamond$) is an Intuitionistic Fuzzy metric space. Define f, g : X $\to$ X as

$$f(x) = \begin{cases} 2, & \text{if} \quad x = 2 \text{ or } x > 5 \\ 6, & \text{if} \quad 2 < x \leq 5 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2, & \text{if} \quad x = 2 \\ x+4, & \text{if} \quad 2 < x \leq 5 \\ \dfrac{4x+10}{15}, & \text{if} \quad x > 5. \end{cases}$$

Clearly, f and g satisfy all the conditions of Theorem 3.1 and have common fixed point at x = 2.

Also, $f(X) \subset g(X)$ and f and g are pointwise R-weakly commuting mappings and satisfy the (E.A.) property.

Setting k = 1 in the above theorem, we get the following theorem :

Theorem 3.2. Let f and g be pointwise R-weakly commuting self mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ satisfying the property (E.A) and :

(3.4) $f(X) \subset g(X)$

(3.5) $M(fx, fy, t) \geq M(gx, gy, t)$ and $N(fx, fy, t) \leq N(gx, gy, t)$,

(3.6) $M(fx, f^2x, t) > \max\{M(gx, gfx, t), M(fx, gx,t), M(f^2x, gfx, t),$

$$M(fx, gfx, t), M(gx, f^2x, t)\}$$ and

$$N(fx, f^2x, t) < \min\{N(gx, gfx, t), N(fx, gx,t), N(f^2x, gfx, t),$$

$$N(fx, gfx, t), N(gx, f^2x, t)\},$$ whenever $fx \neq f^2x$.

If the range of f or g is a complete subspace of X, then f and g have a common fixed point.

Theorem 3.1 has been proved by using the concept of (E.A.) property which has been introduced in a recent work by Aamri and Moutawakil [1]. They have shown that (E.A.) property is more general than the notion of non-compatibility. It may, however, be observed that by using the notion of non-compatible maps in place of (E.A.) property, we cannot only prove a Theorem 3.1 above, but, in addition, we are able to show also that maps are discontinuous at their common fixed points. We do this in our next theorem and thus find out an answer in Intuitionistic Fuzzy metric space to the problem of Rhoades [7].

Theorem 3.3. Let f and g be noncompatible pointwise R-weakly commuting self mappings of type (A_g) of an Intuitionistic Fuzzy metric space $(X, M, N, *, \Diamond)$ satisfying :

(3.7) $f(X) \subset g(X)$

(3.8) $M(fx, fy, kt) \geq M(gx, gy, t)$ and $N(fx, fy, kt) \leq N(gx, gy, t)$, $k \geq 0$

(3.9) $M(fx, f^2x, t) > \max\{M(gx, gfx, t), M(fx, gx,t), M(f^2x, gfx, t),$

$$M(fx, gfx, t), M(gx, f^2x, t)\}$$ and

$$N(fx, f^2x, t) < \min\{N(gx, gfx, t), N(fx, gx, t), N(f^2x, gfx, t),$$

$$N(fx, gfx, t), N(gx, f^2x, t)\}, \text{ whenever } fx \neq f^2x.$$

If the range of f or g is a complete subspace of X, then f and g have a common fixed point and the fixed point is the point of discontinuity.

Proof. Since f and g are noncompatible maps, then there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = p \text{ and } \lim_{n\to\infty} gx_n = p, \text{ for some } p \in X, \tag{1}$$

but either $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or the limit does not exist.

Since $p \in f(X)$ and $f(X) \subset g(X)$, there exists some point u in X such that $p = gu$, where $p = \lim_{n\to\infty} gx_n$. If $fu \neq gu$, then

$$M(fx_n, fu, kt) \geq M(gx_n, gu, t) \text{ and } N(fx_n, fu, kt) \leq N(gx_n, gu, t).$$

Letting $n \to \infty$,

$$M(gu, fu, kt) \geq (M(gu, gu, t) \text{ and } N(gu, fu, kt) \leq N(gu, gu, t)$$

Hence $fu = gu$.

Since f and g are R-weak commuting of type (A_g), there exists $R > 0$ such that

$$M(ffu, gfu, t) \geq (M(fu, gu, t/R) = 1 \text{ and } N(ffu, gfu, t) \leq N(fu, gu, t/R) = 0,$$

that is, $fu = gfu$ and $ffu = fgu = gfu = ggu$.

If $fu \neq ffu$, using (3.9),we get

$$M(fu, ffu, t) > \max\{M(gu, gfu, t), M(fu, gu, t), M(ffu, gfu, t), M(ffu, gfu, t), M(gu, ffu, t)\}$$

$$= M(fu, ffu, t)$$

and

$$N(fu, ffu, t) < \min\{N(gu, gfu, t), N(fu, gu, t), N(ffu, gfu, t), N(ffu, gfu, t), N(gu, ffu, t)\}$$

$$= N(fu, ffu, t),$$

a contradiction.

Hence, fu = ffu and fu = ffu = fgu = gfu = ggu.

Hence fu is a common fixed point of f and g. The case when f(X) is a complete subspace of X is similar to the above case since $f(X) \subset g(X)$. We now show that f and g are discontinuous at the common fixed point p = fu = gu. If possible, suppose f is continuous. Then considering the sequence $\{x_n\}$ of (1) we get $\lim_{n\to\infty} ffx_n = fp = p$. R-weak commutativity of type (A_g) implies that $M(ffx_n, gfx_n, t) \geq M(fx_n, gx_n, t/R) = 1$ and $N(ffx_n, gfx_n, t) \leq N(fx_n, gx_n, t/R) = 0$, which on letting $n \to \infty$ this yields $\lim_{n\to\infty} gfx_n = fp = p$. This, in turn, yields $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ and $\lim_{n\to\infty} N(fgx_n, gfx_n, t) = 0$. This contradicts the fact that $\lim_{n\to\infty} M(fgx_n, gfx_n, t)$ is either nonzero or nonexistent for the sequence $\{x_n\}$ of (1). Hence f is discontinuous at fixed point. Similarly, g is also discontinuous at fixed point.

Thus, both f and g are discontinuous at their common fixed point. Hence we have the theorem.

We now give an example to illustrate the above theorem.

Example 3.2. Let X = [2, 20] and and d be the usual metric on X. For each $t \in [0, \infty)$, define (M, N) for $x, y \in X$ by

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0, \\ 0 & t = 0 \end{cases} \quad \text{and} \quad N(x,y,t) = \begin{cases} \dfrac{|x-y|}{t+|x-y|}, & t > 0, \\ 1 & t = 0. \end{cases}$$

Clearly $(X, M, N, *, \lozenge)$ is an Intuitionistic Fuzzy metric space. Define $f, g : X \to X$ as

$$f(x) = \begin{cases} 2, & \text{if} \quad x = 2 \text{ or } x > 5 \\ 6, & \text{if} \quad 2 < x \leq 5 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2, & \text{if} \quad x = 2 \\ 7, & \text{if} \quad 2 < x \leq 5 \\ \dfrac{4x+10}{15}, & \text{if} \quad x > 5. \end{cases}$$

Clearly, f and g satisfy all the conditions of Theorem 3.3 and have common fixed point at x = 2. Also, $f(X) \subset g(X)$ and f and g are pointwise R-weakly commuting mappings of type (A_g).

Consider a sequence $x_n = 5 + \frac{1}{n}$, $n > 1$, then

$$\lim_{n\to\infty} fx_n = 2,\ \lim_{n\to\infty} gx_n = 2,\ \lim_{n\to\infty} fgx_n = 6 \text{ and } \lim_{n\to\infty} gfx_n = 2.$$

Hence, f and g are noncompatible.

Remark. Aamri and Moutawakil [1] have shown that the property (E.A.) introduced by them is more general than the notion of noncompatibility. It is, however, worth to mention here that if we take the noncompatibility aspect instead of property (E.A.) we can show, in addition, that the mappings are discontinuous at the common fixed point. Aforesaid results illustrate our assertion in the Intuitionisitic Fuzzy metric fixed point theory. This is, however, also true for the study of fixed points in metric space.

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