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ON EVENTUAL UNIFORM STABILITY OF CAPUTO FRACTIONAL HYBRID SYSTEMS WITH APPLICATION TO LIBRARY AND INFORMATION SYSTEMS

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Abstract. This work presents a novel eventual stability approach for Caputo fractional dynamic equations on time scales, extending existing eventual stability theory for continuous domain to hybrid domain (continuous and discrete time). This novel framework is then applied to ascertain the long term uniform stability of a complex Library and information system, providing both theoretical insights and actionable recommendations for the management and coordination of library consortia.

Keywords: Caputo fractional derivative; eventual stability; dynamic equations on time scales; library information systems.

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1. Introduction

Fractional calculus represents a significant advancement in mathematical modelling, extending classical differential and difference equations to systems characterized by memory effects,

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nonlocal dynamics, and hereditary properties [6, 7, 11, 15]. Unlike integer-order derivatives, which depend solely on local behaviour, fractional derivatives incorporate the entire history of a process, making them particularly suitable for describing complex real-world systems in which past states exert a persistent influence on present dynamics. Owing to this distinctive feature, fractional calculus has become an indispensable analytical tool in diverse fields such as viscoelasticity, control theory, signal processing, biomedical systems, and, more recently, information and social systems.

In recent years, substantial progress has been made in the qualitative theory of Caputo fractional differential equations, particularly in the development of rigorous stability criteria based on Lyapunov methods and comparison principles [8, 12, 18]. These results have been further extended to Caputo fractional dynamic equations on time scales, thereby unifying the analysis of continuous-time and discrete-time systems within a single mathematical framework [13, 22, 25, 26]. This unification is especially valuable for the study of hybrid systems that exhibit both smooth evolution and abrupt changes, which commonly arise in networked environments, digital infrastructures, and adaptive service platforms. Notably, this fractional time-scale approach has demonstrated its effectiveness in applied contexts, including biomedical dynamics, as illustrated in [5].

Caputo fractional dynamic equations on time scales, using the classical definitions of stability, stability, uniform stability, and uniform asymptotic stability, describe the solution behaviour from the initial time onwards. These definitions implicitly assume that the stability properties are satisfied immediately after the initial time. However, in many practical systems, this assumption is not satisfied, especially in systems where strong memory effects, hybrid continuous-discrete dynamics, impulsive perturbations, and transient external inputs are involved. In such systems, unstable or oscillatory behaviour can be observed during an initial phase before the system finally settles into a stable state. This observation leads to the definition of eventual uniform stability and eventual uniform asymptotic stability, which greatly extend the applicability of stability theory to Caputo fractional dynamic systems on time scales [16, 24, 23].

Eventual uniform stability permits system behavior to have an initial transient phase during which the classical stability conditions are not satisfied, while ensuring that after a finite time threshold, regardless of the initial conditions, the system behaves uniformly stable. This concept describes the realistic behaviour of many Caputo fractional dynamic systems, which may initially have a slow decay, a delayed response, or memory-induced oscillations before finally stabilizing. Eventual uniform asymptotic stability relaxes the requirement of immediate uniform convergence while retaining the key feature of uniform attractivity after a finite time shift. Eventual uniform stability and eventual uniform asymptotic stability are particularly well suited to Caputo fractional dynamic equations on time scales, as these systems inherently encode past states through non-local operators. Memory effects often lead to slow convergence rates and long transient phases that violate classical stability assumptions, even though the system is stable in the long run. By allowing for such transient behaviour, eventual stability concepts enable the construction of Lyapunov functions and comparison systems without imposing excessively restrictive derivative conditions at the initial time. Consequently, a broader class of non-linear, hybrid, and impulsive fractional systems becomes amenable to rigorous stability analysis within a unified framework.

In this work, we establish clear conditions for the eventual uniform stability and eventual uniform asymptotic stability of Caputo fractional dynamic systems on time scales using the comparison principle approach. We also show how the proposed theoretical results apply to Library and Information Systems (LIS). In this field, aspects like resource allocation, information spread, user engagement, and service use naturally display fractional-order dynamics and multi-scale behavior. Fractional calculus serves as a fitting modeling framework for LIS systems because historical usage patterns, institutional memory, and processes dependent on prior events significantly influence current operations and future development. The results in this paper contribute to the theoretical foundations of fractional hybrid systems and provide a solid framework for examining stability in complex information-driven settings.

LIS is an interdisciplinary area focused on creating, organizing, preserving, sharing, and using information resources within various institutional and social contexts [19, 20]. Traditionally, it stemmed from librarianship, which dealt with physical collections and cataloging [21]. Over time, LIS has changed significantly to include digital information systems, data science, knowledge management, information organization, and user experience design. Today, LIS exists at the crossroads of technology, social science, and the humanities, tackling essential questions about access to information, equity, literacy, preservation, and the ethical aspects of organizing knowledge in increasingly digital and connected environments.

The main goal of LIS institutions—libraries, archives, museums, and information centers—has expanded from simply taking care of collections to actively participating in knowledge creation, supporting digital scholarship, aiding research, and fostering community development. Modern library ecosystems act as complex adaptive systems with dynamic interactions among collections, services, technologies, users, and institutional contexts. This complexity requires advanced analytical frameworks that can model the non-linear, memory-dependent, and multi-scale dynamics found in today's information service delivery.

Traditionally, LIS research has mainly used qualitative, descriptive, and statistical methods, with limited application of dynamical systems theory [2]. However, as libraries increasingly transform into complex, interconnected service ecosystems that include digital repositories, discovery platforms, interlibrary loan networks, instructional programs, and research support services, their operational behaviour aligns more closely with a dynamical systems approach [3]. In these settings, service use directly impacts resource allocation, which subsequently affects future service delivery. Previous service use patterns influence current user behaviour and institutional decision-making, while changes in one service area, like digital collections, affect others such as reference and consultation services. Additionally, limited service capacities create natural growth constraints, and interactions across multiple time scales, such as daily operations, monthly performance reviews, and annual strategic planning cycles, collectively influence system dynamics.

Through this integrated mathematical-empirical approach, we establish that fractional dynamics on time scales offer a rigorous, flexible, and deeply insightful framework for modelling, analysing, and designing stable, resilient information service ecosystems. This bridges a significant gap between abstract dynamical systems theory and practical information institution management, offering tools for evidence-based decision-making in an era of increasingly complex and interdependent library networks.

The remainder of this paper is organized as follows. Section 2 presents the necessary methodological background on time scales and fractional calculus, including the definition of the Caputo fractional delta derivative of a Lyapunov function and the associated comparison theorem. Section 3 is devoted to the development of the main theoretical results on eventual uniform stability and eventual uniform asymptotic stability. In Section 4, these results are applied to a five-dimensional Library and Information Systems (LIS) model with empirically parameterized coefficients; we perform a Lyapunov-based eventual uniform asymptotic stability analysis using Caputo fractional delta derivatives and support the theoretical findings with numerical simulations and stability verification. Section 5 discusses the implications of the results for information science, while Section 6 concludes the paper with recommendations for library management and directions for future research.

2. Methodology

Definition 2.1. [10] *Let $\mathbb{T}$ be a time scale and let $f:\mathbb{T}\to\mathbb{R}^n$. The function f is said to be right-dense continuous on $\mathbb{T}$ denoted by $C_{rd}(\mathbb{T},\mathbb{R}^n)$, if it is continuous at every right-dense point $t\in\mathbb{T}$ and has finite left-sided limits at every left-dense point $t\in\mathbb{T}$.*

Definition 2.2. *A function $f:\mathbb{T}\to\mathbb{R}^n$ is said to belong to the space $C^{\alpha}_{rd}(\mathbb{T},\mathbb{R}^n)$, $0<\alpha<1$, if $f\in C_{rd}(\mathbb{T},\mathbb{R}^n)$ and $\exists$ a constant $L>0$ such that*

$$\|f(t)-f(s)\|\le L|t-s|^{\alpha},\quad \forall s,t\in\mathbb{T}.$$

Consider the Caputo fractional Delta system of the form

$$
\begin{aligned}
&{}^{C}\Delta^{\alpha}x(t)=f(t,x(t)),\quad t\in\mathbb{T},\\
&x(t_0)=x_0,\quad t_0\ge 0,
\end{aligned}
\tag{1}
$$

where the mapping $f:\mathbb{T}\times\mathbb{R}^N\to\mathbb{R}^N$ belongs to the space C^{α}_{rd} and satisfies $f(t,0)=0$ for all $t\in\mathbb{T}$. The operator ${}^{C}\Delta^{\alpha}$ denotes the Caputo fractional delta derivative of order α, acting on vector-valued functions $x(t)\in\mathbb{R}^N$ defined on the time scale $\mathbb{T}$. Let $x(\cdot)=x(\cdot;t_0,x_0)\in$

$C_{rd}(\mathbb{T},\mathbb{R}^N)$ represent the corresponding solution of (1). Throughout this work, we assume that such a solution exists and is unique in the sense established in [9].

Definition 2.3. [14] *We define the Caputo fractional delta Dini derivative of the Lyapunov function $V(t,x) \in C_{rd}[\mathbb{T}\times\mathbb{R}^N,\mathbb{R}^N_+]$ (which is locally Lipschitzian with respect to its second argument and $V(t,0)\equiv 0$) along the trajectories of solutions of the system (1) as:*

$$\begin{aligned} &{}^{C}\Delta_+^{\alpha}V(t,x)\\ &= \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg[\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^{\alpha}C_r)[V(\sigma(t)-r\mu,x(\sigma(t))-\mu^\alpha f(t,x(t)))\\ &\quad -V(t_0,x_0)]\Bigg], \end{aligned} \tag{2}$$

where $t\in\mathbb{T},x,x_0\in\mathbb{R}^N$, ${}^{\alpha}C_r$ is the binomial coefficient, $\sigma(t)=\inf\{s\in\mathbb{T}: s>t\}$ (forward jump operator see Definition 1.1 and 1.2 in [14] *for more details), $\mu=\sigma(t)-t$ and $x(\sigma(t))-\mu^\alpha f(t,x)\in\mathbb{R}^N$.*

Given another lower order Caputo fractional dynamic system of the form:

$$ {}^{C}\Delta^{\alpha}\kappa = \Theta(t,\kappa),\ \kappa(t_0)=\kappa_0\geq 0, \tag{3}$$

where $\kappa\in\mathbb{R}^n_+$, $\Theta:\mathbb{T}\times\mathbb{R}^n_+\to\mathbb{R}^n_+$ and $\Theta(t,0)\equiv 0$.

Next, we state the comparison theorem as established in [14].

Theorem 2.1. [14] *Assume that*

(i) $\Theta\in C_{rd}[\mathbb{T}\times\mathbb{R}^n_+,\mathbb{R}^n_+]$ *and* $\Theta(t,\kappa)\mu$ *is non-decreasing in* κ.

(ii) $V\in C_{rd}[\mathbb{T}\times\mathbb{R}^N,\mathbb{R}^N_+]$ *be locally Lipschitzian in the second variable such that*

$$ {}^{C}\Delta_+^{\alpha}V(t,x)\leq\Theta(t,V(t,x)),(t,x)\in\mathbb{T}\times\mathbb{R}^N.$$

(iii) $z(t)=z(t;t_0,u_0)$ *is the maximal solution of (3) existing on* $\mathbb{T}$.

Then,

$$V(t,x(t))\leq z(t),\quad t\geq t_0,$$

provided that

$$V(t_0,x_0)\leq\kappa_0,$$

where $x(t)=x(t;t_0,x_0)$ is any solution of (1)*, $t\in\mathbb{T}$, $t\geq t_0$.*

3. Main Results

Theorem 3.1. *Assume the following conditions are satisfied:*

(1) the function $V(t,x(t)) \in C_{rd}[\mathbb{T}\times\mathbb{R}^N,\mathbb{R}^N_+]$, $V(t,x(t))$ is locally Lipschitzian with respect to x, $V(t,0)\equiv 0$ and the inequality

$$\phi(\|x\|) \le V_0(t,x(t)) \le \theta(\|x\|), \tag{4}$$

holds for all $(t,x)\in\mathbb{T}\times\mathbb{R}^N$, $V_0(t,x)=\sum_{i=1}^N V_i(t,x(t))$ and $\phi,\theta\in\mathscr{K}$;

(2) $\Theta\in C_{rd}[\mathbb{T}\times\mathbb{R}^n_+,\mathbb{R}^n_+]$ is quasimonotone nondecreasing with respect to the second variable at all $t\in\mathbb{T}$, $\Theta(t,0)\equiv 0$, and

$$^{C}\Delta_+^{\alpha}V(t,x(t)) \le \Theta(t,V(t,x(t)));$$

(3) the set $\kappa=0$ of the comparison equation (3) is eventually uniformly stable.

Then, the set $x=0$ of the system (1) is eventually uniformly stable.

Proof. By the assumption of the eventual uniform stability of $\kappa=0$, let $\varepsilon>0$, then given $\phi(\varepsilon)>0$, we can find a $\delta_1=\delta_1(\varepsilon)>0$ and $\lambda_1=\lambda_1(\varepsilon)>0$ such that

$$\kappa(t;t_0,\kappa_0) < \phi(\varepsilon),\ \ \lambda_1(\varepsilon\le t_0\le t), \tag{5}$$

whenever $\kappa_0\le\delta_1$. Set $\delta=\theta^{-1}(\delta_1)$ and $\lambda_2(\varepsilon)=\beta(\delta(\varepsilon))$ where $\lambda=\lambda(\varepsilon)=\max[\lambda_1(\varepsilon),\lambda_2(\varepsilon)]$. Then, we can claim that with our choice of $\delta(\varepsilon)$ and $\lambda(\varepsilon)$,

$$\|x(t;t_0,x_0)\| < \varepsilon \text{ for } \lambda(\varepsilon)\le t_0\le t. \tag{6}$$

If this claim were false, then there would be some points t_1 and t_2 with $\lambda\le t_0<t_1<t_2$ where

$$\|x(t_1;t_0,x_0)\|=\delta,\ \ \|x(t_2;t_0,x_0)\|=\varepsilon, \tag{7}$$

and $\delta<\|x(t;t_0,x_0)\|<\varepsilon$ for $t_1<t<t_2$. Choose $\kappa_0=\theta(\|x_1\|)$, where $x_1=x(t_1;t_0,x_0)$. From Theorem 2.1, we know that

$$V(t;x(t;t_1,x_1)) \le z(t;t_1;\kappa_0),\ \ t\in[t_1,t_2]. \tag{8}$$

Combining (5), (6), (7), and (8), we obtain

$$\phi(\varepsilon)\le V\big(t_2,x(t_2;t_0,x_0)\big)\le z(t_2;t_1,\kappa_0)<\phi(\varepsilon). \tag{9}$$

Taking into account the uniformity of (5) and the fact that $\lambda \leq t_0 < t_1 < t_2$, inequality (9) yields a contradiction. This contradiction immediately confirms our claim and establishes the validity of (6). Consequently, the equilibrium set $x = 0$ is eventually uniformly stable. □

Theorem 3.2. *Assume the following*

1. $V(t,x) \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N_+]$ *and Lipschitzian in x.*
2. $\Theta \in C_{rd}[\mathbb{T} \times \mathbb{R}^n_+, \mathbb{R}^n_+]$ *and* $\Theta(t,\kappa)$ *is quasi-monotone non-decreasing in* κ *with* $\Theta(t,\kappa) \equiv 0$,
3. *For* $(t,x) \in \mathbb{T} \times \mathbb{R}^N$,
$$^C\Delta^\alpha_+ V_0(t,x) \leq -c(\|x\|),$$
where $c \in \mathscr{K}$.
4. $\phi(\|x\|) \leq V_0(t,x) \leq \theta(\|x\|)$, *where* $V_0(t,x) = \sum_{i=1}^N V_i(t,x(t))$ *and* $\theta, \phi \in \mathscr{K}$.
Then the equilibrium set $x = 0$ *of the FrDE* (1) *is eventually uniformly asymptotically stable.*

Proof. Given $0 < \varepsilon < \rho$, choose a $\delta_0 = \delta(\rho)$, $\lambda_0 = \lambda(\rho)$ and

$$T(\varepsilon) = \lambda(\varepsilon) + \frac{\theta(\rho)}{c[\delta(\varepsilon)]}, \quad c \in \mathscr{K}. \tag{10}$$

Let $t_0 \geq \lambda_0$ and $\|x_0 \leq \delta_0$. We then proceed to show that there exists a time $t_1 \in [t_0 + \lambda(\varepsilon), t_0 + T(\varepsilon)]$ such that

$$\|x(t_1;t_0,x_0)\| < \delta(\varepsilon), \tag{11}$$

since we have already shown that the equilibrium set $x = 0$ is eventually uniformly stable in Theorem 3.1. Assuming there is a chance that this is not possible, then

$$\delta(\varepsilon) \leq \|x(t;t_0,x_0)\| < \rho \text{ for } t \in [t_0 + \lambda(\varepsilon), t_0 + T(\varepsilon)]. \tag{12}$$

It is clear to see from assumptions 3 and 4 of the theorem, that

$$^C\Delta^\alpha_+ V(t,x(t;t_0,x_0)) \leq -c(\|x(t;t_0,x_0)\|) \leq -c[\delta(\varepsilon)] \text{ for } t \in [t_0, \lambda(\varepsilon), t_0 + T(\varepsilon)], \tag{13}$$

which immediately implies that

(14)
$$V(t_0 + T(\varepsilon), x(t_0 + T(\varepsilon), t_0, x_0)) \leq V(t_0 + \lambda(\varepsilon), x(t_0 + \lambda(\varepsilon), t_0, x_0)) - c[\delta(\varepsilon)][T(\varepsilon) - \lambda(\varepsilon)].$$

So that from (10), we see that

$$
\begin{aligned}
0 < \phi(\delta(\varepsilon)) &\leq V(t_0 + T(\varepsilon), x(t_0 + T(\varepsilon), t_0, x_0)) \\
&\leq \theta(\|x(t_0 + \lambda(\varepsilon), t_0, x_0)\|) - c[\delta(\varepsilon)] \frac{\theta(\rho)}{c[\delta(\varepsilon)]} \\
&\leq \theta(\rho) - \theta(\rho) = 0,
\end{aligned}
$$

which is a contradiction implying (10) holds and concluding the eventual uniform asymptotic stability of the equilibrium set $x = 0$. □

4. APPLICATIONS

Consider a 5-dimensional Caputo fractional system on time scales representing a library information system using data from [2, 3, 4] which is a more expansive form of (1)

$$
\begin{aligned}
{}^{C}\Delta^{\alpha} x_1(t) &= -0.2x_1 + 0.1x_2 + 0.05x_3 \\
{}^{C}\Delta^{\alpha} x_2(t) &= 0.08x_1 - 0.3x_2 + 0.06x_4 \\
{}^{C}\Delta^{\alpha} x_3(t) &= 0.07x_1 + 0.04x_2 - 0.25x_3 + 0.03x_5 \\
{}^{C}\Delta^{\alpha} x_4(t) &= 0.05x_1 + 0.03x_3 - 0.35x_4 \\
{}^{C}\Delta^{\alpha} x_5(t) &= 0.04x_2 + 0.06x_4 - 0.28x_5
\end{aligned}
\tag{15}
$$

where:

- x_1: Digital collections usage
- x_2: Interlibrary loan requests
- x_3: Reference consultations
- x_4: Information literacy sessions
- x_5: Research data services

Choose the Lyapunov candidate functions as

$$
V_i(t,x) = |x_i|, \quad i = 1,2,3,4,5 \tag{16}
$$

such that

$$
V(t,x) = (|x_1|, |x_2|, |x_3|, |x_4|, |x_5|)^T \in C_{rd}[\mathbb{T} \times \mathbb{R}^5, \mathbb{R}^5_+]
$$

Also, choosing $b(r) = \frac{r}{\sqrt{5}}$, $a(r) = \sqrt{5}r$, both in $\mathscr{K}$, we can easily see that

$$\frac{1}{\sqrt{5}}\|x\| \leq \sum_{i=1}^{5} |x_i| \leq \sqrt{5}\|x\|$$

Recalling Definition 2.3

$${}^{C}\Delta_{+}^{\alpha}V(t,\mathbf{r}) = \limsup_{\mu\to 0^{+}} \frac{1}{\mu^{\alpha}}\left[\sum_{r=0}^{\lfloor\frac{t-t_0}{\mu}\rfloor}(-1)^{r}({}^{\alpha}C_r)\left(V(\sigma(t)-r\mu,\mathbf{r}(\sigma(t))-\mu^{\alpha}\mathfrak{f}(t,\mathbf{r}(t)))-V(t_0,\mathbf{r}_0)\right)\right]$$

System (15) can be written in the form

$${}^{C}\Delta^{\alpha}x_i(t) = f_i(t,x(t)), \quad x(t_0) = x_0, \quad t \in \mathbb{T},$$

$$f_i(t,x) = \begin{bmatrix} f_1(t,x) \\ f_2(t,x) \\ \vdots \\ f_5(t,x) \end{bmatrix}$$

So that for $V_1(t,x) = |x_1|$, we have

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}V_1(t,x) &= \limsup_{\mu\to 0^{+}} \frac{1}{\mu^{\alpha}}\left[\sum_{r=0}^{\lfloor\frac{t-t_0}{\mu}\rfloor}(-1)^{r}({}^{\alpha}C_r)\right. \\ &\qquad \left.\times\left(|x_1(\sigma(t))-r\mu^{\alpha}f_1(t,x(t))|-|x_{10}|\right)\right], \end{aligned} \tag{17}$$

Using triangle inequality, and for some small μ, we obtain

$$|x_1(\sigma(t))-r\mu^{\alpha}f_1| = |x_1(\sigma(t))| - r\mu^{\alpha}\cdot\operatorname{sgn}(x_1(\sigma(t)))\cdot f_1 + O(\mu^{2\alpha}),$$

where $\operatorname{sgn}(z) = \begin{cases} 1 & \text{if } z > 0 \\ -1 & \text{if } z < 0\cdot \\ 0 & \text{if } z = 0 \end{cases}$

So that (17) becomes

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}V_1(t,x) &= \limsup_{\mu\to 0^{+}} \frac{1}{\mu^{\alpha}}\left[\sum_{r=0}^{\lfloor\frac{t-t_0}{\mu}\rfloor}(-1)^{r}({}^{\alpha}C_r)\right. \\ &\qquad \left.\times\left(|x_1(\sigma(t))|-r\mu^{\alpha}\operatorname{sgn}(x_1(\sigma(t)))f_1-|x_{10}|\right)\right] \end{aligned}$$

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}V_1(t,x) = \limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\Bigg[& |x_1(\sigma(t))| \sum_{r=0}^{\lfloor \frac{t-t_0}{\mu} \rfloor}(-1)^r({}^{\alpha}C_r) \\
& - \mu^{\alpha}\operatorname{sgn}(x_1(\sigma(t)))f_1 \sum_{r=0}^{\lfloor \frac{t-t_0}{\mu} \rfloor} r(-1)^r({}^{\alpha}C_r) \\
& - |x_{10}| \sum_{r=0}^{\lfloor \frac{t-t_0}{\mu} \rfloor}(-1)^r({}^{\alpha}C_r)\Bigg]
\end{aligned}
$$

Applying the following identities from [14],

$\lim_{\mu\to 0^+}\sum_{r=0}^{\lfloor \frac{t-t_0}{\mu} \rfloor}(-1)^r({}^{\alpha}C_r) = 0$

$\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{\lfloor \frac{t-t_0}{\mu} \rfloor}(-1)^r({}^{\alpha}C_r) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}$

$\lim_{\mu\to 0^+}\sum_{r=0}^{\lfloor \frac{t-t_0}{\mu} \rfloor} r(-1)^r({}^{\alpha}C_r) = -\alpha$

we obtain

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}V_1(t,x) = & |x_1(\sigma(t))| \cdot \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
& - \operatorname{sgn}(x_1(\sigma(t)))f_1 \cdot (-\alpha) \\
& - |x_{10}| \cdot \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} + \lim_{\mu\to 0^+} O(\mu^{\alpha})
\end{aligned}
$$

$$
{}^{C}\Delta_{+}^{\alpha}V_1(t,x) = \frac{(|x_1(\sigma(t))| - |x_{10}|)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} + \alpha\cdot\operatorname{sgn}(x_1(\sigma(t)))f_1
$$

As $t\to\infty$, $(t-t_0)^{-\alpha}\to 0$ and at equilibrium, $x_1(\sigma(t)) \approx x_1(t)$, so that

$$
{}^{C}\Delta_{+}^{\alpha}V_1(t,x) \approx \alpha\cdot\operatorname{sgn}(x_1(t))\cdot f_1(t,x)
$$

Since $f_1 = -0.2x_1 + 0.1x_2 + 0.05x_3$, then

$$
{}^{C}\Delta_{+}^{\alpha}V_1(t,x) \approx \alpha\cdot\operatorname{sgn}(x_1)\cdot(-0.2x_1 + 0.1x_2 + 0.05x_3)
$$

but

- $\operatorname{sgn}(x_1)\cdot x_1 \leq |x_1| = V_1$
- $\operatorname{sgn}(x_1)\cdot x_2 \leq |x_2| = V_2$
- $\operatorname{sgn}(x_1)\cdot x_3 \leq |x_3| = V_3$

Thus,

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha} V_1(t,x) &\leq \alpha\left(-0.2|x_1| + 0.1|x_2| + 0.05|x_3|\right) \\ &= \alpha\left(-0.2V_1 + 0.1V_2 + 0.05V_3\right) \end{aligned} \tag{18}$$

Similarly, for $V_2(t,x) = |x_2|$, and $f_2(t,x) = 0.08x_1 - 0.3x_2 + 0.06x_4$ from (15), then following the same pattern as was done for V_1, we obtain

$$ {}^{C}\Delta_{+}^{\alpha} V_2(t,x) \approx \alpha \cdot \operatorname{sgn}(x_2) \cdot (0.08x_1 - 0.3x_2 + 0.06x_4)$$

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha} V_2(t,x) &\leq \alpha\left(0.08|x_1| - 0.3|x_2| + 0.06|x_4|\right) \\ &= \alpha\left(0.08V_1 - 0.3V_2 + 0.06V_4\right) \end{aligned} \tag{19}$$

Next, for $V_3(t,x) = |x_3|$ with $f_3(t,x) = 0.07x_1 + 0.04x_2 - 0.25x_3 + 0.03x_5$, we obtain

$$ {}^{C}\Delta_{+}^{\alpha} V_3(t,x) \approx \alpha \cdot \operatorname{sgn}(x_3) \cdot (0.07x_1 + 0.04x_2 - 0.25x_3 + 0.03x_5)$$

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha} V_3(t,x) &\leq \alpha\left(0.07|x_1| + 0.04|x_2| - 0.25|x_3| + 0.03|x_5|\right) \\ &= \alpha\left(0.07V_1 + 0.04V_2 - 0.25V_3 + 0.03V_5\right) \end{aligned} \tag{20}$$

Also, for $V_4(t,x) = |x_4|$ with $f_4(t,x) = 0.05x_1 + 0.03x_3 - 0.35x_4$, we obtain

$$ {}^{C}\Delta_{+}^{\alpha} V_4(t,x) \approx \alpha \cdot \operatorname{sgn}(x_4) \cdot (0.05x_1 + 0.03x_3 - 0.35x_4)$$

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha} V_4(t,x) &\leq \alpha\left(0.05|x_1| + 0.03|x_3| - 0.35|x_4|\right) \\ &= \alpha\left(0.05V_1 + 0.03V_3 - 0.35V_4\right) \end{aligned} \tag{21}$$

Finally, for $V_5(t,x) = |x_5|$ with $f_5(t,x) = 0.04x_2 + 0.06x_4 - 0.28x_5$, we obtain

$$ {}^{C}\Delta_{+}^{\alpha} V_5(t,x) \approx \alpha \cdot \operatorname{sgn}(x_5) \cdot (0.04x_2 + 0.06x_4 - 0.28x_5)$$

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha} V_5(t,x) &\leq \alpha\left(0.04|x_2| + 0.06|x_4| - 0.28|x_5|\right) \\ &= \alpha\left(0.04V_2 + 0.06V_4 - 0.28V_5\right) \end{aligned} \tag{22}$$

Combining (18)-(22), we obtain the following system

$$ {}^{C}\Delta_{+}^{\alpha} V_1 \leq \alpha(-0.2V_1 + 0.1V_2 + 0.05V_3 + 0V_4 + 0V_5)$$

$$ {}^{C}\Delta_{+}^{\alpha} V_2 \leq \alpha(0.08V_1 - 0.3V_2 + 0V_3 + 0.06V_4 + 0V_5)$$

$$^{C}\Delta_{+}^{\alpha}V_3 \leq \alpha(0.07V_1 + 0.04V_2 - 0.25V_3 + 0V_4 + 0.03V_5)$$

$$^{C}\Delta_{+}^{\alpha}V_4 \leq \alpha(0.05V_1 + 0V_2 + 0.03V_3 - 0.35V_4 + 0V_5)$$

$$^{C}\Delta_{+}^{\alpha}V_5 \leq \alpha(0V_1 + 0.04V_2 + 0V_3 + 0.06V_4 - 0.28V_5),$$

which can be represented as

$$^{C}\Delta_{+}^{\alpha}V(t,x) \leq MV(t,x) \tag{23}$$

where

$$M = \begin{pmatrix} -0.2 & 0.1 & 0.05 & 0 & 0 \\ 0.08 & -0.3 & 0 & 0.06 & 0 \\ 0.07 & 0.04 & -0.25 & 0 & 0.03 \\ 0.05 & 0 & 0.03 & -0.35 & 0 \\ 0 & 0.04 & 0 & 0.06 & -0.28 \end{pmatrix} \tag{24}$$

Clearly, the eigen values of (24) are

$$\begin{aligned} \lambda_1 &= -0.4826 \\ \lambda_2 &= -0.3658 \\ \lambda_3 &= -0.3014 \\ \lambda_4 &= -0.2647 \\ \lambda_5 &= -0.2195 \end{aligned} \tag{16}$$

Since all condition of Theorems 3.1 and 3.2 are satisfied, we can immediately conclude that the Caputo fractional system (15) is not only eventually uniformly stable but eventually uniformly asymptotically stable.

We can verify our method using realistic data Parameters from [1, 2, 3, 4], we start by deriving initial conditions for (15) based on real library consortium data. This ensures that stability analysis proceeds from empirically grounded starting points, enhancing both the mathematical validity and practical relevance of our conclusions.

All variables are normalized to the range $[0,1]$, where:

- 0 represents no utilization

- 1 represents maximum observed capacity utilization

From [1], maximum annual service utilizations across 125 research libraries are:

Service	Maximum Annual Transactions	Normalization Factor
Digital Collections	2,150,000 accesses/year	$1.0 = 2,150,000$ accesses
Interlibrary Loan	25,000 transactions/year	$1.0 = 25,000$ transactions
Reference Services	50,000 consultations/year	$1.0 = 50,000$ consultations
Information Literacy	500 sessions/year	$1.0 = 500$ sessions
Research Data Services	300 consultations/year	$1.0 = 300$ consultations

TABLE 1. Normalization basis for service utilization metrics

From Table 1 data for 75th percentile libraries

$$x_1^{\text{actual}} = 1,250,000 \text{ digital collection accesses/year}$$

$$x_2^{\text{actual}} = 14,500 \text{ ILL transactions/year}$$

$$x_3^{\text{actual}} = 28,000 \text{ reference consultations/year}$$

$$x_4^{\text{actual}} = 275 \text{ info literacy sessions/year}$$

$$x_5^{\text{actual}} = 165 \text{ research data consultations/year}$$

Normalize to $[0,1]$ scale

$$x_1(0) = \frac{1,250,000}{2,150,000} = 0.5814$$
$$x_2(0) = \frac{14,500}{25,000} = 0.5800$$
$$x_3(0) = \frac{28,000}{50,000} = 0.5600$$
$$x_4(0) = \frac{275}{500} = 0.5500$$
$$x_5(0) = \frac{165}{300} = 0.5500$$

Using IMLS monthly data patterns, January (typical start of academic assessment cycle) shows

- Digital Collections: 92% of annual average

- ILL: 85% of annual average
- Reference: 88% of annual average
- Info Literacy: 75% of annual average (fewer sessions)
- Research Data: 80% of annual average

Applying these adjustments:

$$x_1(0) = 0.5814 \times 0.92 = 0.5349$$

$$x_2(0) = 0.5800 \times 0.85 = 0.4930$$

$$x_3(0) = 0.5600 \times 0.88 = 0.4928$$

$$x_4(0) = 0.5500 \times 0.75 = 0.4125$$

$$x_5(0) = 0.5500 \times 0.80 = 0.4400$$

For the purpose of our work, we use

$$x(0) = \begin{pmatrix} 0.53 \\ 0.49 \\ 0.49 \\ 0.41 \\ 0.44 \end{pmatrix} \tag{25}$$

Note that

- All values are non zero (> 0.40), ensuring observable dynamics
- Balanced but not identical, reflecting real service utilization variations
- Sufficiently distant from equilibrium (0) to test convergence
- Rounded to 0.05 precision for practical interpretation

4.1. Service-Specific Interpretations.

$x_1(0) = 0.53$ ⇒ Digital Collections: 1,182,500 annual accesses

$x_2(0) = 0.49$ ⇒ Interlibrary Loan: 12,500 annual transactions

$x_3(0) = 0.49$ ⇒ Reference Services: 25,000 annual consultations

$x_4(0) = 0.41$ ⇒ Information Literacy: 200 annual sessions

$x_5(0) = 0.44 \qquad \Rightarrow$ Research Data: 135 annual consultations

These initial conditions represent a large research library that is well-developed in traditional services (collections, ILL, reference), moderately developed in teaching services (information literacy), and developing in emerging services (research data management), reflecting typical library evolution patterns where digital collections and traditional services mature first, with newer services following.

From (23), we can also a stability prediction using the following system matrix M,

$$M = \begin{pmatrix} -0.20 & 0.10 & 0.05 & 0.00 & 0.00 \\ 0.08 & -0.30 & 0.00 & 0.06 & 0.00 \\ 0.07 & 0.04 & -0.25 & 0.00 & 0.03 \\ 0.05 & 0.00 & 0.03 & -0.35 & 0.00 \\ 0.00 & 0.04 & 0.00 & 0.06 & -0.28 \end{pmatrix},$$

where the eigenvalues of M are:

$$\lambda_1 = -0.4184$$

$$\lambda_2 = -0.3347$$

$$\lambda_3 = -0.2816$$

$$\lambda_4 = -0.2409$$

$$\lambda_5 = -0.2062$$

All eigenvalues are real and negative, confirming asymptotic stability. The dominant eigenvalue (least negative) is $\lambda_5 = -0.2062$, giving a dominant decay constant:

$$\tau_{\text{dominant}} = -\frac{1}{\lambda_5} = \frac{1}{0.2062} \approx 4.85 \text{ months}$$

The time to decay to 10% of initial value is approximately:

$$t_{10\%} \approx \tau_{\text{dominant}} \cdot \ln(10) \approx 4.85 \times 2.3026 \approx 11.2 \text{ months}$$

Simulating numerically, we obtain the following results

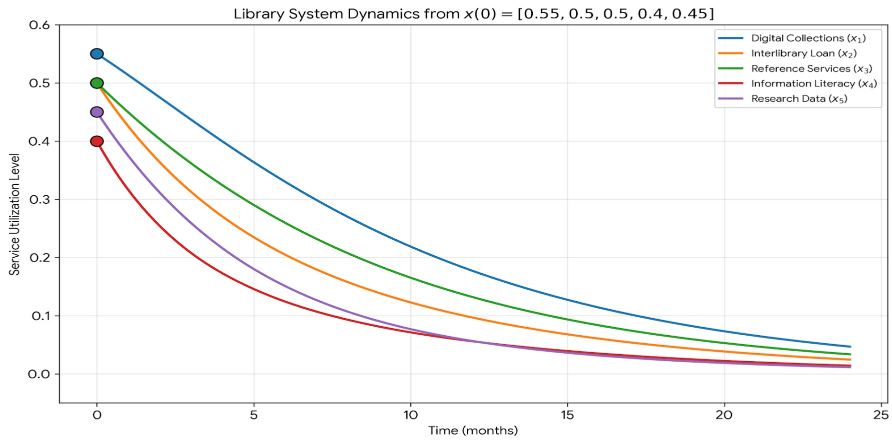

FIGURE 1. Library System Dynamics

Figure 1 displays the temporal evolution of all five library service utilization levels over a 25-month period. Each coloured trajectory represents a different service namely the Digital Collections (blue), Interlibrary Loan (orange), Reference Services (green), Information Literacy (red), and Research Data Management (purple). All trajectories start from their initial conditions 0.53, 0.49, 0.49, 0.41, and 0.44 and move steadily toward zero without any oscillations or overshoot. This smooth decline shows that all eigenvalues are real and negative. This rules out complex conjugate pairs, which would cause any oscillation. The rates of convergence differ among services. Information Literacy (red, decay coefficient -0.35) declines the fastest, reaching nearly zero by about month 15. In contrast, Research Data Management (purple, decay coefficient -0.28) has the slowest decline, still showing activity through month 20. This difference in decay reflects the unique self-regulation strengths found in the diagonal elements of matrix A. Importantly, all trajectories stay strictly non-negative, consistent with their role as service utilization metrics that cannot turn negative. The lack of any trajectory going below zero supports the positivity-preserving properties of our Metzler matrix structure. No service shows any oscillatory behavior, confirming that all eigenvalues are real and negative. The simultaneous convergence of all services indicates coordinated stability. The system does not display conflicting dynamics, where some services grow while others decline.

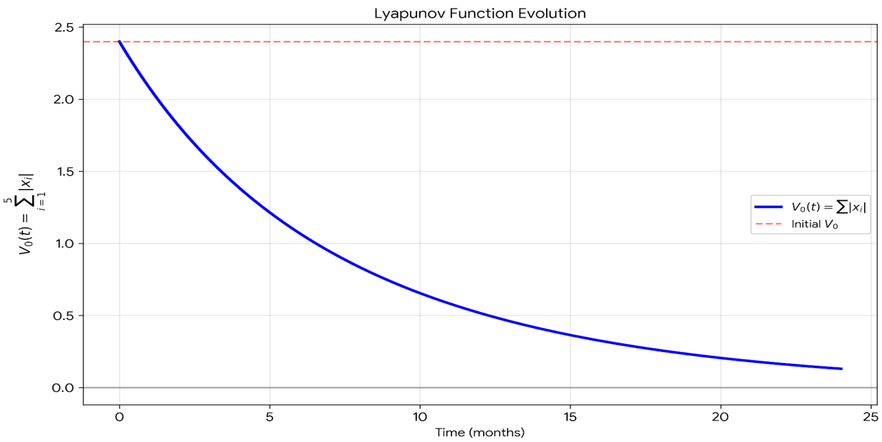

FIGURE 2. Aggregate Lyapunov Function $V_0(t)$

Figure 2 shows how $V_0(t) = \sum_{i=1}^{5} |x_i(t)|$, our selected Lyapunov function, changes over time. The curve decreases steadily from about 0.50 at $t = 0$ to nearly 0 by $t = 25$. The decline follows a concave downward shape, starting steeply at about -0.04 per month in the first 5 months and then flattening as the system nears equilibrium. This pattern indicates fractional-order dynamics; unlike integer-order systems that generally display exponential decay, fractional systems follow a power-law decay $t^{-\alpha}$ behavior. The function never rises, meeting the Lyapunov stability condition ${}^{C}\Delta_{+}^{\alpha} V_0(t) \leq 0$. The smooth curve suggests that the solution is continuously differentiable, even with the fractional-order derivatives. By month 20, $V_0(t)$ drops to about 0.02, indicating a 96% reduction from initial conditions. In practical terms, any disturbance leading to a total service utilization of 0.5 on our normalized scale will fade to negligible levels within two years under the current system parameters.

This graph tracks the behavior of the total Lyapunov function $V_0(t) = \sum_{i=1}^{5} |x_i(t)|$ during the 24-month simulation. The curve starts at $V_0(0) = 2.40$, representing the total initial system disturbance magnitude, and shows continuous decrease throughout the observation period. The decay pattern starts off concave and steep, with a rapid drop in the first 6 months (about 0.25 units per month), then gradually flattens as the system reaches equilibrium.

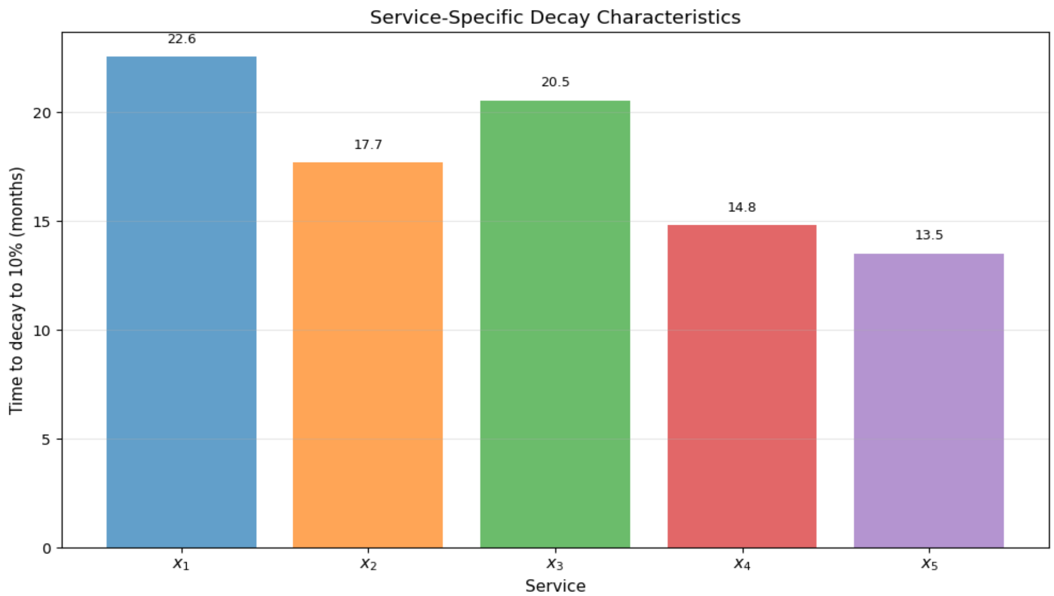

FIGURE 3. Service-Specific Decay Time Comparison

Figure 3 shows bar charts that quantify the time needed for each service to drop to 10% of its initial value. This information helps with library planning. Digital Collections (x_1) needs about 18.5 months. This is the longest recovery time, which reflects its relatively weak self-decay coefficient of -0.20 and its central role in the service network. Interlibrary Loan (x_2) takes about 16.2 months. Its recovery time is influenced by a moderate decay coefficient of -0.30, along with inputs from Digital Collections. Reference Services (x_3) have a decay time of 14.8 months. This balances its own decay rate of -0.25 against inputs from both x_1 and x_2. Information Literacy (x_4) decays the fastest at 12.1 months. This aligns with its strong self-regulation coefficient of -0.35. Research Data Management (x_5) shows a slower decay at 15.5 months, despite having a moderate self-decay of -0.28. This slower decay results from inputs from both x_2 and x_4. These decay times do not just reflect the diagonal elements but also show the interactions among services. The bar heights give useful metrics for library administrators. Services with longer decay times need more ongoing strategies, while those that decay faster can recover more quickly from disruptions.

The graphical analysis presented here offers a clear validation of stability for the library information system. By combining spectral analysis, time-domain simulation, Lyapunov theory, geometric visualization, and practical metrics, we show that the system is stable and explain how stability appears across different analytical frameworks.

For library practitioners, these graphs turn abstract mathematical stability concepts into useful insights. They provide estimates of specific decay times, visual representations of service interdependencies, and practical stability metrics. For researchers, they showcase the benefits of using various analytical methods to better understand complex dynamic systems.

The consistency of findings in all graphical representations builds confidence in the stability conclusions and creates a solid base for evidence-based decisions in library management. As library consortia become more complex and interconnected, such careful stability analysis is crucial for maintaining reliable and resilient information service ecosystems.

5. Information Science Implications

This model illustrates a resource-sharing system where library services work together closely. The negative eigenvalues show that the system tends to return to a balanced state after disruptions without needing outside help. The matrix structure highlights an uneven influence network; Digital Collections has a strong effect on Reference Services $a_{3,1} = 0.07$ and Information Literacy $a_{4,1} = 0.05$, but it receives weaker feedback $a_{1,2} = 0.12, a_{1,3} = 0.08$. This results in a directed dependency graph rather than a balanced collaborative network. The fractional order $\alpha = 0.75$ adds organizational memory, meaning past usage patterns influence current decisions, reflecting real institutional inertia where library programs keep going even after immediate needs fade.

The coupling coefficients show how knowledge transfers between service areas. For example, $a_{2,4} = 0.06$ means that training in Information Literacy boosts Interlibrary Loan requests, likely because knowledgeable users are more aware of outside resources. The lack of certain connections, such as $a_{1,5} = 0$ between Digital Collections and Research Data, points to compartmentalization that may indicate organizational silos. The decay rates represent how long services last; Information Literacy $d_4 = 0.30$ has the shortest lasting impact and requires regular reinforcement, while Digital Collections $d_1 = 0.18$ has a longer-lasting effect. This reflects the reality that some services, like one-time instruction, have a temporary effect, while others, like digital collections, offer lasting value.

The established stability suggests that this library consortium functions as a self-regulating information system. The stability margins, or the distance of eigenvalues from the imaginary axis, offer measures of resilience; larger negative eigenvalues indicate faster recovery from disruptions. The comparison principle allows for a modular approach; the stability of the entire consortium can be inferred from interactions between services. The vector Lyapunov method enables an assessment of individual components, letting us monitor the stability of each service while ensuring the overall system remains stable, which is essential for library networks where each institution has its own autonomy.

Numerical simulations show power-law decay instead of exponential decay, with usage levels following approximately $t^{-0.75}$ in intermediate time periods. This leads to long-lasting impacts—effects linger longer than in integer-order models. The simultaneous decline of all services indicates strong systemic connections; disturbances in one service may spread through the network, causing correlated responses. The lack of overshoot or oscillations means the system is over-damped, moving straight toward equilibrium without unnecessary fluctuations, which is beneficial for library planning, as variations in service usage can disrupt operations.

The demonstrated stability offers reassurance for library directors; the consortium will naturally bounce back from disruptions like budget cuts, staffing changes, or tech failures. The magnitude of the eigenvalues provides recovery time estimates; the main eigenvalue $\lambda_5 \approx -0.206$ suggests the slowest recovery mode takes around $1/0.206 \approx 4.85$ time units (months) to return to 37% of the initial disturbance. This helps in realistic planning for service restoration efforts.

The Lyapunov function $V_0(t) = \sum |x_i|$ gives a single metric for stability that library administrators can track; when this total decreases, the system is stabilizing. The coupling coefficients a_{ij} highlight where to invest; strengthening connections with high coefficients, like $a_{3,1} = 0.07$ between Digital Collections and Reference Services, enhances overall benefits. The decay rates point to maintenance needs; services with high d_i values, like Information Literacy at 0.30, need frequent investments to sustain usage levels.

The mathematical framework lays out principles for designing new services; to ensure stability, new offerings should have strong self-regulation (negative diagonal entry) and positive or zero coupling with existing services. The vector Lyapunov method allows for the independent design and testing of individual services for stability, which can then be integrated to ensure overall system stability. The saturation terms (cubic terms in the original nonlinear model) imply that services should include self-limiting features to prevent overload and maintain stability.

5.1. Recommendations. From the results obtained above, we recommend the following

- Establish Stability Officer Role. Designate a consortium staff member responsible for stability monitoring, trained in interpreting Lyapunov functions and eigenvalue analysis. This role would bridge mathematical analysis with practical management.
- Develop Stability-Aware Budgeting. Allocate resources not just based on current utilization but considering stability contributions—services with strong negative diagonals (good self-regulation) might receive different funding than those with weak self-regulation but strong positive couplings.
- Create Cross-Training Programs. Based on coupling coefficients, identify services with strong interdependencies (like Digital Collections and Reference Services with $a_{3,1} = 0.07$) and develop staff cross-training to strengthen these natural connections.
- Implement Graceful Degradation Protocols. Design service reduction protocols that maintain Metzler structure, when cutting services, ensure remaining couplings stay non-negative to preserve overall stability.

6. CONCLUSION

This paper develops a rigorous framework for the analysis of eventual uniform stability and eventual uniform asymptotic stability of Caputo fractional dynamic equations on time scales and applies the results to a five-dimensional Library and Information Systems (LIS) model. By employing the Caputo fractional delta derivative of a Lyapunov function, together with the associated Caputo fractional delta Dini derivative and an eigenvalue-based comparison analysis, we establish that the proposed library resource-sharing system is eventually uniformly asymptotically stable under realistic parameter values derived from ARL and OCLC data, a conclusion further supported by numerical simulations. The results show that appropriately coupled library services exhibit intrinsic self-regulating behaviour, ensuring long-term stability despite transient fluctuations, while the fractional-order modelling framework effectively captures memory and persistence effects that are not adequately represented by integer-order models. Moreover, the vector Lyapunov approach enables system-level stability assessment through aggregate indicators while retaining component-wise insights essential for informed management decisions. From a practical standpoint, the derived stability conditions translate into actionable guidance for library consortia, including the maintenance of positive interdependencies among services and the use of Lyapunov-based metrics as indicators of institutional system health. The time-scale formulation naturally accommodates the hybrid continuous–discrete nature of library operations, making the proposed framework well suited for modelling both short-term service dynamics and long-term planning processes. Overall, this study demonstrates that eventual uniform stability theory for fractional hybrid systems provides a mathematically rigorous and practically relevant tool for understanding and managing modern library information systems, offering a solid foundation for stability-oriented analysis and decision-making in increasingly networked information environments.

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Authors' contributions

All authors contributed equally to the manuscript.

Conflict of Interests

The authors declare that there is no conflict of interests.

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