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SOME APPROXIMATE BEST PROXIMITY POINT RESULTS ON METRIC SPACES AND ITS APPLICATIONS

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Abstract. The aim of this paper is to extend the concept of determining an approximate fixed points (*AFP*) to determine approximate best proximity points (*ABPP*) using several contraction mappings, such as weak contraction, Zamfirescu contraction, Ćirić-Reich-Rus contraction and the related consequences in metric spaces. In particular, we study the existence (qualitative results) and the diameter (quantitative results) of *ABPP* on metric spaces. Moreover, a few examples are provided to illustrate our results. Furthermore, suitable applications of the main findings are discussed in the domain of differential equations.

Keywords: metric spaces; weak contraction; Zamfirescu contraction; approximate best proximity point.

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1. INTRODUCTION

In recent decades, nonlinear functional analysis-particularly best proximity point (*BPP*) theory-has been extended to a variety of abstract spaces. It has been widely applied to numerical scientific problems, bridging both applied and pure mathematical methods, especially in relation to computational complexities. Further, *BPP* theory has played a typical role in modeling and analyzing a broad range of real-world phenomena and applications, including the study of integral and differential equations, as well as in physics, economics, social sciences, biology, and the field of engineering.

In 1922, one of the most influential and fundamental results in this area, known as Banach contraction principle [*BCP*] was established by Stefan Banach [1]. Since then, the field of fixed point [*FP*] theory has witnessed significant developments and wide-ranging applications. The requirements of *FP*'s existence are, of course, extremely stringent. Therefore, it is not certain that *FP*'s will always exist. Because the *FP* approaches have too stringent constraints, the *BPP* may be employed in the absence of an accurate *FP*. This is the main motivation behind trying to find *BPP* on metric spaces. When a direct solution is not feasible, especially for non-self mappings, *BPP* theory, a generalization of *FP* theory, is essential for locating the best approximation solutions to the equation $\nabla\mathbb{k} = \mathbb{k}$, where $\nabla : \nabla^* \rightarrow \nabla^*$ are disjoint subsets of a metric space. In the similar way, the *AFP*, was also developed and the approximate solutions (only ε -differences) are determined in many complicated situations. Thus, over the years, a number of academics have studied *AFP* and *ABPP* theory in kinds of extended metric spaces.

$$FP's \implies AFP's \implies BPP's \implies ABPP's$$

Notably, the author Berinde [2] proved existence and the diameter of *AFP* results using various contraction-type operators (Kannan, Chatterjea, Zamfirescu, and weak contractions) on metric spaces. Later, Dey and Saha [3] extended these results, and they found the diameter of the *AFP* for the Reich operator tends to zero when ε approaches zero. In the same manner, S. A. M. Mohsenialhosseini (see, [4],[5],[6]) derived some new *AFP* results for cyclical contraction mappings and extended these results to a family of contraction mappings and found a common *FP* for the Mohseni-Saheli contraction mapping. Following that, Theivaraman et al. extended the concept and determined more fruitful *AFP* and *ABPP* results on metric spaces,

b -metric spaces, G -metric spaces and the related subsequences using several contraction mappings (refer to [8],[9],[10],[11], [12], [13], [14], [15]). In addition to that, there has been interest in extending the Banach FP theorem to address the convergence of measurable functions with respect to measure, thereby enhancing the relevance of FP theory in the broader context of analysis and applied mathematics.

The paper is structured in the following manner. An overview of the study and its motivation are given in Section 1. In Section 2, we revisit fundamental concepts and relevant results from the existing literature that form the basis for our work. Section 3 is devoted to the presentation of our main contributions, where we establish new results concerning $ABPP$ results in the setting of metric spaces. These results are derived using various types of contraction mappings, including Zamfirescu contractions, weak contractions, and others. A particular focus is placed on analyzing the diameter of $ABPP$'s for a given pair $(\mathfrak{S}, \mathfrak{R})$, drawing upon and extending the frameworks provided in [11]. In Section 4, we illustrate the application of our major findings by examining their consequences in connection to differential equations. The paper is finally concluded in Section 5 with a review of the results and future directions for upcoming studies.

2. PRELIMINARIES

This section reviews essential definitions and lemmas from earlier studies, which serve as foundational tools for the main results presented in the remainder of this manuscript.

Definition 2.1. [4],[6] *Let \mathfrak{S} and \mathfrak{R} be two nonempty subsets of a metric space \mathfrak{D} and $\nabla : \mathfrak{S} \cup \mathfrak{R} \rightarrow \mathfrak{R} \cup \mathfrak{S}$ such that $\nabla(\mathfrak{S}) \subseteq \mathfrak{R}$ and $\nabla(\mathfrak{R}) \subseteq \mathfrak{S}$. Then \hbar is said to be an approximate best proximity point of the pair $(\mathfrak{S}, \mathfrak{R})$, if*

$$d(\hbar, \nabla \hbar) \leq d(\mathfrak{S}, \mathfrak{R}) + \varepsilon.$$

Remark 2.2. [4],[6] *Let*

$$BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) = \{\hbar \in BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) : d(\hbar, \nabla \hbar) < d(\mathfrak{S}, \mathfrak{R}) + \varepsilon, \text{ for some } \varepsilon > 0\}$$

be denotes the set of all approximate best proximity point of pair $(\mathfrak{S}, \mathfrak{R})$ for a given $\varepsilon > 0$. Also, the pair $(\mathfrak{S}, \mathfrak{R})$ is said to be an approximate best proximity pair property, if

$$d(\hbar, \nabla \hbar) \leq d(\mathfrak{S}, \mathfrak{R}) \neq 0.$$

Example 2.3. Let us take $\mathfrak{D} = \mathbb{R}^2$ and $\mathfrak{S} = \{(\hbar, \iota) \in \mathfrak{D} : (\hbar - \iota)^2 + \iota^2 \leq 1\}$ and $\mathfrak{R} = \{(\hbar, \iota) \in \mathfrak{D} : (\hbar + \iota)^2 + \iota^2 \leq 1\}$ with $B(\hbar, \iota) = (-\hbar, \iota)$ for $(\hbar, \iota) \in \mathfrak{D}$. Then

$$d((\hbar, \iota), B(\hbar, \iota)) \leq d(\mathfrak{S}, \mathfrak{R}) + \varepsilon \text{ for some } \varepsilon > 0.$$

Hence,

$$BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) \neq \emptyset.$$

Theorem 2.4. [4],[6] Let \mathfrak{S} and \mathfrak{R} be two nonempty subsets of a metric space (\mathfrak{D}, d) . Suppose that the mapping $\nabla : \mathfrak{S} \cup \mathfrak{R} \rightarrow \mathfrak{S} \cup \mathfrak{R}$ satisfying $\nabla(\mathfrak{S}) \subseteq \mathfrak{R}$ and $\nabla(\mathfrak{R}) \subseteq \mathfrak{S}$ and

$$\lim_{n \rightarrow \infty} d(\nabla^n \mathfrak{S}, \nabla^{n+1} \mathfrak{S}) = d(\mathfrak{S}, \mathfrak{R}), \text{ for some } \hbar \in (\mathfrak{S} \cup \mathfrak{R}).$$

Then the pair $(\mathfrak{S}, \mathfrak{R})$ is called an approximate best proximity pair.

Definition 2.5. [4],[6] Let $\nabla : \mathfrak{S} \cup \mathfrak{R} \rightarrow \mathfrak{S} \cup \mathfrak{R}$ be a continuous map such that $\nabla(\mathfrak{S}) \subseteq \mathfrak{R}$, $\nabla(\mathfrak{R}) \subseteq \mathfrak{S}$ and $\varepsilon > 0$. Then, we define the diameter $\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}))$, i.e.,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) = \sup\{d(\hbar, \iota) : \hbar, \iota \in BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})\}.$$

Theorem 2.6. [4],[6] Let \mathfrak{S} and \mathfrak{R} be two non-empty subsets of a metric space (\mathfrak{D}, d) . Suppose that a mapping $\nabla : \mathfrak{S} \cup \mathfrak{R} \rightarrow \mathfrak{S} \cup \mathfrak{R}$ satisfying $\nabla(\mathfrak{S}) \subseteq \mathfrak{R}$, $\nabla(\mathfrak{R}) \subseteq \mathfrak{S}$ is a α -contraction and $\varepsilon > 0$. Suppose that:

(i) $BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) \neq \emptyset$;

(ii) for every $\varphi > 0$, there exists $\theta(\varphi) > 0$ such that $d(\hbar, \iota) - d(\nabla \hbar, \nabla \iota) \leq \varphi$ implies that

$$d(\hbar, \iota) \leq \theta(\varphi), \text{ for every } \hbar, \iota \in BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) \neq \emptyset.$$

Then,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq \theta(2d(\mathfrak{S}, \mathfrak{R}) + \varepsilon).$$

Remark 2.7. Consider \mathfrak{S} and \mathfrak{R} be two non-empty subsets of a metric space (\mathfrak{D}, d) and a selfmap $\nabla : \mathfrak{S} \cup \mathfrak{R} \rightarrow \mathfrak{S} \cup \mathfrak{R}$ satisfying $\nabla(\mathfrak{S}) \subseteq \mathfrak{R}$, $\nabla(\mathfrak{R}) \subseteq \mathfrak{S}$.

Definition 2.8. [2] Remark 2.7 is a weak contraction if there exists $\alpha \in [0, 1)$ and $k \geq 0$ such that

$$d(\nabla \hbar, \nabla \iota) \leq \alpha d(\hbar, \iota) + k d(\iota, \nabla \hbar), \text{ for all } \hbar, \iota \in \mathfrak{S} \cup \mathfrak{R}.$$

Definition 2.9. [2] *Remark 2.7 is said to be a Zamfirescu contraction mapping if there exists $\alpha \in [0, 1)$, $\beta \in [0, \frac{1}{2})$, $\gamma \in [0, \frac{1}{2})$ and for all $\hbar, \iota \in \mathfrak{S} \cup \mathfrak{R}$ such that one of the following is true:*

- (i) $d(\nabla\hbar, \nabla\iota) \leq \alpha d(\hbar, \iota)$;
- (ii) $d(\nabla\hbar, \nabla\iota) \leq \beta[d(\hbar, \nabla\hbar) + d(\iota, \nabla\iota)]$;
- (iii) $d(\nabla\hbar, \nabla\iota) \leq \gamma[d(\hbar, \nabla\iota) + d(\iota, \nabla\hbar)]$.

Definition 2.10. *Remark 2.7 is said to be a Ćirić-Reich-Rus contraction if there exists $\alpha, \beta \in [0, 1)$ with $\alpha + 2\beta < 1$ such that*

$$d(\nabla\hbar, \nabla\iota) \leq \alpha d(\hbar, \iota) + \beta[d(\hbar, \nabla\hbar) + d(\iota, \nabla\iota)], \text{ for all } \hbar, \iota \in \mathfrak{S} \cup \mathfrak{R}.$$

3. MAIN RESULT

In this section, we establish several *ABPP* theorems within the framework of metric spaces, utilizing a variety of contraction mappings such as weak contractions, Zamfirescu contractions, and Ćirić–Reich–Rus contractions, along with their associated consequences. The proofs of these theorems are structured in two parts: the first focuses on qualitative aspects, while the second addresses quantitative estimates. Both components are centered around the behavior of *ABPP*'s for the pair $(\mathfrak{S}, \mathfrak{R})$ in metric spaces.

Theorem 3.1. *By using the Definition 2.8, the given ∇ is a weak contraction mapping then for every $\varepsilon > 0$, $BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) \neq \emptyset$ and the diameter,*

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq \frac{(k+2)d(\mathfrak{S}, \mathfrak{R}) + (k+1)\varepsilon}{1 - \alpha - k}, \text{ for all } \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$ and $\hbar \in \mathfrak{S} \cup \mathfrak{R}$. Consider,

$$\begin{aligned} d(\nabla^n \hbar, \nabla^{n+1} \hbar) &= d(\nabla(\nabla^{n-1} \hbar), \nabla(\nabla^n \hbar)) \\ &\leq \alpha d(\nabla^{n-1} \hbar, \nabla^n \hbar) + k d(\nabla^n \hbar, \nabla^n \hbar) \\ &\leq \alpha d(\nabla^{n-1} \hbar, \nabla^n \hbar) \end{aligned}$$

Since $\alpha \in (0, 1)$ implies that $\lim_{n \rightarrow \infty} d(\nabla^n \hbar, \nabla^{n+1} \hbar) = 0$, for all $\hbar \in \mathfrak{S} \cup \mathfrak{R}$. Then, by condition (i) of Theorem 2.6, it follows that

$$BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

To determine the diameter, utilize condition (ii) of Theorem 2.6. Since $\hbar, \iota \in BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})$ implies that

$$d(\hbar, \nabla \hbar) \leq d(\mathfrak{S}, \mathfrak{R}) + \varepsilon_1$$

And

$$d(\iota, \nabla \iota) \leq d(\mathfrak{S}, \mathfrak{R}) + \varepsilon_2$$

Now, choose $\varepsilon = \max \{\varepsilon_1, \varepsilon_2\}$. Therefore,

$$\begin{aligned} d(\hbar, \iota) &\leq d(\nabla \hbar, \nabla \iota) + \varphi \\ &\leq \alpha d(\hbar, \iota) + \mathfrak{k}d(\iota, \hbar) + \mathfrak{k}d(\hbar, \nabla \hbar) + \varphi \\ &= \alpha d(\hbar, \iota) + \mathfrak{k}d(\hbar, \iota) + \mathfrak{k}[d(\mathfrak{S}, \mathfrak{R}) + \varepsilon] + \varphi \\ &= (\alpha + \mathfrak{k})d(\hbar, \iota) + \mathfrak{k}d(\mathfrak{S}, \mathfrak{R}) + \mathfrak{k}\varepsilon + \varphi \\ &= \frac{\mathfrak{k}d(\mathfrak{S}, \mathfrak{R}) + \mathfrak{k}\varepsilon + \varphi}{1 - \alpha - \mathfrak{k}} \\ &= \theta(\varphi) \end{aligned}$$

Thus, for every $\varphi > 0$ there exists $\theta(\varphi) > 0$ such that $d(\hbar, \iota) - d(\nabla \hbar, \nabla \iota) \leq \varphi$ implies that $d(\hbar, \iota) \leq \theta(\varphi)$. Then, by Theorem 2.6, the diameter

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq \theta(2d(\mathfrak{S}, \mathfrak{R}) + \varepsilon), \text{ for all } \varepsilon > 0.$$

This means exactly,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq \frac{\mathfrak{k}d(\mathfrak{S}, \mathfrak{R}) + \mathfrak{k}\varepsilon + 2d(\mathfrak{S}, \mathfrak{R}) + \varepsilon}{1 - \alpha - \mathfrak{k}}$$

Hence,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq \frac{(\mathfrak{k} + 2)d(\mathfrak{S}, \mathfrak{R}) + (\mathfrak{k} + 1)\varepsilon}{1 - \alpha - \mathfrak{k}}, \text{ for all } \varepsilon > 0.$$

□

Theorem 3.2. *By using the Definition 2.9, the given ∇ is a Zamfirescu contraction mapping then for every $\varepsilon > 0$, $BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) \neq \emptyset$ and the diameter,*

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq \frac{2(\gamma + 1)d(\mathfrak{S}, \mathfrak{R}) + 3\varepsilon}{1 - 2\gamma}, \text{ for all } \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$ and $\hbar \in \mathfrak{S} \cup \mathfrak{R}$. Consider,

$$d(\nabla^n \hbar, \nabla^{n+1} \hbar) = d(\nabla(\nabla^{n-1} \hbar), \nabla(\nabla^n \hbar)).$$

Case 1. Suppose (i) of Definition 2.8 holds. Then,

$$(3.1) \quad d(\nabla^n \hbar, \nabla^{n+1} \hbar) \leq \alpha d(\nabla(\nabla^{n-1} \hbar), \nabla(\nabla^n \hbar))$$

Case 2. Suppose (ii) of Definition 2.8 holds. Then,

$$d(\nabla^n \hbar, \nabla^{n+1} \hbar) \leq \beta [d(\nabla(\nabla^{n-1} \hbar), \nabla(\nabla^n \hbar)) + d(\nabla(\nabla^n \hbar), \nabla(\nabla^{n+1} \hbar))]$$

That is,

$$(3.2) \quad d(\nabla^n \hbar, \nabla^{n+1} \hbar) \leq \left(\frac{\beta}{1-\beta} \right) d(\nabla(\nabla^{n-1} \hbar), \nabla(\nabla^n \hbar))$$

Case 3. Suppose (iii) of Definition 2.8 holds. Then,

$$\begin{aligned} d(\nabla^n \hbar, \nabla^{n+1} \hbar) &\leq \gamma [d(\nabla^{n-1} \hbar, \nabla^{n+1} \hbar) + d(\nabla^n \hbar, \nabla^{n+1} \hbar)] \\ &= \gamma d(\nabla^{n-1} \hbar, \nabla^n \hbar) + d(\nabla^n \hbar, \nabla^{n+1} \hbar) \end{aligned}$$

That is,

$$(3.3) \quad d(\nabla^n \hbar, \nabla^{n+1} \hbar) \leq \left(\frac{\gamma}{1-\gamma} \right) d(\nabla(\nabla^{n-1} \hbar), \nabla(\nabla^n \hbar))$$

From the equations (3.1), (3.2) and (3.3), choose

$$\lambda = \max \left\{ \alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma} \right\}.$$

Since $\alpha, \beta, \gamma \in (0, 1)$ implies that $\lambda \in (0, 1)$. Therefore,

$$\lim_{n \rightarrow \infty} d(\nabla^n \hbar, \nabla^{n+1} \hbar) = 0, \text{ for all } \hbar \in \mathfrak{S} \cup \mathfrak{R}.$$

Then by Theorem 2.6, it follows that

$$BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

For diameter, using the same procedure as mentioned in Theorem 3.1 and choose $\varepsilon = \max \{ \varepsilon_1, \varepsilon_2 \}$. Therefore, by using the equation (3.3), we get

$$\begin{aligned}
d(\tilde{h}, \iota) &= d(\nabla \tilde{h}, \nabla \iota) + \varphi \\
&= \gamma[d(\tilde{h}, \nabla \iota) + d(\iota, \nabla \tilde{h})] + \varphi \\
&= \gamma[d(\tilde{h}, \iota) + d(\iota, \nabla \iota) + d(\iota, \tilde{h}) + d(\tilde{h}, \nabla \tilde{h})] + \varphi \\
&= 2\gamma d(\tilde{h}, \iota) + 2\gamma d(\mathfrak{S}, \mathfrak{R}) + 2\varepsilon + \varphi \\
&= \frac{2\gamma d(\mathfrak{S}, \mathfrak{R}) + 2\varepsilon + \varphi}{1 - 2\gamma} \\
&= \theta(\varphi)
\end{aligned}$$

Thus, for every $\varphi > 0$ there exists $\theta(\varphi) > 0$ such that $d(\tilde{h}, \iota) - d(\nabla \tilde{h}, \nabla \iota) \leq \varphi$ implies that $d(\tilde{h}, \iota) \leq \theta(\varphi)$. Then, by Theorem 2.6, the diameter

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq \theta(2d(\mathfrak{S}, \mathfrak{R}) + \varepsilon), \text{ for all } \varepsilon > 0.$$

This means that,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq \frac{2\gamma d(\mathfrak{S}, \mathfrak{R}) + 2\varepsilon + 2d(\mathfrak{S}, \mathfrak{R}) + \varepsilon}{1 - 2\gamma b}, \text{ for all } \varepsilon > 0.$$

Hence,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq \frac{2(\gamma + 1)d(\mathfrak{S}, \mathfrak{R}) + 3\varepsilon}{1 - 2\gamma}, \text{ for all } \varepsilon > 0.$$

□

Remark 3.3. (1) Suppose **case 1** holds independently in Theorem 3.2 then this is called α -contraction mapping on metric space $(\tilde{\mathfrak{D}}, d)$. Hence, we have $BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) \neq \emptyset$ and the diameter,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq \frac{2d(\mathfrak{S}, \mathfrak{R}) + \varepsilon}{1 - \alpha}, \text{ for all } \varepsilon > 0.$$

(2) Suppose **case 2** holds independently in Theorem 3.2, then this is called Kannan contraction mapping on metric space $(\tilde{\mathfrak{D}}, d)$. Hence, we have $BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) \neq \emptyset$ and the diameter,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq 2(\beta + 1)d(\mathfrak{S}, \mathfrak{R}) + \varepsilon(2\beta + 1), \text{ for all } \varepsilon > 0.$$

(3) Suppose **case 3** holds independently in Theorem 3.2, then this is called Chatterjea contraction mapping on metric space (\mathfrak{D}, d) . Hence, we have $BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) \neq \emptyset$ and the diameter,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq \frac{2(\gamma+1)d(\mathfrak{S}, \mathfrak{R}) + 3\varepsilon}{1-2\gamma}, \text{ for all } \varepsilon > 0.$$

Theorem 3.4. By using the Definition 2.10, the given ∇ is a Ciric-Reich-Rus contraction mapping then for every $\varepsilon > 0$, $BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) \neq \emptyset$ and the diameter,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})) \leq \frac{2(\beta+1)d(\mathfrak{S}, \mathfrak{R}) + \varepsilon(2\beta+1)}{1-\alpha}, \text{ for all } \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$ and $\hbar \in \mathfrak{S} \cup \mathfrak{R}$. Consider,

$$\begin{aligned} d(\nabla^n \hbar, \nabla^{n+1} \hbar) &= d(\nabla(\nabla^{n-1} \hbar), \nabla(\nabla^n \hbar)) \\ &\leq \alpha d(\nabla^{n-1} \hbar, \nabla^n \hbar) + \beta [d(\nabla^{n-1} \hbar, \nabla^n \hbar) + d(\nabla^n, \nabla^{n+1} \hbar)] \\ &= \left(\frac{\alpha + \beta}{1 - \beta} \right) d(\nabla^{n-1} \hbar, \nabla^n \hbar) \end{aligned}$$

Since $\alpha, \beta \in (0, 1)$ implies that $\lim_{n \rightarrow \infty} d(\nabla^n \hbar, \nabla^{n+1} \hbar) = 0$, for all $\hbar \in \mathfrak{S} \cup \mathfrak{R}$. Then, by Theorem 2.6, it follows that

$$BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}) \neq \emptyset, \text{ for all } \varepsilon > 0.$$

For diameter, using the same procedure as mentioned in Theorem 3.1. Since $\hbar, \iota \in BPP^\varepsilon(\mathfrak{S}, \mathfrak{R})$ implies that

$$d(\hbar, \nabla \hbar) \leq d(\mathfrak{S}, \mathfrak{R}) + \varepsilon_1$$

And

$$d(\iota, \nabla \iota) \leq d(\mathfrak{S}, \mathfrak{R}) + \varepsilon_2$$

Now, choose $\varepsilon = \max \{\varepsilon_1, \varepsilon_2\}$. Therefore,

$$\begin{aligned} d(\hbar, \iota) &= d(\nabla \hbar, \nabla \iota) + \varphi \\ &= \alpha d(\hbar, \iota) + \beta [d(\iota, \nabla \iota) + d(\hbar, \nabla \hbar)] + \varphi \\ &= \alpha d(\hbar, \iota) + \beta [2d(\mathfrak{S}, \mathfrak{R}) + 2\varepsilon] + \varphi \\ &= \frac{2\beta d(\mathfrak{S}, \mathfrak{R}) + 2\beta\varepsilon + \varphi}{1-\alpha} \end{aligned}$$

$$= \theta(\varphi)$$

Then, by using condition (ii) of Theorem 2.6, the diameter

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{K})) \leq \theta(2d(\mathfrak{S}, \mathfrak{K}) + \varepsilon), \text{ for all } \varepsilon > 0.$$

This means that,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{K})) \leq \frac{2\beta d(\mathfrak{S}, \mathfrak{K}) + 2\beta\varepsilon + 2d(\mathfrak{S}, \mathfrak{K}) + \varepsilon}{1 - \alpha}, \text{ for all } \varepsilon > 0.$$

Hence,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{K})) \leq \frac{2(\beta + 1)d(\mathfrak{S}, \mathfrak{K}) + \varepsilon(2\beta + 1)}{1 - \alpha}, \text{ for all } \varepsilon > 0.$$

□

Theorem 3.5. *Let \mathfrak{S} and \mathfrak{K} are two non-empty subsets of a metric space (\mathfrak{D}, d) . Suppose that a mapping $\nabla : \mathfrak{S} \cup \mathfrak{K} \rightarrow \mathfrak{S} \cup \mathfrak{K}$ satisfying $\nabla(\mathfrak{S}) \subseteq \mathfrak{K}$ and $\nabla(\mathfrak{K}) \subseteq \mathfrak{S}$ is a Ciric contraction then for every $\varepsilon > 0$, $BPP^\varepsilon(\mathfrak{S}, \mathfrak{K}) \neq \emptyset$ and the diameter,*

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{K})) \leq \frac{(\beta + \gamma + 2\delta + 2)d(\mathfrak{S}, \mathfrak{K}) + (\beta + \gamma + 2\delta + 1)\varepsilon}{1 - \alpha - 2\delta}, \text{ for all } \varepsilon > 0.$$

Proof. The first part of the proof is easy to verify. For diameter, to show condition (ii) of Theorem 2.6 holds. For that, take $\varphi > 0$ and $\hbar, \iota \in BPP^\varepsilon(\mathfrak{S}, \mathfrak{K})$. Also, $d(\hbar, \iota) - d(\nabla\hbar, \nabla\iota) \leq \varphi$ implies that $d(\hbar, \iota) \leq d(\nabla\hbar, \nabla\iota) + \varphi$. Since $\hbar, \iota \in BPP^\varepsilon(\mathfrak{S}, \mathfrak{K})$ implies that

$$d(\hbar, \nabla\hbar) \leq d(\mathfrak{S}, \mathfrak{K}) + \varepsilon_1$$

And

$$d(\iota, \nabla\iota) \leq d(\mathfrak{S}, \mathfrak{K}) + \varepsilon_2$$

Now, choose $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$. Therefore,

$$\begin{aligned} d(\hbar, \iota) &= d(\nabla\hbar, \nabla\iota) + \varphi \\ &\leq \alpha d(\hbar, \iota) + \beta d(\hbar, \nabla\hbar) + \gamma d(\iota, \nabla\iota) + \delta [d(\hbar, \nabla\iota) + d(\iota, \nabla\hbar)] + \varphi \\ &= (\alpha + 2\delta\beta)d(\hbar, \iota) + (\beta + \gamma + 2\delta)d(\mathfrak{S}, \mathfrak{K}) + (\beta + \gamma + 2\delta)\varepsilon + \varphi \\ &= \frac{(\beta + \gamma + 2\delta)d(\mathfrak{S}, \mathfrak{K}) + (\beta + \gamma + 2\delta)\varepsilon + \varphi}{1 - \alpha - 2\delta} \end{aligned}$$

$$= \theta(\varphi)$$

Thus, for every $\varphi > 0$ there exists $\theta(\varphi) > 0$ such that $d(\hbar, \iota) - d(\nabla\hbar, \nabla\iota) \leq \varphi$ implies that $d(\hbar, \iota) \leq \theta(\varphi)$. Then, by Theorem 2.6, the diameter

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{K})) \leq \theta(2d(\mathfrak{S}, \mathfrak{K}) + \varepsilon), \text{ for all } \varepsilon > 0.$$

Hence,

$$\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{K})) \leq \frac{(\beta + \gamma + 2\delta + 2)d(\mathfrak{S}, \mathfrak{K}) + (\beta + \gamma + 2\delta + 1)\varepsilon}{1 - \alpha - 2\delta}, \text{ for all } \varepsilon > 0.$$

□

Example 3.6. Let $\mathfrak{D} = [0, 1]$ and consider the closed subsets $\nabla_1 = [0, 3/6]$, $\nabla_2 = [2/6, 3/6]$ and $\nabla_3 = [5/6, 1]$ of a metric space (\mathfrak{D}, d) and $\nabla : \nabla_1 \cup \nabla_2 \cup \nabla_3 \rightarrow \nabla_1 \cup \nabla_2 \cup \nabla_3$ is defined by:

$$\nabla\hbar = \begin{cases} \frac{2}{6} + \hbar & \text{when } \hbar \in \left[0, \frac{3}{6}\right] \\ \frac{3}{6} + \hbar & \text{when } \hbar \in \left[\frac{2}{6}, \frac{3}{6}\right] \\ 1 - \frac{3}{6} & \text{when } \hbar \in \left[\frac{5}{6}, 1\right] \end{cases}$$

This clearly shows that $\nabla(\nabla_1) \subseteq \nabla_2$, $\nabla(\nabla_2) \subseteq \nabla_3$ and $\nabla(\nabla_3) \subseteq \nabla_1$. Also for every $\hbar, \iota \in \nabla_1 \cup \nabla_2 \cup \nabla_3 \subseteq \nabla$ satisfies the Definitions 2.8 2.9 and 2.10. Hence, ∇ satisfies all the conditions of the Theorems 3.1, 3.2 and 3.4.

In the similar manner, we have proved many *ABPP* results by using various operators on metric spaces. The diameters of several contraction operators are shown in the table below.

S. No	Operator(s)	Diameter, for every $\varepsilon > 0$, $\Delta(BPP^\varepsilon(\mathfrak{S}, \mathfrak{R}))$
1	Contraction	$\leq \frac{2d(\mathfrak{S}, \mathfrak{R}) + \varepsilon}{1 - \alpha}$
2	Kannan	$\leq 2(\beta + 1)d(\mathfrak{S}, \mathfrak{R}) + \varepsilon(2\beta + 1)$
3	Chatterjea	$\leq \frac{2(\gamma + 1)d(\mathfrak{S}, \mathfrak{R}) + \varepsilon(2\gamma + 1)}{1 - 2\gamma}$
4	B-contraction	$\leq \frac{2(\alpha + \gamma + 1)d(\mathfrak{S}, \mathfrak{R}) + \varepsilon(2\alpha + 2\gamma + 1)}{1 - \beta - 2\gamma}$
5	Bianchini	$\leq (\alpha + 2)d(\mathfrak{S}, \mathfrak{R}) + \varepsilon(\alpha + 1)$
6	Hardy-Rogers	$\leq \frac{(\beta + \gamma + \delta + \nu + 2)d(\mathfrak{S}, \mathfrak{R}) + \varepsilon(\beta + \gamma + \delta + \nu + 1)}{1 - \alpha - \delta - \nu}$
7	Ćirić	$\leq \frac{(\beta + \gamma + 2\delta + 2)d(\mathfrak{S}, \mathfrak{R}) + (\beta + \gamma + 2\delta + 1)\varepsilon}{1 - \alpha - 2\delta}$
8	Ćirić-Reich-Rus	$\leq \frac{2(\beta + 1)d(\mathfrak{S}, \mathfrak{R}) + \varepsilon(2\beta + 1)}{1 - \alpha}$
9	Reich	$\leq \frac{(\beta + \gamma + 2)d(\mathfrak{S}, \mathfrak{R}) + (\beta + \gamma + 1)\varepsilon}{1 - \alpha}$
10	Zamfirescu	$\leq \frac{(2\gamma + 2)d(\mathfrak{S}, \mathfrak{R}) + 3\varepsilon}{1 - 2\gamma}$
11	Mohseni-saheli	$\leq \frac{(2\alpha + 2)d(\mathfrak{S}, \mathfrak{R}) + \varepsilon(2\alpha + 1)}{1 - \alpha - \alpha}$
12	Mohseni-semi	$\leq \frac{(\alpha + 2)d(\mathfrak{S}, \mathfrak{R}) + \varepsilon(\alpha + 1)}{1 - \alpha}$
13	Weak contraction	$\leq \frac{(\mathfrak{k} + 2)d(\mathfrak{S}, \mathfrak{R}) + (\mathfrak{k} + 1)\varepsilon}{1 - \alpha - \mathfrak{k}}$

4. APPLICATIONS

The *ABPP* covers a wide range of applications in the domain of mathematics, particularly in differential equations, Fourier series, numerical analysis, and so on. By reading [7] and references therein, one can find a variety of applications involving *ABPP* results in differential equations. The examples below demonstrate how to apply *ABPP* results in differential equations.

Example 4.1. Let $\mathfrak{D} = C([0, 1], \mathbb{R})$ and χ is metric space defined by $d(\mathfrak{h}, \mathfrak{l}) = \sup_{\tau \in [0, 1]} |\mathfrak{h}, \mathfrak{l}|^2$. Also, consider $\mathfrak{x}''(\tau) = 3\mathfrak{x}^2(\tau)/2$, $0 \leq \tau \leq 1$ and the initial conditions $\mathfrak{x}(0) = 4$, $\mathfrak{x}(1) = 1$. Here, the exact solution is $\mathfrak{x}(\tau) = 4/(1 + \tau)^2$. We have, $\mathfrak{x}_0(\tau) = c_1\tau + c_2$. By using the initial conditions, we get $\mathfrak{x}_0(\tau) = 4 - 3\tau$. Now, define the integral operator,

$$(4.1) \quad \mathfrak{S}(\mathfrak{x}) = \mathfrak{x} + \int_0^1 G(\tau, \omega) [\mathfrak{x}'' - f(\omega, \mathfrak{x}, \mathfrak{x}')] d\omega \mathfrak{h}$$

where

$$G(\tau, \omega) = \begin{cases} \omega(1-\tau) & 0 \leq \omega \leq \tau \\ \tau(1-\tau) & \tau \leq \omega \leq 1 \end{cases}$$

Then, the equation (4.1) becomes

$$\begin{aligned} \mathfrak{S}(\varkappa) &= \varkappa(\tau) + \int_0^1 G(\tau, \omega) \varkappa''(s) ds - \int_0^1 G(\tau, \omega) f(\omega, \varkappa, \varkappa') d\omega \\ &= (4-3\tau) - \int_0^1 G(\tau, \omega) \left[-\frac{3\varkappa^2(\omega)}{2} \right] d\omega \\ &= 4-3\tau + \frac{3}{2} \left\{ \int_0^1 G(\tau, \omega) \varkappa^2(\omega) d\omega \right\} \end{aligned}$$

So, we have

$$\begin{aligned} d(\varkappa\hbar, \varkappa\iota) &= \sup_{\tau \in [0,1]} |\varkappa\hbar - \varkappa\iota|^2 \\ &= \sup_{\tau \in [0,1]} \left| \frac{3}{2} \int_0^1 G(\tau, \omega) \hbar^2(\omega) d\omega - \frac{3}{2} \int_0^1 G(\tau, \omega) \iota^2(\omega) d\omega \right|^2 \\ &\leq (2.25) \left(\int_0^1 |G(\tau, \omega)|^2 ds \right) \left(\int_0^1 |\hbar^2(\omega) - \iota^2(\omega)|^2 d\omega \right) \\ &\leq (0.75) \frac{\tau^2(1-\tau)^2}{3} \int_0^1 |\hbar^2(\omega) - \iota^2(\omega)|^2 d\omega \\ &\leq (0.046875) \int_0^1 |\hbar^2(\omega) - \iota^2(\omega)|^2 d\omega \\ &\leq (0.046875) \sup_{\tau \in [0,1]} |\hbar(\omega) - \iota(\omega)|^2 \\ &\leq (0.046875) d(\hbar, \iota) \end{aligned}$$

Hence, it satisfies all the conditions of Theorem 3.1 and Theorem 3.2. Also, by Theorem 2.6, \mathfrak{S} has ABPP in $\mathfrak{D} = C([0, 1], \mathbb{R})$. Therefore, the given bounded value problem has ABPP in \mathfrak{D} .

5. CONCLUSION

In this paper, some ABPP results are established on metric spaces by utilizing various types of contraction mappings. It is worth observing that in the limiting case $\varepsilon \rightarrow 0$, all the results established in the present paper produce more restricted ABPP's. Furthermore, ABPP's are

consequently not less important than BPP' 's. As various future results can be demonstrated in a smaller setting to ensure the existence of the $ABPP'$'s.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

- [1] S. Banach, Sur les Opérations dans les Ensembles Abstraites et Leur Application aux Équations Intégrales, *Fundam. Math.* 3 (1922), 133–181. <https://doi.org/10.4064/fm-3-1-133-181>.
- [2] M. Berinde, Approximate Fixed Point Theorems, *Stud. Univ. "Babes-Bolyai", Math.* 51 (2006), 11–25.
- [3] D. Dey, M. Saha, Approximate Fixed Point of Reich Operator, *Acta Math. Univ. Comenianae* 82 (2017), 119–123.
- [4] S.A.M. Mohsenalhosseini, Approximate Best Proximate Pairs in Metric Space for Contraction Maps, *Fixed Point Theory* 4 (2014), 310–324.
- [5] S.A.M. Mohsenalhosseini, M. Saheli, Some of Family of Contractive Type Maps and Approximate Common Fixed Point, *J. Fixed Point Theory* 2021 (2021), 1–12.
- [6] S.A.M. Mohsenalhosseini, H. Mazaheri, M.A. Dehghan, Approximate Best Proximity Pairs in Metric Space, *Abstr. Appl. Anal.* 2011 (2011), 596971. <https://doi.org/10.1155/2011/596971>.
- [7] S. Khuri, I. Louhichi, A Novel Ishikawa–Green's Fixed Point Scheme for the Solution of BVPs, *Appl. Math. Lett.* 82 (2018), 50–57. <https://doi.org/10.1016/j.aml.2018.02.016>.
- [8] R. Theivaraman, P.S. Srinivasan, S. Thenmozhi, S. Radenovic, Some Approximate Fixed Point Results for Various Contraction Type Mappings, *Adv. Fixed Point Theory* 13 (2023), 9. <https://doi.org/10.28919/afpt/8080>.
- [9] R. Theivaraman, P.S. Srinivasan, M. Marudai, S. Thenmozhi, A. Herminau Jothy, G-Metric Spaces and the Related Approximate Fixed Point Results, *Adv. Fixed Point Theory* 13 (2023), 17. <https://doi.org/10.28919/afpt/8178>.
- [10] R. Theivaraman, P.S. Srinivasan, S. Radenovic, C. Park, New Approximate Fixed Point Results for Various Cyclic Contraction Operators on E-Metric Spaces, *J. Korean Soc. Ind. Appl. Math.* 27 (2023), 160–179. <https://doi.org/10.12941/jksiam.2023.27.160>.
- [11] R. Theivaraman, P.S. Srinivasan, A.H. Jothy, Approximate Best Proximity Pair Results on Metric Spaces Using Contraction Operators, *Korean J. Math.* 31 (2023), 373–383. <https://doi.org/10.11568/kjm.2023.31.373>.

- [12] D. Sujatha, S.R. Ananthalakshmi, R. Theivaraman, Existence of Approximate Best Proximity Point of the Triplets (Q_1, Q_2, Q_3) on G-Metric Spaces and Its Applications, *Adv. Fixed Point Theory* 15 (2025), 48. <https://doi.org/10.28919/afpt/9548>.
- [13] V. Anbukkarasi, M. Marudai, R. Theivaraman, Best Proximity Points for Contractive Mappings in Generalized Modular Metric Spaces, *Korean J. Math.* 31 (2023), 123–131. <https://doi.org/10.11568/kjm.2023.31.2.123>.
- [14] K. Saravanan, V. Piramanantham, R. Theivaraman, Some Fixed Point Results on E-Metric Spaces Using Contraction Operators, *Adv. Fixed Point Theory* 14 (2024), 3. <https://doi.org/10.28919/afpt/8333>.
- [15] A.M.P. Dharsini, K. Maheshwaran, T.C. Mujeeburahman, P.J. Lavino, R. Theivaraman, Existence of Approximate Best Proximity Point Results on Extended b-Metric Spaces and Its Applications, *Adv. Fixed Point Theory* 16 (2026), 2. <https://doi.org/10.28919/afpt/9683>.