Available online at http://scik.org
Adv. Fixed Point Theory, 3 (2013), No. 4, 608-628
ISSN: 1927-6303

EXISTENTIAL AND UNIQUENESS RESULTS FOR INITIAL VALUE PROBLEMS ASSOCIATED WITH n^{th} ORDER NONLINEAR SINGULAR INTERFACE PROBLEMS ON TIME SCALES USING FIXED POINT THEOREMS

D. K. K. VAMSI*, K. N. V. S. D. DWARAKANATH, I. ADITYA, P. K. BARUAH

Department of Mathematics and Computer Science, Sri Sathya Sai Institute of Higher Learning,

Prasanthinilayam, Puttaparthi, Andhra-Pradesh, India

Copyright © 2013 D. K. K. Vamsi et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this paper, we present existence and uniqueness results for IVPs associated with general nonlinear singular interface problems on Time Scales. We discuss the existential results for a n^{th} order IVP associated with nonlinear singular interface problems using the classical Schauder fixed point theorem.

Keywords: regular problems; singular problems; singular interface problems; fixed point theorems.

2000 AMS Subject Classification: 47H10, 65R20.

1. Introduction

In the literature, we find a class of problems wherein two different differential equations are defined on adjacent intervals with a common point of interface. We term these problems as interface problems.

If the interface problem has a well defined boundary, we call the problem to be a *regular boundary value problem* (RBVP). The interface problem with a boundary that has singularity at the end points is called a *singular boundary value problem* (SBVP). If there is a singularity at the point of interface, we term the problem to be

*Corresponding author

E-mail address: dkkvamsi@sssihl.edu.in (D. K. K. Vamsi)

Received April 2, 2013

a *singular interface problem* (SIP). Solving these type of boundary value problems with singularities remains a challenge for mathematicians.

While regular boundary value problems, those over finite intervals with well-behaved coefficients pose no difficulties, the problems wherein the domain of the problem is not well defined, or the continuity and/or smoothness of the functions, coefficients involved are not guaranteed in some parts of the domain, sometimes in the boundary or parts of the boundary are difficult to tackle. There are quite a number of different approaches that we come across in the literature to tackle these singular problems [1],[4],[6],[8],[9],[10].

In the literature, we see that work has been done on initial and boundary value problems associated with a pair of linear differential operators with conditions at the interface for both regular and singular cases; see [15]-[22],[26]-[31] and the references therein.

The singular interface problem requires a special mention. In this case existing theory based on the conventional analysis may not come handy.

We feel that the new framework of dynamic equations on time scales(an arbitrary closed subset of real numbers)[2] with facilities of the two jump operators with various definitions of continuity and derivatives makes one's job simple to study these singular interface problems. These dynamic equations are nothing but the differential equations when $\mathbb{T} = \mathbb{R}$ and are difference equations when $\mathbb{T} = \mathbb{Z}$.

Our preliminary investigation about the feasibility of this study for linear second order interface problems has resulted in the work [16, 17, 25].

From the above we observe that substantial amount of work has been done for regular and singular boundary value problems involving linear differential operators. It is clear that there is a need for these singular interface problems to be discussed for the case where the problem involves nonlinear differential operators.

A systematic study of Initial Value Problems, Boundary Value Problems and eigen Value problems associated with these nonlinear singular interface problems involving nonlinear second order pair of dynamic equations is done in [34]-[41].

In this paper we study the existence of solution for a n^{th} order Initial Value Problem associated with these nonlinear singular interface problems. Schauder fixed point theorem is used for proving the existential result.

2. Preliminaries

Definition 0.1. Let $\mathbb{T}$ be a time scale(an arbitrary closed subset of real numbers). For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is *right-scattered,* while $\rho(t) < t$ we say that t is *left-scattered.* Points that are right-scattered and left-scattered at the same time are called *isolated.* Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense,* and if $t >$ inf $\mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense.* Points that are right-dense and left-dense at the same time are called *dense.* Finally, the *graininess function* $\mu : \mathbb{T} \to [0,\infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Definition 0.2. $\mathbb{T}^\kappa = \left\{ \begin{array}{l} \mathbb{T} - \{m\} \text{ if sup } \mathbb{T} < \infty \\ \mathbb{T} \text{ if sup } \mathbb{T} = \infty \end{array} \right\}$ where m is the left scattered maximum of $\mathbb{T}$.

Definition 0.3. Let f be a function defined on $\mathbb{T}$. We say that f is *delta differentiable* at $t \in \mathbb{T}^\kappa$ provided there exists an α such that for all $\varepsilon > 0$ there is a neighborhood $\mathscr{N}$ around t with

$$|f(\sigma(t) - f(s) - \alpha(\sigma(t) - s)| \leq \varepsilon\, |\sigma(t) - s| \text{ for all } s \in \mathscr{N}.$$

Definition 0.4. For a function $f : \mathbb{T} \to \mathbb{R}$ we shall talk about the second derivative $f^{\Delta\Delta}$ provided f^Δ is differentiable on $\mathbb{T}^{\kappa 2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta\Delta} = (f^\Delta)^\Delta : \mathbb{T}^{\kappa 2} \to \mathbb{R}$. Similarly we define the higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa n} \to \mathbb{R}$.

Theorem 0.5. *(Banach Contraction Mapping Theorem)*

If $T : X \to X$ is contractive on a complete metric space X then T has a unique fixed point in X.

Theorem 0.6. *(Schauder's Fixed Point Thorem)*

Let L be a convex subset of a normed linear space E. Then each compact map $T : L \to L$ has a fixed point.

Let $I = [c,d]$ with $c < \rho(d)$. We define $I_c = [c,\infty)$ in case $sup\mathbb{T} = +\infty$. By $\mathscr{C}^B_{TS}(I_a)$ we mean the linear space of all continuous functions $f : I_c \to \mathbb{R}$ such that $sup_{t\in I_c}|f(t)| < \infty$.

Now we quote the time scales version of the Arzela-Ascoli theorem [1].

Theorem 0.7. *(Arzela-Ascoli Theorem)*

Let X be a subset of $\mathscr{C}^B_{TS}(I_a)$ having the following properties.

(i) X is bounded.

(ii) On every compact subinterval J of $[c,\infty)$ we have: For any $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$ implies $|f(t_1) - f(t_2)| < \varepsilon$ for all $f \in X$.

(iii) For every $\varepsilon > 0$ there exists $b \in I_c$ such that $t_1, t_2 \in [b,\infty)$ implies $|f(t_1) - f(t_2)| < \varepsilon$ for all $f \in X$.

Then X is relatively compact.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales. Let $C(\mathbb{T})$ denote the space of all continuous functions on the time scale $\mathbb{T}$.

Definition 0.8. By $(t_1,t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we mean that $t_1 \in \mathbb{T}_1$ and $t_2 \in \mathbb{T}_2$ with the product topology on $\mathbb{T}_1 \times \mathbb{T}_2$.

Definition 0.9. Let $Z(\mathbb{T})$ be a function space on the time scale $\mathbb{T}$. For $x \in Z(\mathbb{T})$ we define

$$\|x\| = sup_{t\in\mathbb{T}}|x(t)|.$$

Definition 0.10. By $(x_1,x_2) \in X(\mathbb{T}_1)\times Y(\mathbb{T}_2)$ where X and Y are function spaces, we mean that $x_1 \in X(\mathbb{T}_1)$ and $x_2 \in Y(\mathbb{T}_2)$ with the product topology on $X(\mathbb{T}_1)\times Y(\mathbb{T}_2)$. For $(t_1,t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we define

$$(x_1,x_2)(t_1,t_2) = (x_1(t_1),x_2(t_2)).$$

Definition 0.11. For $(t_1,t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$, we define

$$\|(t_1,t_2)\| = \|t_1\| + \|t_2\| = |t_1| + |t_2|.$$

Definition 0.12. For $(x_1,x_2) \in C(\mathbb{T}_1)\times C(\mathbb{T}_2)$, we define

$$\begin{aligned} \|(x_1,x_2)\| &= \|x_1\| + \|x_2\| \\ &= sup_{t_1\in\mathbb{T}_1}|x_1(t_1)| + sup_{t_2\in\mathbb{T}_2}|x_2(t_2)|. \end{aligned}$$

Definition 0.13. Let $(y_{11},y_{12}) \in X(\mathbb{T}_1)\times Y(\mathbb{T}_2)$. We say that (y_{11},y_{12}) is continuous on $\mathbb{T}_1 \times \mathbb{T}_2$ if for $\varepsilon > 0$ there exists $\delta > 0$ such that for arbitrarily fixed $(t_{01},t_{02}) \in \mathbb{T}_1 \times \mathbb{T}_2$ and $(t_1,t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that

$$\begin{aligned} \| & (t_1,t_2) - (t_{01},t_{02})\| < \delta \\ \Rightarrow & \|(y_{11},y_{12})(t_1,t_2) - (y_{11},y_{12})(t_{01},t_{02})\| < \varepsilon. \end{aligned} \tag{0.1}$$

Definition 0.14. A sequence $(y_{n1},y_{n2}) \in C(\mathbb{T}_1)\times C(\mathbb{T}_2)$ is said to be cauchy sequence if for every $\varepsilon > 0$ there exists N such that $\forall n1,n2,m1,m2 > N$ implies

$$\|(x_{n1},x_{n2}) - (x_{m1},x_{m2})\| < \varepsilon.$$

Definition 0.15. A sequence $(y_{n1},y_{n2}) \in X(\mathbb{T}_1)\times Y(\mathbb{T}_2)$ is said to be equicontinuous if for every $\varepsilon > 0$ there is a $\delta > 0$, depending only on ε, such that for all (y_{n1},y_{n2}) and all $(t_1,t_2),(t_1{'},t_2{'}) \in \mathbb{T}_1 \times \mathbb{T}_2$ satisfying

$$\begin{aligned} \| & (t_1,t_2) - (t_1{'},t_2{'})\| < \delta \\ \Rightarrow & \|(y_{n1},y_{n2})(t_1,t_2) - (y_{n1},y_{n2})(t_1{'},t_2{'})\| < \varepsilon. \end{aligned} \tag{0.2}$$

Definition 0.16.

$$(a^n - b^n) = (a-b)(a^{n-1} + ba^{n-2} + \ldots + b^{n-2}a + b^{n-1}).$$

Definition 0.17. The space $X(\mathbb{T}_1)\times Y(\mathbb{T}_2)$ is said to convex if for every $(y_{11},y_{12}),(y_{21},y_{22})\in X(\mathbb{T}_1)\times Y(\mathbb{T}_2)$, we have

$$\alpha(y_{11},y_{12})+(1-\alpha)(y_{21},y_{22})\in X(\mathbb{T}_1)\times Y(\mathbb{T}_2) \text{ for } 0<\alpha<1.$$

3. Definition and Equivalent Operator Integral Equation of a General n^{th} order Initial Value Problem associated with Nonlinear Singular Interface Problem

Initial Value Problem associated with Nonlinear Singular Interface Problem

Let $\mathbb{T}_1=[0,a]_{\mathbb{T}},\mathbb{T}_2=[\sigma(a),l]_{\mathbb{T}}$ where $a,\sigma(a),l<+\infty$. Also let (f_1,f_2) be nonlinear function tuple in $\mathscr{C}(\mathbb{T}_1\times R^n)\times\mathscr{C}(\mathbb{T}_2\times R^n)$. We define the problem to be

$$y_1^{\Delta^n}(t) = f_1(t,y_1,\dots,y_1^{\Delta^{n-1}}),\ t\in\mathbb{T}_1 \tag{0.3}$$

$$y_2^{\Delta^n}(t) = f_2(t,y_2,\dots,y_2^{\Delta^{n-1}}),\ t\in{\mathbb{T}_2}^{\kappa^n} \tag{0.4}$$

with the initial conditions

$$y_1(0) = 0 \tag{0.5}$$

$$y_1^{\Delta}(0) = 0 \tag{0.6}$$

$$\vdots \tag{0.7}$$

$$y_1^{\Delta^{n-1}}(0) = 0 \tag{0.8}$$

followed by the matching interface conditions

$$\rho_1 y_1(a) = \rho_2 y_2(\sigma(a)), \tag{0.9}$$

$$\vdots \tag{0.10}$$

$$\rho_{2n-5}y_1^{\Delta^{n-2}}(a) = \rho_{2n-4}y_2^{\Delta^{n-2}}(\sigma(a)), \tag{0.11}$$

$$\rho_{2n-3}y_1^{\Delta^{n-3}}(a) = \rho_{2n-2}y_2^{\Delta^{n-3}}(\sigma(a)), \tag{0.12}$$

$$\rho_{2n-1}y_1^{\Delta^{n-1}}(a) = \rho_{2n}y_2^{\Delta^{n-1}}(\sigma(a)),\ \rho_i>0,\ i=1\dots\ 2n. \tag{0.13}$$

4. Existence of Solution for the n^{th} order

Initial Value Problem associated with Nonlinear Singular Interface Problem using Schauder's Fixed Point Theorem

In this section we prove the existence of solution for the IVP (0.3) -(0.13) using Schauder's fixed point theorem.

Theorem 0.18. *If (f_1, f_2) is continuous and bounded, then there exists atleast one solution for the nth order IVP-SIP (0.3) -(0.13).*

Proof. Case I Let $t \in \mathbb{T}_1$

Then,

$$\begin{aligned}
y_1^{\Delta^n}(t) &= f_1(t, y_1, \ldots, y_1^{\Delta^{n-1}}) \\
y_1^{\Delta^{n-1}}(t) &= \int_0^t f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s + c_{11} \\
y_1^{\Delta^{n-2}}(t) &= \int_0^t \int_0^{t_{n-1}} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s \Delta t_{n-1} + \int_0^t c_{11}\Delta s + c_{12} \\
&\vdots \\
y_1(t) &= \int_0^t \int_0^{t_{n-1}} \ldots \int_0^{t_2} \int_0^{t_1} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s \Delta t_1 \ldots \Delta t_{n-1} \\
&+ \int_0^t \int_0^{t_{n-2}} \ldots \int_0^{t_1} c_{11}\Delta s \ldots \Delta t_{n-2} + \ldots + \int_0^t c_{1(n-1)}\Delta s + c_{1n}
\end{aligned}$$

where c_{11} , c_{12} ... c_{1n} are constants to be determined. By using the initial conditions (0.5),(0.6),...(0.8) we get

$$\begin{aligned}
y_1(0) = 0 &\Rightarrow c_{1n} = 0 \\
y_1^{\Delta}(0) = 0 &\Rightarrow c_{1(n-1)} = 0 \\
&\vdots \\
y_1^{\Delta^{n-1}}(0) = 0 &\Rightarrow c_{11} = 0 \\
\Rightarrow y_1(t) &= \int_0^t \int_0^{t_{n-1}} \ldots \int_0^{t_2} \int_0^{t_1} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s \Delta t_1 \ldots \Delta t_{n-1}.
\end{aligned}$$

Case II Let $t \in \mathbb{T}_2$

$$\begin{aligned}
y_2^{\Delta^n}(t) &= f_2(t, y_2, \ldots, y_2^{\Delta^{n-1}}) \\
y_2^{\Delta^{n-1}}(t) &= \int_{\sigma(a)}^t f_2(s, y_2, \ldots, y_2^{\Delta^{n-1}})\Delta s + c_{21} \\
y_2^{\Delta^{n-2}}(t) &= \int_{\sigma(a)}^t \int_{\sigma(a)}^{t'_{n-1}} f_2(s, y_2, \ldots, y_2^{\Delta^{n-1}})\Delta s \Delta t'_{n-1} + \int_{\sigma(a)}^t c_{21}\Delta s + c_{22} \\
&\vdots \\
y_2(t) &= \int_{\sigma(a)}^t \int_{\sigma(a)}^{t'_{n-1}} \ldots \int_{\sigma(a)}^{t'_2} \int_{\sigma(a)}^{t'_1} f_2(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s \Delta t'_1 \ldots \Delta t'_{n-1} \\
&+ \int_{\sigma(a)}^t \int_{\sigma(a)}^{t'_{n-2}} \ldots \int_{\sigma(a)}^{t'_2} \int_{\sigma(a)}^{t'_1} c_{21}\Delta s \Delta t'_1 \ldots \Delta t'_{n-2} + \ldots + c_{2n}
\end{aligned}$$

where c_{21} , $c_{22}, \ldots,$ c_{2n} are constants to be determined.

Now, by (0.9), we get

$$\begin{aligned} \rho_1 y_1(a) &= \rho_2 y_2(\sigma(a)) \\ \Rightarrow c_{2n} &= \frac{\rho_1}{\rho_2}\left(\int_0^a \int_0^{t_{n-1}} \ldots \int_0^{t_2}\int_0^{t_1} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s \Delta t_1 \ldots \Delta t_{n-1}\right). \end{aligned}$$

Also, by (0.11), we get

$$\begin{aligned} \rho_{2n-5} y_1^{\Delta^{n-2}}(a) &= \rho_{2n-4} y_2^{\Delta^{n-2}}(\sigma(a)) \\ \Rightarrow c_{23} &= \frac{\rho_{2n-5}}{\rho_{2n-4}}\left(\int_0^a \int_0^{t_{n-1}} \int_0^{t_{n-2}} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s \Delta t_{n-2} \Delta t_{n-1}\right). \end{aligned}$$

Also, by (0.12), we get

$$\begin{aligned} \rho_{2n-3} y_1^{\Delta^{n-3}}(a) &= \rho_{2n-2} y_2^{\Delta^{n-3}}(\sigma(a)) \\ \Rightarrow c_{22} &= \frac{\rho_{2n-3}}{\rho_{2n-2}}\left(\int_0^a \int_0^{t_{n-1}} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s \Delta t_{n-1}\right). \end{aligned}$$

Also, by (0.13), we get

$$\begin{aligned} \rho_{2n-1} y_1^{\Delta^{n-1}}(a) &= \rho_{2n} y_2^{\Delta^{n-1}}(\sigma(a)) \\ \Rightarrow c_{21} &= \frac{\rho_{2n-1}}{\rho_{2n}}\left(\int_0^a f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s\right). \end{aligned}$$

Hence,

$$\begin{aligned} y_2(t) &= \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{t'_{n-1}} \ldots \int_{\sigma(a)}^{t'_2}\int_{\sigma(a)}^{t'_1} f_2(s, y_2, \ldots, y_2^{\Delta^{n-1}})\Delta s \Delta t'_1 \ldots \Delta t'_{n-1} \\ &+ \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{t'_{n-2}} \ldots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-1}}{\rho_{2n}}\left(\int_0^a f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s\right)\Delta t \Delta t'_1 \ldots \Delta t'_{n-2} \\ &+ \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{t'_{n-3}} \ldots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-3}}{\rho_{2n-2}}\left(\int_0^a \int_0^{t_{n-1}} f_1(s, y_1, \ldots,\right. \\ &\left. y_1^{\Delta^{n-1}})\Delta s \Delta t_{n-1}\right)\Delta t \Delta t'_1 \ldots \Delta t'_{n-3} \\ &+ \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{t'_{n-4}} \ldots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-5}}{\rho_{2n-4}}\left(\int_0^a \int_0^{t_{n-1}} \int_0^{t_{n-2}} f_1(s, y_1, \ldots,\right. \\ &\left. y_1^{\Delta^{n-1}})\Delta s \Delta t_{n-2} \Delta t_{n-1}\right)\Delta t \Delta t'_1 \ldots \Delta t'_{n-4} \\ &+ \ldots + \frac{\rho_1}{\rho_2}\left(\int_0^a \int_0^{t_{n-1}} \ldots \int_0^{t_1} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s \Delta t_1 \ldots \Delta t_{n-1}\right) \end{aligned}$$

We now define the integral operator $T : C(\mathbb{T}_1) \times C(\mathbb{T}_2) \to C(\mathbb{T}_1) \times C(\mathbb{T}_2)$.

$$\begin{aligned}
T(y_1, y_2) &= \Bigg(\int_0^t \int_0^{t_{n-1}} \dots \int_0^{t_2} \int_0^{t_1} f_1(s, y_1, \dots, y_1^{\Delta^{n-1}}) \Delta s \Delta t_1 \dots \Delta t_{n-1}, \\
& \int_{\sigma(a)}^{t'} \int_{\sigma(a)}^{t'_{n-1}} \dots \int_{\sigma(a)}^{t'_2} \int_{\sigma(a)}^{t'_1} f_2(s, y_2, \dots, y_2^{\Delta^{n-1}}) \Delta s \Delta t'_1 \dots \Delta t'_{n-1} \\
+ & \int_{\sigma(a)}^{t'} \int_{\sigma(a)}^{t'_{n-2}} \dots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-1}}{\rho_{2n}} \Bigg(\int_0^a f_1(s, y_1, \dots, \\
& y_1^{\Delta^{n-1}}) \Delta s \Bigg) \Delta t \Delta t'_1 \dots \Delta t'_{n-2} \\
+ & \int_{\sigma(a)}^{t'} \int_{\sigma(a)}^{t'_{n-3}} \dots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-3}}{\rho_{2n-4}} \Bigg(\int_0^a \int_0^{t_{n-1}} f_1(s, y_1, \dots, \\
& y_1^{\Delta^{n-1}}) \Delta s \Delta t_{n-1} \Bigg) \Delta t \Delta t'_1 \dots \Delta t'_{n-3} \\
+ & \int_{\sigma(a)}^{t'} \int_{\sigma(a)}^{t'_{n-4}} \dots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-5}}{\rho_{2n-4}} \Bigg(\int_0^a \int_0^{t_{n-1}} \int_0^{t_{n-2}} f_1(s, y_1, \dots, \\
& y_1^{\Delta^{n-1}}) \Delta s \Delta t_{n-2} \Delta t_{n-1} \Bigg) \Delta t \Delta t'_1 \dots \Delta t'_{n-4} \\
+ & \dots + \frac{\rho_1}{\rho_2} \Bigg(\int_0^a \int_0^{t_{n-1}} \dots \int_0^{t_1} f_1(s, y_1, \dots, y_1^{\Delta^{n-1}}) \Delta s \Delta t_1 \dots \Delta t_{n-1} \Bigg) \Bigg)
\end{aligned}$$

where $t, t_1, t_2, \dots, t_n \in \mathbb{T}_1$ and $t', t'_1, t'_2 \dots, t'_n \in \mathbb{T}_2$. It is clear that (y_1, y_2) is a solution of IVP-SIP iff (y_1, y_2) solves the operator equation $(y_1, y_2) = T(y_1, y_2)$. In other words a fixed point for the operator $(y_1, y_2) = T(y_1, y_2)$ is a solution for the IVP-SIP.

We use Schauder's fixed point theorem to show the existence of a solution.

From [40] it can be seen that the space $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is convex.

Claim 0.19. T is a completely continuous map.

We first show that T is continuous. We prove it by showing that T preserves convergence. Indeed let (y_{n1}, y_{n2}) be a sequence of functions in $\mathscr{C}(\mathbb{T}_1) \times \mathscr{C}(\mathbb{T}_2)$ such that

$$lim_{n\to\infty} \|(y_{n1}, y_{n2}) - (y_1, y_2)\| \to 0.$$

The above equation implies that

$$\begin{aligned}
lim_{n\to\infty} \|(y_{n1} - y_1, y_{n2} - y_2)\| &\to 0 \\
i.e., lim_{n\to\infty} sup_{t_1 \in \mathbb{T}_1} |(y_{n1} - y_1)(t_1)| &\to 0 \\
\text{and } lim_{n\to\infty} sup_{t_2 \in \mathbb{T}_2} |(y_{n2} - y_2)(t_2)| &\to 0.
\end{aligned}$$

Let us consider

$\|T(y_{n1},y_{n2})-T(y_1,y_2)\|$

$$\begin{aligned}
&= sup_{t_1\in\mathbb{T}_1}\Bigg|\int_0^t\int_0^{t_{n-1}}\dots\int_0^{t_2}\int_0^{t_1} f_1(s,y_{n1},\dots,y_{n1}^{\Delta^{n-1}})\Delta s\Delta t_1\dots\Delta t_{n-1}\\
&- \int_0^t\int_0^{t_{n-1}}\dots\int_0^{t_2}\int_0^{t_1} f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})\Delta s\Delta t_1\dots\Delta t_{n-1}\Bigg|\\
&+ sup_{t_2\in\mathbb{T}_2}\Bigg|\Bigg(\int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-1}}\dots\int_{\sigma(a)}^{t'_2}\int_{\sigma(a)}^{t'_1} f_2(s,y_{n2},\dots,y_{n2}^{\Delta^{n-1}})\Delta s\Delta t'_1\dots\Delta t'_{n-1}\\
&- \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-1}}\dots\int_{\sigma(a)}^{t'_2}\int_{\sigma(a)}^{t'_1} f_2(s,y_2,\dots,y_2^{\Delta^{n-1}})\Delta s\Delta t'_1\dots\Delta t'_{n-1}\Bigg)\\
&+ \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-2}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-1}}{\rho_{2n}}\left(\int_0^a f_1(s,y_{n1},\dots,y_{n1}^{\Delta^{n-1}})\Delta s\right)\Delta t\Delta t'_1\dots\Delta t'_{n-2}\\
&- \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-2}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-1}}{\rho_{2n}}\left(\int_0^a f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})\Delta s\right)\Delta t\Delta t'_1\dots\Delta t'_{n-2}\\
&+ \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-3}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-3}}{\rho_{2n-2}}\Bigg(\int_0^a\int_0^{t_{n-1}} f_1(s,y_{n1},\dots,\\
&\quad y_{n1}^{\Delta^{n-1}})\Delta s\Delta t_{n-1}\Bigg)\Delta t\Delta t'_1\dots\Delta t'_{n-3}\\
&- \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-3}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-3}}{\rho_{2n-2}}\Bigg(\int_0^a\int_0^{t_{n-1}} f_1(s,y_1,\dots,\\
&\quad y_1^{\Delta^{n-1}})\Delta s\Delta t_{n-1}\Bigg)\Delta t\Delta t'_1\dots\Delta t'_{n-3}
\end{aligned}$$

$$\begin{aligned}
&+ \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-4}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-5}}{\rho_{2n-4}}\Bigg(\int_0^a\int_0^{t_{n-1}}\int_0^{t_{n-2}} f_1(s,y_{n1},\dots,\\
&\quad y_{n1}^{\Delta^{n-1}})\Delta s\Delta t_{n-2}\Delta t_{n-1}\Bigg)\Delta t\Delta t'_1\dots\Delta t'_{n-4}\\
&- \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-4}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-5}}{\rho_{2n-4}}\Bigg(\int_0^a\int_0^{t_{n-1}}\int_0^{t_{n-2}} f_1(s,y_1,\dots,\\
&\quad y_1^{\Delta^{n-1}})\Delta s\Delta t_{n-2}\Delta t_{n-1}\Bigg)\Delta t\Delta t'_1\dots\Delta t'_{n-4}\\
&+ \dots+\frac{\rho_1}{\rho_2}\left(\int_0^a\int_0^{t_{n-1}}\dots\int_0^{t_1} f_1(s,y_{n1},\dots,y_{n1}^{\Delta^{n-1}})\Delta s\Delta t_1\dots\Delta t_{n-1}\right)\\
&- \dots+\frac{\rho_1}{\rho_2}\left(\int_0^a\int_0^{t_{n-1}}\dots\int_0^{t_1} f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})\Delta s\Delta t_1\dots\Delta t_{n-1}\right)\Bigg|.
\end{aligned}$$

Since (f_1,f_2) is continuous on $\mathscr{C}(\mathbb{T}_1)\times\mathscr{C}(\mathbb{T}_2)$ we have

$$\begin{aligned}
lim_{n\to\infty}|f_1(s,y_{n1},\dots,y_{n1}^{\Delta^{n-1}})-f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})| &\to 0,\\
lim_{n\to\infty}|f_2(s,y_{n2},\dots,y_{n2}^{\Delta^{n-1}})-f_2(s,y_2,\dots,y_2^{\Delta^{n-1}})| &\to 0.
\end{aligned}$$

Now $\|T(y_{n1},y_{n2})-T(y_1,y_2)||$

$$
\begin{aligned}
\leq\ & sup_{t_1\in\mathbb{T}_1}\int_0^t\int_0^{t_{n-1}}\dots\int_0^{t_2}\int_0^{t_1}|f_1(s,y_{n1},\dots,y_{n1}^{\Delta^{n-1}})\\
-\ & f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})|\Delta s\Delta t_1\dots\Delta t_{n-1}\\
+\ & sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-1}}\dots\int_{\sigma(a)}^{t'_2}\int_{\sigma(a)}^{t'_1}|f_2(s,y_{n2},\dots,y_{n2}^{\Delta^{n-1}})\\
-\ & f_2(s,y_2,\dots,y_2^{\Delta^{n-1}})|\Delta s\Delta t'_1\dots\Delta t'_{n-1}\\
+\ & \left(\frac{\rho_{2n-1}}{\rho_{2n}}\right)sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-2}}\dots\int_{\sigma(a)}^{t'_1}\int_0^a|f_1(s,y_{n1},\dots,y_{n1}^{\Delta^{n-1}})\\
-\ & f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})|\Delta s\Delta t\Delta t'_1\dots\Delta t'_{n-2}\\
+\ & \left(\frac{\rho_{2n-3}}{\rho_{2n-2}}\right)sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-3}}\dots\int_{\sigma(a)}^{t'_1}\int_0^a\int_0^{t_{n-1}}|f_1(s,y_{n1},\dots,y_{n1}^{\Delta^{n-1}})\\
-\ & f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})|\Delta s\Delta t_{n-1}\Delta t\Delta t'_1\dots\Delta t'_{n-3}\\
+\ & \left(\frac{\rho_{2n-5}}{\rho_{2n-4}}\right)sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-4}}\dots\int_{\sigma(a)}^{t'_1}\int_0^a\int_0^{t_{n-1}}\int_0^{t_{n-2}}|f_1(s,y_{n1},\dots,y_{n1}^{\Delta^{n-1}})\\
-\ & f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})|\Delta s\Delta t_{n-2}\Delta t_{n-1}\Delta t\Delta t'_1\dots\Delta t'_{n-4}\\
+\ & \dots+\\
+\ & \left(\frac{\rho_1}{\rho_2}\right)\int_0^a\int_0^{t_{n-1}}\dots\int_0^{t_1}|f_1(s,y_{n1},\dots,y_{n1}^{\Delta^{n-1}})\\
-\ & f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})|\Delta s\Delta t_1\dots\Delta t_{n-1}
\end{aligned}
$$

Hence, $lim_{n\to\infty}\|T(y_{n1},y_{n2})-T(y_1,y_2)\|\to 0$ proving that T is continuous. Let

$$
\begin{aligned}
f_1(s,y_1,\dots,y_1^{\Delta^{n-1}}) &\leq M_1,\ \text{for some } M_1>0,\forall s\in\mathbb{T}_1,\\
f_2(s,y_2,\dots,y_2^{\Delta^{n-1}}) &\leq M_2,\ \text{for some } M_2>0,\forall s\in\mathbb{T}_2.
\end{aligned}
$$

We now show that $T(\mathscr{C}(\mathbb{T}_1)\times\mathscr{C}(\mathbb{T}_2))$ is bounded and equicontinuous subset of $\mathscr{C}(\mathbb{T}_1)\times\mathscr{C}(\mathbb{T}_2)$. Let us assume that $\|(y_1,y_2)\|\leq M$. Then

$$\begin{aligned}
\|T(y_1,y_2)\| \leq\ & sup_{t_1\in\mathbb{T}_1}\int_0^t\int_0^{t_{n-1}}\dots\int_0^{t_2}\int_0^{t_1}|f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})|\Delta s\Delta t_1\dots\Delta t_{n-1}\\
+\ & sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-1}}\dots\int_{\sigma(a)}^{t'_2}\int_{\sigma(a)}^{t'_1}|f_2(s,y_2,\dots,y_2^{\Delta^{n-1}})|\Delta s\Delta t'_1\dots\Delta t'_{n-1}\\
+\ & sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-2}}\dots\int_{\sigma(a)}^{t'_1}\left(\frac{\rho_{2n-1}}{\rho_{2n}}\right)\Bigg(\int_0^a|f_1(s,y_1,\dots,\\
& y_1^{\Delta^{n-1}})|\Delta s\Bigg)\Delta t\Delta t'_1\dots\Delta t'_{n-2}\\
+\ & sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-3}}\dots\int_{\sigma(a)}^{t'_1}\left(\frac{\rho_{2n-3}}{\rho_{2n-2}}\right)\Bigg(\int_0^a\int_0^{t_{n-1}}|f_1(s,y_1,\dots,\\
& y_1^{\Delta^{n-1}})|\Delta s\Delta t_{n-1}\Bigg)\Delta t\Delta t'_1\dots\Delta t'_{n-3}\\
+\ & sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-4}}\dots\int_{\sigma(a)}^{t'_1}\left(\frac{\rho_{2n-5}}{\rho_{2n-4}}\right)\Bigg(\int_0^a\int_0^{t_{n-1}}\int_0^{t_{n-2}}|f_1(s,y_1,\dots,\\
& y_1^{\Delta^{n-1}})|\Delta s\Delta t_{n-2}\Delta t_{n-1}\Bigg)\Delta t\Delta t'_1\dots\Delta t'_{n-4}\\
+\ & \dots+\frac{\rho_1}{\rho_2}\left(\int_0^a\int_0^{t_{n-1}}\dots\int_0^{t_1}|f_1(s,y_{n1},\dots,y_{n1}^{\Delta^{n-1}})|\Delta s\Delta t_1\dots\Delta t_{n-1}\right)
\end{aligned}$$

Since (f_1,f_2) is bounded we can conclude that there exists a $K>0$ independent of choice of (y_1,y_2) such that $\|T(y_1,y_2)\|\leq K$. Hence, $T(\mathscr{C}(\mathbb{T}_1)\times\mathscr{C}(\mathbb{T}_2))$ is bounded.

Claim 0.20.

$$\int_a^{t_n}\int_a^{t_{n-1}}\dots\int_a^{t_1}dt_0dt_1\dots dt_{n-1}=\sum_{i=0}^{n}\frac{(-1)^i a^i {t_n}^{n-i}}{i!(n-i)!} \tag{0.14}$$

for all n $\geq$1 and for all a in $\mathbb{R}^+$.

We'll Prove By induction on n.

For n=1 ,

$$\int_a^{t_1}dt_0=t_1-a$$

Hence (0.14) is true for n=1.

Now, let (0.14) be true for n=k. i.e,

$$\int_a^{t_k}\int_a^{t_{k-1}}\dots\int_a^{t_1}dt_0dt_1\dots dt_{k-1}=\sum_{i=0}^{k}\frac{(-1)^i a^i {t_k}^{k-i}}{i!(k-i)!}$$

Now for n= k+1,

$$\int_a^{t_{k+1}}\int_a^{t_k}\dots\int_a^{t_1} dt_0 dt_1\dots dt_k = \int_a^{t_{k+1}}\sum_{i=0}^{k}\frac{(-1)^i a^i {t_k}^{k-i}}{i!(k-i)!}dt_k$$

from above case.

$$\begin{aligned}
&= \sum_{i=0}^{k}\frac{(-1)^i a^i}{i!(k-i)!}\int_a^{t_{k+1}} {t_k}^{k-i}dt_k\\
&= \sum_{i=0}^{k}\frac{(-1)^i a^i}{i!(k-i)!}\left(\frac{{t_{k+1}}^{(k+1)-i}}{(k+1)-i}-\frac{a^{(k+1)-i}}{(k+1)-i}\right)\\
&= \sum_{i=0}^{k}\frac{(-1)^i a^i}{i!(k-i)!}\left(\frac{{t_{k+1}}^{(k+1)-i}}{(k+1)-i}\right)-\left(\sum_{i=0}^{k}\frac{(-1)^i a^i}{i!(k-i)!}\frac{a^{(k+1)-i}}{(k+1)-i}\right)\\
&= \sum_{i=0}^{k}\frac{(-1)^i a^i}{i!(k-i)!}\left(\frac{{t_{k+1}}^{(k+1)-i}}{(k+1)-i}\right)+\left(\sum_{i=0}^{k}\frac{(-1)^{i+1} a^i}{i!(k-i)!}\frac{a^{(k+1)-i}}{(k+1)-i}\right)
\end{aligned}$$

Let us consider the Second term above

$$\begin{aligned}
&\sum_{i=0}^{k}\frac{(-1)^{i+1} a^i}{i!(k-i)!}\frac{a^{(k+1)-i}}{(k+1)-i}\\
&= \sum_{i=0}^{k}\frac{(-1)^{(i+1)}a^{(k+1)}}{i!(k+1-i)!} = a^{(k+1)}\left[\frac{-1}{(k+1)!}+\frac{1}{1!k!}-\frac{1}{2!(k-1)!}+\dots+\frac{(-1)^{(k+1)}}{k!}\right]\\
&= -\frac{a^{k+1}}{(k+1)!}\left[1-(k+1)+\frac{k(k+1)}{2!}-\dots+(-1)^k k+1\right]\\
&= -\frac{a^{k+1}}{(k+1)!}\left[1-(k+1)+\frac{k(k+1)}{2!}-\dots+(-1)^k k+1+(-1)^{(k+1)}-(-1)^{(k+1)}\right]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{a^{k+1}}{(k+1)!}\left[\sum_{i=0}^{k+1}\left(\binom{k+1}{i}(-1)^i\right)-(-1)^{k+1}\right]\\
&= -\frac{a^{k+1}}{(k+1)!}\left[\sum_{i=0}^{k+1}\left(\binom{k+1}{i}(-1)^i(1)^{k+1-i}\right)-(-1)^{k+1}\right]\\
&= -\frac{a^{k+1}}{(k+1)!}\left[(1-1)^{k+1}-(-1)^{k+1}\right] = \frac{(-1)^{k+1}a^{(k+1)}}{(k+1)!}
\end{aligned}$$

So,

$$\begin{aligned}
&\int_a^{t_{k+1}}\int_a^{t_k}\dots\int_a^{t_1} dt_0 dt_1\dots dt_k\\
&= \sum_{i=0}^{k}\left(\frac{(-1)^i a^i {t_{k+1}}^{(k+1)-i}}{i!((k+1)-i)!}\right)+\frac{(-1)^{k+1}a^{k+1}}{(k+1)!}\\
&= \sum_{i=0}^{(k+1)}\frac{(-1)^i a^i {t_{(k+1)}}^{(k+1)-i}}{i!((k+1)-i)!}.
\end{aligned}$$

Hence (0.14) holds for the case n=k+1. Therefore (0.14)is proved by Mathematical induction on n.

We next prove that $T(\mathscr{C}(\mathbb{T}_1)\times\mathscr{C}(\mathbb{T}_2))$ is equicontinuous subset of $\mathscr{C}(\mathbb{T}_1)\times\mathscr{C}(\mathbb{T}_2)$. We need to show that $\forall\varepsilon>0\ \exists\ \delta>0$ such that whenever

$$\begin{aligned} \| & \quad (t,t')-(t^*,t'')\|<\delta \\ \Rightarrow & \quad \|T(y_1(t),y_2(t'))-T(y_1(t^*),y_2(t''))\|<\varepsilon. \end{aligned}$$

Let us assume that $|t-t^*|+|t'-t''|<\delta$. We see that $T(y_1(t),y_2(t'))-T(y_1(t^*),y_2(t''))$

$$\begin{aligned}
= & \Bigg(\int_0^t\int_0^{t_{n-1}}\dots\int_0^{t_2}\int_0^{t_1} f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})\Delta s\Delta t_1\dots\Delta t_{n-1} \\
- & \int_0^{t^*}\int_0^{t_{n-1}}\dots\int_0^{t_2}\int_0^{t_1} f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})\Delta s\Delta t_1\dots\Delta t_{n-1}, \\
& \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-1}}\dots\int_{\sigma(a)}^{t'_2}\int_{\sigma(a)}^{t'_1} f_2(s,y_{n2},\dots,y_{n2}^{\Delta^{n-1}})\Delta s\Delta t'_1\dots\Delta t'_{n-1} \\
- & \int_{\sigma(a)}^{t''}\int_{\sigma(a)}^{t'_{n-1}}\dots\int_{\sigma(a)}^{t'_2}\int_{\sigma(a)}^{t'_1} f_2(s,y_{n2},\dots,y_{n2}^{\Delta^{n-1}})\Delta s\Delta t'_1\dots\Delta t'_{n-1} \\
+ & \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-2}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-1}}{\rho_{2n}}\left(\int_0^a f_1(s,y_{n1},\dots,y_{n1}^{\Delta^{n-1}})\Delta s\right)\Delta t\Delta t'_1\dots\Delta t'_{n-2} \\
- & \int_{\sigma(a)}^{t''}\int_{\sigma(a)}^{t'_{n-2}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-1}}{\rho_{2n}}\left(\int_0^a f_1(s,y_{n1},\dots,y_{n1}^{\Delta^{n-1}})\Delta s\right)\Delta t\Delta t'_1\dots\Delta t'_{n-2} \\
+ & \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-3}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-3}}{\rho_{2n-2}}\Bigg(\int_0^a\int_0^{t_{n-1}} f_1(s,y_{n1},\dots, \\
& y_{n1}^{\Delta^{n-1}})\Delta s\Delta t_{n-1}\Bigg)\Delta t\Delta t'_1\dots\Delta t'_{n-3}
\end{aligned}$$

$$\begin{aligned}
- & \int_{\sigma(a)}^{t''}\int_{\sigma(a)}^{t'_{n-3}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-3}}{\rho_{2n-2}}\Bigg(\int_0^a\int_0^{t_{n-1}} f_1(s,y_{n1},\dots, \\
& y_{n1}^{\Delta^{n-1}})\Delta s\Delta t_{n-1}\Bigg)\Delta t\Delta t'_1\dots\Delta t'_{n-3} \\
+ & \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-4}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-5}}{\rho_{2n-4}}\Bigg(\int_0^a\int_0^{t_{n-1}}\int_0^{t_{n-2}} f_1(s,y_{n1},\dots, \\
& y_{n1}^{\Delta^{n-1}})\Delta s\Delta t_{n-2}\Delta t_{n-1}\Bigg)\Delta t\Delta t'_1\dots\Delta t'_{n-4} \\
- & \int_{\sigma(a)}^{t''}\int_{\sigma(a)}^{t'_{n-4}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-5}}{\rho_{2n-4}}\Bigg(\int_0^a\int_0^{t_{n-1}}\int_0^{t_{n-2}} f_1(s,y_{n1},\dots, \\
& y_{n1}^{\Delta^{n-1}})\Delta s\Delta t_{n-2}\Delta t_{n-1}\Bigg)\Delta t\Delta t'_1\dots\Delta t'_{n-4}\Bigg)
\end{aligned}$$

Now

$$\int_0^t \int_0^{t_{n-1}} \dots \int_0^{t_2} \int_0^{t_1} f_1(s, y_1, \dots, y_1^{\Delta^{n-1}}) \Delta s \Delta t_1 \dots \Delta t_{n-1}$$
$$- \int_0^{t^*} \int_0^{t_{n-1}} \dots \int_0^{t_2} \int_0^{t_1} f_1(s, y_1, \dots, y_1^{\Delta^{n-1}}) \Delta s \Delta t_1 \dots \Delta t_{n-1}.$$

$$\begin{aligned}
&\leq \int_0^t \int_0^{t_{n-1}} \dots \int_0^{t_2} \int_0^{t_1} M_1 \Delta s \Delta t_1 \dots \Delta t_{n-1} \\
&- \int_0^{t^*} \int_0^{t_{n-1}} \dots \int_0^{t_2} M_1 \Delta s \Delta t_1 \dots \Delta t_{n-1} \\
&= \int_0^t \int_0^{t_{n-1}} \dots \int_0^{t_2} M_1 t_1 \Delta t_1 \dots \Delta t_{n-1} \\
&- \int_0^{t^*} \int_0^{t_{n-1}} \dots \int_0^{t_2} M_1 t_1 \Delta t_1 \dots \Delta t_{n-1} \\
&= \int_0^t \int_0^{t_{n-1}} \dots \int_0^{t_3} \frac{M_1 t_2^2}{2} \Delta t_2 \dots \Delta t_{n-1} \\
&- \int_0^{t^*} \int_0^{t_{n-1}} \dots \int_0^{t_3} \frac{M_1 t_2^2}{2} \Delta t_2 \dots \Delta t_{n-1} \\
&\vdots \\
&= M_1 \left(\frac{t^n}{n!} - \frac{t^{*n}}{n!} \right)
\end{aligned}$$

Using (0.16) we obtain

$$= \frac{1}{n!} M_1 (t - t^*) \left(t^{(n-1)} + t^* t^{(n-2)} + \dots + t^{*(n-2)} t + t^{*(n-1)} \right)$$

Hence, whenever $|t - t^*| < \delta$ we have

$$\begin{aligned}
\Bigg| & \int_0^t \int_0^{t_{n-1}} \dots \int_0^{t_2} \int_0^{t_1} f_1(s, y_1, \dots, y_1^{\Delta^{n-1}}) \Delta s \Delta t_1 \dots \Delta t_{n-1} \\
- & \int_0^{t^*} \int_0^{t_{n-1}} \dots \int_0^{t_2} \int_0^{t_1} f_1(s, y_1, \dots, y_1^{\Delta^{n-1}}) \Delta s \Delta t_1 \dots \Delta t_{n-1} \Bigg| < \frac{\varepsilon}{n+1}.
\end{aligned}$$

Also

$$\int_{\sigma(a)}^{t'} \int_{\sigma(a)}^{t'_{n-1}} \dots \int_{\sigma(a)}^{t'_2} \int_{\sigma(a)}^{t'_1} f_2(s, y_2, \dots, y_2^{\Delta^{n-1}}) \Delta s \Delta t'_1 \dots \Delta t'_{n-1}$$
$$- \int_{\sigma(a)}^{t''} \int_{\sigma(a)}^{t'_{n-1}} \dots \int_{\sigma(a)}^{t'_2} \int_{\sigma(a)}^{t'_1} f_2(s, y_2, \dots, y_2^{\Delta^{n-1}}) \Delta s \Delta t'_1 \dots \Delta t'_{n-1}.$$

$$\begin{aligned} \leq & \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-1}}\dots\int_{\sigma(a)}^{t'_2}\int_{\sigma(a)}^{t'_1} M_2\Delta s\Delta t'_1\dots\Delta t'_{n-1} \\ - & \int_{\sigma(a)}^{t''}\int_{\sigma(a)}^{t'_{n-1}}\dots\int_{\sigma(a)}^{t'_2}\int_{\sigma(a)}^{t'_1} M_2\Delta s\Delta t'_1\dots\Delta t'_{n-1} \end{aligned}$$

Using (0.14) we obtain

$$= M_2\left(\sum_{i=0}^{n}\frac{\sigma(a)^i t'^{(n-i)}}{i!(n-i)!} - \sum_{i=0}^{n}\frac{\sigma(a)^i t''^{(n-i)}}{i!(n-i)!}\right)$$

$$\begin{aligned} = & M_2\sum_{i=0}^{n}\frac{\sigma(a)^i}{i!(n-i)!}\left(t'^{(n-i)} - t''^{(n-i)}\right) \\ = & M_2\sum_{i=0}^{n}\frac{\sigma(a)^i}{i!(n-i)!}(t'-t'')\left(t'^{(n-i-1)} + t''t'^{(n-i-2)} + \dots\right. \\ + & \left. t''^{(n-i-2)}t' + t''^{(n-i-1)}\right) \text{ Using (0.16)} \end{aligned}$$

Hence, whenever $|t' - t''| < \delta$ we have

$$\begin{aligned} \Bigg| & \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-1}}\dots\int_{\sigma(a)}^{t'_2}\int_{\sigma(a)}^{t'_1} f_2(s,y_2,\dots,y_2^{\Delta^{n-1}})\Delta s\Delta t'_1\dots\Delta t'_{n-1} \\ - & \int_{\sigma(a)}^{t''}\int_{\sigma(a)}^{t'_{n-1}}\dots\int_{\sigma(a)}^{t'_2}\int_{\sigma(a)}^{t'_1} f_2(s,y_2,\dots,y_2^{\Delta^{n-1}})\Delta s\Delta t'_1\dots\Delta t'_{n-1}\Bigg| < \frac{\varepsilon}{n+1}. \end{aligned}$$

Now

$$\begin{aligned} & \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-2}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-1}}{\rho_{2n}}\left(\int_0^a f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})\Delta s\right)\Delta t\Delta t'_1\dots\Delta t'_{n-2} \\ - & \int_{\sigma(a)}^{t''}\int_{\sigma(a)}^{t'_{n-2}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-1}}{\rho_{2n}}\left(\int_0^a f_1(s,y_1,\dots,y_1^{\Delta^{n-1}})\Delta s\right)\Delta t\Delta t'_1\dots\Delta t'_{n-2}. \end{aligned}$$

$$\begin{aligned} \leq & \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-2}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-1}}{\rho_{2n}}\left(\int_0^a M_1\Delta s\right)\Delta t\Delta t'_1\dots\Delta t'_{n-2} \\ - & \int_{\sigma(a)}^{t''}\int_{\sigma(a)}^{t'_{n-2}}\dots\int_{\sigma(a)}^{t'_1}\frac{\rho_{2n-1}}{\rho_{2n}}\left(\int_0^a M_1\Delta s\right)\Delta t\Delta t'_1\dots\Delta t'_{n-2} \end{aligned}$$

Using (0.14) we obtain

$$\begin{aligned}
&= \frac{\rho_{2n-1}}{\rho_{2n}} M_1 a \left(\sum_{i=0}^{n-1} \frac{\sigma(a)^i t'^{(n-i-1)}}{i!(n-i-1)!} - \sum_{i=0}^{n-1} \frac{\sigma(a)^i t''^{(n-i-1)}}{i!(n-i-1)!} \right) \\
&= \frac{\rho_{2n-1}}{\rho_{2n}} M_1 a \sum_{i=0}^{n-1} \frac{\sigma(a)^i}{i!(n-i-1)!} \left(t'^{(n-i-1)} - t''^{(n-i-1)} \right) \\
&= \frac{\rho_{2n-1}}{\rho_{2n}} M_1 a \sum_{i=0}^{n-1} \frac{\sigma(a)^i}{i!(n-i-1)!} (t' - t'') \left(t'^{(n-i-2)} + t'' t'^{(n-i-3)} + \dots \right. \\
&+ \left. t''^{(n-i-3)} t' + t''^{(n-i-2)} \right) \text{ Using (0.16)}
\end{aligned}$$

Hence, whenever $|t' - t''| < \delta$ we have

$$\begin{aligned}
&\Bigg| \int_{\sigma(a)}^{t'} \int_{\sigma(a)}^{t'_{n-2}} \dots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-1}}{\rho_{2n}} \left(\int_0^a f_1(s, y_1, \dots, y_1^{\Delta^{n-1}}) \Delta s \right) \Delta t \Delta t'_1 \dots \Delta t'_{n-2} \\
&- \int_{\sigma(a)}^{t''} \int_{\sigma(a)}^{t'_{n-2}} \dots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-1}}{\rho_{2n}} \left(\int_0^a f_1(s, y_1, \dots, y_1^{\Delta^{n-1}}) \Delta s \right) \Delta t \Delta t'_1 \dots \Delta t'_{n-2} \Bigg| < \frac{\varepsilon}{n+1}.
\end{aligned}$$

Also

$$\begin{aligned}
&\int_{\sigma(a)}^{t'} \int_{\sigma(a)}^{t'_{n-3}} \dots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-3}}{\rho_{2n-2}} \left(\int_0^a \int_0^{t'_{n-1}} f_1(s, y_1, \dots, y_1^{\Delta^{n-1}}) \Delta s \Delta t'_{n-1} \right) \Delta t \Delta t'_1 \dots \Delta t'_{n-3} \\
&- \int_{\sigma(a)}^{t''} \int_{\sigma(a)}^{t'_{n-3}} \dots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-3}}{\rho_{2n-2}} \left(\int_0^a \int_0^{t'_{n-1}} f_1(s, y_1, \dots, y_1^{\Delta^{n-1}}) \Delta s \Delta t'_{n-1} \right) \Delta t \Delta t'_1 \dots \Delta t'_{n-3}.
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\sigma(a)}^{t'} \int_{\sigma(a)}^{t'_{n-3}} \dots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-3}}{\rho_{2n-2}} \left(\int_0^a \int_0^{t'_{n-1}} M_1 \Delta s \Delta t'_{n-1} \right) \Delta t \Delta t'_1 \dots \Delta t'_{n-3} \\
&- \int_{\sigma(a)}^{t''} \int_{\sigma(a)}^{t'_{n-3}} \dots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-3}}{\rho_{2n-2}} \left(\int_0^a \int_0^{t'_{n-1}} M_1 \Delta s \Delta t'_{n-1} \right) \Delta t \Delta t'_1 \dots \Delta t'_{n-3}
\end{aligned}$$

Using (0.14) we obtain

$$\begin{aligned}
&= \frac{\rho_{2n-3}}{\rho_{2n-2}} \frac{M_1 a^2}{2} \left(\sum_{i=0}^{n-2} \frac{\sigma(a)^i t'^{(n-i-2)}}{i!(n-i-2)!} - \sum_{i=0}^{n-2} \frac{\sigma(a)^i t''^{(n-i-2)}}{i!(n-i-2)!} \right) \\
&= \frac{\rho_{2n-3}}{\rho_{2n-2}} \frac{M_1 a^2}{2} \sum_{i=0}^{n-2} \frac{\sigma(a)^i}{i!(n-i-2)!} \left(t'^{(n-i-2)} - t''^{(n-i-2)} \right) \\
&= \frac{\rho_{2n-3}}{\rho_{2n-2}} \frac{M_1 a^2}{2} \sum_{i=0}^{n-2} \frac{\sigma(a)^i}{i!(n-i-2)!} (t' - t'') \left(t'^{(n-i-3)} + t'' t'^{(n-i-4)} + \dots \right. \\
&+ \left. t''^{(n-i-4)} t' + t''^{(n-i-3)} \right) \text{ Using (0.16)}
\end{aligned}$$

Hence, whenever $|t^{'} - t^{''}| < \delta$ we have

$$\Bigg| \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-3}} \cdots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-3}}{\rho_{2n-2}}\left(\int_0^a \int_0^{t'_{n-1}} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s \Delta t'_{n-1}\right)\Delta t \Delta t_1^{'} \ldots \Delta t_{n-3}^{'}$$

$$- \int_{\sigma(a)}^{t''}\int_{\sigma(a)}^{t'_{n-3}} \cdots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-3}}{\rho_{2n-2}}\left(\int_0^a \int_0^{t'_{n-1}} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s \Delta t'_{n-1}\right)\Delta t \Delta t_1^{'} \ldots \Delta t_{n-3}^{'}\Bigg| < \frac{\varepsilon}{n+1}.$$

Also

$$\int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-4}} \cdots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-5}}{\rho_{2n-4}}\left(\int_0^a \int_0^{t'_{n-1}} \int_0^{t'_{n-2}} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s \Delta t'_{n-2} \Delta t'_{n-1}\right)\Delta t \Delta t_1^{'} \ldots \Delta t_{n-4}^{'}$$

$$- \int_{\sigma(a)}^{t''}\int_{\sigma(a)}^{t'_{n-4}} \cdots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-5}}{\rho_{2n-4}}\left(\int_0^a \int_0^{t'_{n-1}} \int_0^{t'_{n-2}} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}})\Delta s \Delta t'_{n-2} \Delta t'_{n-1}\right)\Delta t \Delta t_1^{'} \ldots \Delta t_{n-4}^{'}$$

$$\leq \int_{\sigma(a)}^{t'}\int_{\sigma(a)}^{t'_{n-4}} \cdots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-5}}{\rho_{2n-4}}\left(\int_0^a \int_0^{t'_{n-1}} \int_0^{t'_{n-2}} M_1 \Delta s \Delta t'_{n-2} \Delta t'_{n-1}\right)\Delta t \Delta t_1^{'} \ldots \Delta t_{n-4}^{'}$$

$$- \int_{\sigma(a)}^{t''}\int_{\sigma(a)}^{t'_{n-4}} \cdots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-5}}{\rho_{2n-4}}\left(\int_0^a \int_0^{t'_{n-1}} \int_0^{t'_{n-2}} M_1 \Delta s \Delta t'_{n-2} \Delta t'_{n-1}\right)\Delta t \Delta t_1^{'} \ldots \Delta t_{n-4}^{'}$$

Using (0.14) we obtain

$$= \frac{\rho_{2n-5}}{\rho_{2n-4}} \frac{M_1 a^3}{6}\left(\sum_{i=0}^{n-3} \frac{\sigma(a)^i t'^{(n-i-3)}}{i!(n-i-3)!} - \sum_{i=0}^{n-3} \frac{\sigma(a)^i t''^{(n-i-3)}}{i!(n-i-3)!}\right)$$

$$= \frac{\rho_{2n-5}}{\rho_{2n-4}} \frac{M_1 a^3}{6} \sum_{i=0}^{n-3} \frac{\sigma(a)^i}{i!(n-i-3)!}\left(t'^{(n-i-3)} - t''^{(n-i-3)}\right)$$

$$= \frac{\rho_{2n-5}}{\rho_{2n-4}} \frac{M_1 a^3}{6} \sum_{i=0}^{n-3} \frac{\sigma(a)^i}{i!(n-i-3)!}(t' - t'')\Big(t'^{(n-i-4)} + t''t'^{(n-i-5)} + \ldots + t''^{(n-i-5)}t' + t''^{(n-i-4)}\Big) \text{ Using (0.16)}$$

Hence, whenever $|t^{'} - t^{''}| < \delta$ we have

$$\begin{aligned} \Bigg| \quad & \int_{\sigma(a)}^{t'} \int_{\sigma(a)}^{t'_{n-4}} \cdots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-5}}{\rho_{2n-4}} \left(\int_0^a \int_0^{t'_{n-1}} \int_0^{t'_{n-2}} f_1(s, y_1, \ldots, \right. \\ & \left. y_1^{\Delta^{n-1}}) \Delta s \Delta t'_{n-2} \Delta t'_{n-1} \right) \Delta t \Delta t'_1 \ldots \Delta t'_{n-4} \\ - \quad & \int_{\sigma(a)}^{t''} \int_{\sigma(a)}^{t'_{n-4}} \cdots \int_{\sigma(a)}^{t'_1} \frac{\rho_{2n-5}}{\rho_{2n-4}} \left(\int_0^a \int_0^{t'_{n-1}} \int_0^{t'_{n-2}} f_1(s, y_1, \ldots, \right. \\ & \left. y_1^{\Delta^{n-1}}) \Delta s \Delta t'_{n-2} \Delta t'_{n-1} \right) \Delta t \Delta t'_1 \ldots \Delta t'_{n-4} \Bigg| < \frac{\varepsilon}{n+1} \end{aligned}$$

Similarly it can be shown that the terms involving

$$\frac{\rho_{2n-7}}{\rho_{2n-6}} + \ldots + \frac{\rho_3}{\rho_4} \text{ can be made less than} \frac{\varepsilon}{n+1}$$

Now

$$\begin{aligned} & \frac{\rho_1}{\rho_2} \left(\int_0^a \int_0^{t_{n-1}} \cdots \int_0^{t_1} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}}) \Delta s \Delta t_1 \ldots \Delta t_{n-1} \right) \\ - \quad & \frac{\rho_1}{\rho_2} \left(\int_0^a \int_0^{t_{n-1}} \cdots \int_0^{t_1} f_1(s, y_1, \ldots, y_1^{\Delta^{n-1}}) \Delta s \Delta t_1 \ldots \Delta t_{n-1} \right) \\ = \quad & 0 \end{aligned}$$

So, we see that

$$\|T(y_1(t), y_2(t') - T(y_1(t^*), y_2(t''))\| < \frac{\varepsilon}{n+1} + \frac{\varepsilon}{n+1} + \ldots + \frac{\varepsilon}{n+1} = \frac{\varepsilon(n+1)}{n+1} = \varepsilon$$

whenever

$$\begin{aligned} \| \quad & (t, t') - (t^*, t'')\| < \delta \\ i.e., \quad & |t - t^*| + |t' - t''| < \delta. \end{aligned}$$

So, $T(\mathscr{C}(\mathbb{T}_1) \times \mathscr{C}(\mathbb{T}_2))$ is equicontinuous subset of $\mathscr{C}(\mathbb{T}_1) \times \mathscr{C}(\mathbb{T}_2)$. Condition(iii) of Arzela-Acoli theorem can be seen from that fact that any $g_i(s, z_i) \in \mathscr{C}(\mathbb{T}_i \times \mathscr{C}(\mathbb{T}_i)),\ i = 1, 2, ...n$ is uniformly continuous on $\mathbb{T}_i$ as $\mathbb{T}_i$ is compact(since closed and bounded).Thus T is compact by Arzela-Ascoli theorem. So from Schauder's fixed point theorem(0.6) a fixed point exists for the operator equation $(y_1, y_2) = Ty$. Hence a solution exists for the IVP-SIP. □

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] F. V. Atkinson, *Discrete and continuous boundary problems*, Mathematics in Science and Engineering, Vol. 8 Academic Press, New York-London, 1964.

[2] M. Bohner and A. Peterson, *Dynamic equations on time scales. An introduction with applications*, Birkhuser Boston, Inc., Boston, MA, 2001.

[3] C. Allan Boyles, *Acoustic Waveguides, Applications to Oceanic Sciences*, JohnWiley and Sons, 1978.

[4] C. T. Fulton, *Parametrizations of Titchmarshs $m(\lambda)$-functions in the limit circle case*, Trans. Amer. Math. Soc. **229** (1977), 51-63.

[5] P. K. Ghosh, *The Mathematics of Waves and Vibrations*, MacMillan, 1975.

[6] D. B. Hinton and J. K. Shaw, *Titchmarsh-Weyl theory for Hamiltonian systems. Spectral theory of differential operators (Birmingham, Ala., 1981)*, North-Holland Math. Stud., 219-231, 55, North-Holland, Amsterdam-New York, 1981.

[7] John Von Neumann, *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Ann. **102**(1929), 49–131.

[8] H. G. Kaper, M. Kwong, and A. Zettl, *Anton Characterizations of the Friedrichs extensions of singular Sturm-Liouville expressions*, SIAM J. Math. Anal. **17** (1986), 772-777.

[9] K. Kodaria, *On ordinary differential equations of any even order and the corresponding eigenfunction expansions*, Amer. J. Math. **72** (1950), 502-544.

[10] B. M. Levitan and I. S. Sargsjan, *Introduction to spectral theory: selfadjoint ordinary differential operators*, Translated from the Russian by Amiel Feinstein. Translations of Mathematical Monographs, Vol. 39. American Mathematical Society, Providence, R.I., 1975.

[11] G. L. Lamb, *Elements of Soliton Theory*, JohnWiley and Sons, 1980.

[12] F. Merdivenci Atici and G. Sh. Guseinov, *On Greens functions and positive solutions for boundary value problems on time scales*, J. Comput. Appl. Math.**141**(2002), 75-99.

[13] M. A. Naimark, *Linear differential operators. Part II: Linear differential operators in Hilbert space*, With additional material by the author, and a supplement by V. . Ljance. Translated from the Russian by E. R. Dawson. English translation edited by W. N. Everitt Frederick Ungar Publishing Co., New York 1968.

[14] K. Noda, *Optical Fiber Transmission, Studies in Telecommunications*, North-Holland, 1986.

[15] Pallav Kumar Baruah and Dibya Jyothi Das, *Singular Sturm Liouville problems with an interface*, Int. J. Math. Sci. **3** (2004), 323-340.

[16] Pallav Kumar Baruah and D. K. K. Vamsi, *IVPs for Singular Interface Problems*, to appear.

[17] Pallav Kumar Baruah and Dasu Krishna Kiran Vamsi, *Oscillation Theory for a Pair of Second Order Dynamic Equations with a Singular Interface*, Electron. J. Differential Equations **43** (2008), 1–7.

[18] Pallav Kumar Baruah and M. Venkatesulu, *Characterization of the resolvent of a differential operator generated by a pair of singular ordinary differential expressions satisfying certain matching interface conditions*, Int. J. Mod. Math. **1** (2006), 31–47.

[19] Pallav Kumar Baruah and M. Venkatesulu, *Deficiency indices of a differential operator satisfying certain matching interface conditions*, Electron. J. Differential Equations **38** (2005), 1–9.

[20] Pallav Kumar Baruah and M. Venkatesulu, *Number of linearly independent square integrable solutions of a pair of ordinary differential equations satisfying certain matching interface conditions*, to appear.

[21] Pallav Kumar Baruah and M. Venkatesulu, *Self adjoint boundary value problems associated with a pair of singular ordinary differential expressions with interface spatial conditions*, submitted.

[22] Pallav Kumar Baruah and M. Venkatesulu, *Spectrum of pair of ordinary differential operators with a matching interface conditions*, to appear.

[23] A. Pleijel, *Generalized Weyl circles*, Lecture Notes in Math., Vol. 415, Springer, Berlin, 1974, 211-226.

[24] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations, Part I*, Second Edition Clarendon Press, Oxford, 1962.

[25] D. K. K. Vamsi, *IVP's for a Pair of Dynamic Equations with Matching Interface Conditions*, Sri Sathya Sai Institute of Higher Learning, March 2006.

[26] M. Venkatesulu and Pallav Kumar Baruah, *A classical approach to eigenvalue problems associated with a pair of mixed regular Sturm-Liouville equations-I*, J. Appl. Math. Stoch. Anal. **14** (2001), 205-214.

[27] M. Venkatesulu and Pallav Kumar Baruah, *A classical approach to eigenvalue problems associated with a pair of mixed regular Sturm-Liouville equations-II*, J. Appl. Math. Stoch. Anal. **15** (2002), 197-203.

[28] M. Venkatesulu and T. Gnana Bhaskar, *Computation of Greens matrices for boundary value problems associated with a pair of mixed linear regular ordinary differential operators*, Int. J. Math. Math. Sci. **18** (1995), 789-797.

[29] M. Venkatesulu and T.Gnana Bhaskar, *Fundamental systems and solutions of nonhomogeneous equations for a pair of mixed linear ordinary differential equations*, J. Aust. Math. Soc.(Series A), **49** (1990), 161-173.

[30] M. Venkatesulu and T.Gnana Bhaskar, *Selfadjoint boundary value problems associated with a pair of mixed linear ordinary differential equations*, J. Math. Anal. Appl. **144** (1989), 322-341.

[31] M. Venkatesulu and Pallav Kumar Baruah, *Solutions of initial value problems for a pair of linear first order ordinary differential systems with interface spatial conditions*, J. Appl. Math. Stoch. Anal. **9** (1996), 303-314.

[32] M. Venkatesulu and T.Gnana Bhaskar, *Solutions of initial value problems associated with a pair of mixed linear ordinary differential equations*, J. Math. Anal. Appl. **148** (1990), 63-78.

[33] D.K.K.Vamsi and Pallav Kumar Baruah, *A study of pair of Dynamic equations with singular interface*, M.Phil. Thesis, Sri Sathya Sai Institute of Higher Learning, 2008.

[34] D.K.K.Vamsi and Pallav Kumar Baruah,*Existential Results for Nonlinear Singular Interface Problems Involving Second Order Nonlinear Dynamic Equations Using Picards Iterative Technique*, The Journal of Nonlinear Analysis and Applications, Vol. 4, No. 3 , (2011), 200-209.

[35] D.K.K.Vamsi and Pallav Kumar Baruah,*Disconjugacy(D) and Non-Oscillation(N) Domains for Nonlinear Singular Interface Problems on Semi Infinite Time Scales*,ISRN Mathematical Analysis, Vol. 2011, 23.

[36] D.K.K.Vamsi and Pallav Kumar Baruah, *Existential Results for Nonlinear Sturm-Liouville Singular Interface Problems*,International Journal of Mathematics Research, Vol. 4, No. 1, (2012), 1-14.

[37] D.K.K.Vamsi and Pallav Kumar Baruah,*Existential of Positive Solutions and Eigenvalue Intervals for Nonlinear Sturm Liouville Problems with a Singular Interface*,Electronic Journal of Differential Equations, Vol. (2012), No. 53, (2012), 1-12.

[38] D.K.K.Vamsi and Pallav Kumar Baruah, *Existential of Multiple Positive Solutions Localization of Eigen Values for a Nonlocal Boundary Value Problem Involving a Pair of Second Order Dynamic Equations with a Singular Interface*,Advances in Theoretical and Applied Mathematics , Vol. 7, No. 3, (2012), 281-294.

[39] D.K.K.Vamsi and Pallav Kumar Baruah, *Existential of Multiple Positive Solutions Localization of Eigen Values for Nonlinear Sturm Liouville Problems with a Singular Interface using the Greens Matrix*,International Journal of Mathematical Analysis, Vol. 6, No. 39, (2012), 1891-1910.

[40] D.K.K.Vamsi and Pallav Kumar Baruah, *Existential and Uniqueness Results for IVPs associated with Nonlinear Singular Interface Problems on Time Scales*, International Journal of Nonlinear Analysis and Applications, Vol. 2012.

[41] D.K.K.Vamsi and Pallav Kumar Baruah,*Existential and Uniqueness Results for BVPs associated with Nonlinear Singular Interface Problems on Time Scales*, Advanced Dynamical Systems and Applications, Vol. 6, No. 2, (2011), 271-290.

[42] H. Weyl, *Uber gewhnliche Differentialgleichungen mit Singularitten und die zugehrigen Entwicklungen willkrlicher Funktionen* (German), Math. Ann. **68** (1910), 220-269.

[43] Chi-Teh Wang, *Applied Elasticity*, McGraw Hill, 1953.