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CONVERGENCE OF A GENERAL ITERATIVE SCHEME FOR THREE INFINITE FAMILIES OF UNIFORMLY QUASI-LIPSCHITZIAN MAPPINGS IN CONVEX METRIC SPACES

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Abstract. We consider a new Noor-type iterative procedure with errors for approximating the common fixed point of three infinite families of uniformly quasi-Lipschitzian mappings in convex metric spaces. Under appropriate conditions, some convergence theorems are proved. The results presented in this paper extend, improve and unify some main results in previous work.

Keywords: Uniformly quasi-Lipschitzian mappings, Common fixed points, Convex metric spaces.

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1. Introduction and preliminaries

Takahashi [1] introduced the concept of convex metric space which is a more general space and each linear normed space is a special case of a convex metric space. In 2005, Tian [2] gave some sufficient and necessary conditions such that the Ishikawa iteration sequence for an asymptotically quasi-nonexpansive mapping to converge to a fixed point in convex metric spaces. In 2009, Wang and Liu [3] gave some sufficient and necessary conditions for an Ishikawa iteration sequence with errors to approximate a common fixed point of two uniformly quasi-Lipschitzian mappings in convex metric spaces. Recently, Chang et al.

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[4] and Liu et al. [5] gave some sufficient and necessary conditions for Ishikawa iteration process with errors to approximate common fixed points of infinite families of uniformly quasi-Lipschitzian mappings in convex metric spaces. Later on, many authors discussed the existence of the fixed point and the convergence of the iterative processes for finite and infinite families of asymptotically quasi-nonexpansive mappings and uniformly quasi-Lipschitzian mappings in convex metric spaces (see, for example, [8-13] and the references therein).

First of all, let us list some definitions and notations.

Let (X, d) be a metric space. A mapping $T : X \to X$ is called asymptotically nonexpansive if there exists $k_n \in [1, \infty)$, $\lim_{n\to\infty} k_n = 1$, such that

$$d(T^n x, T^n y) \leq k_n d(x, y)$$

for all $x, y \in X$. Let $F(T) = \{x \in X : Tx = x\}$. if $F(T) \neq \varnothing$, then T is called asymptotically quasi-nonexpansive if there exists $k_n \in [1, \infty)$, $\lim_{n\to\infty} k_n = 1$, such that

$$d(T^n x, p) \leq k_n d(x, p)$$

for all $x \in X$ and $p \in F(T)$. Moreover, it is uniformly quasi-Lipschitzian if there exists $L > 0$ such that

$$d(T^n x, p) \leq L d(x, p)$$

for all $x \in X$ and $p \in F(T)$. From the above definitions, if $F(T) \neq \varnothing$, it follows that an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive, and an asymptotically quasi-nonexpansive mapping must be uniformly quasi-Lipschitzian $\left(L = \sup_{n\geq 1}\{k_n\} < \infty\right)$. However, the inverse does not hold. In recent years, asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings have been studied by many authors.

Definition 1. [6]*Let* (X, d) *be a metric space,* $I = [0, 1]$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *be a real sequences in* $[0, 1]$ *with* $a_n + b_n + c_n = 1$. *A mapping* $W : X^3 x I^3 \to X$ *is said to be a convex structure on* X *if it satisfies the following conditions: For any* $(x, y, z, a_n, b_n, c_n) \in X^3 x I^3$

and $u \in X$,

$$d(W(x,y,z,a_n,b_n,c_n),u) \leq a_n d(x,u) + b_n d(y,u) + c_n d(z,u) \tag{1.1}$$

if (X,d) *is a metric space with a convex structure* W*, then* (X,d) *is called a convex metric space.*

Definition 2. [6]*Let* (X,d) *be a convex metric space. A nonempty subset* E *of* X *is said to be convex if* $W(x,y,z;a,b,c) \in E$*, for all* $(x,y,z,a,b,c) \in E^3 \times [0,1]^3$ *with* $a+b+c=1$.

Definition 3. *Let* (X,d) *be a convex metric space with a convex structure* $W : X^3 \times [0.1]^3 \to X$ *and* E *be a nonempty convex subset of* X. $T_i, R_i, S_i : E \to E$ *be uniformly quasi-Lipschitzian mappings with sequences* L_i, L'_i *and* L''_i *respectively,* $i = 1,2,3,....$ *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{l_n\}$ *be nine sequences in* $[0,1]$ *with*

$$\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + l_n = 1, \ n = 1,2,....$$

For any given $x_1 \in E$, *define a sequence* $\{x_n\}$ *by:*

$$\begin{cases} x_{n+1} = W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \\ y_n = W(R_n^n x_n, S_n^n z_n, v_n; a_n, b_n, c_n), \\ z_n = W(S_n^n x_n, R_n^n x_n, w_n; d_n, e_n, l_n), \ n = 1,2,... \end{cases} \tag{1.2}$$

where $\{u_n\}$, $\{v_n\}$ *and* $\{w_n\}$ *are any given three sequences in* E. *Then* $\{x_n\}$ *is called the Noor-type iterative sequence with errors for three sequences of uniformly quasi-Lipschitzian mappings* T_i, R_i *and* S_i *with* $i = 1,2,....$

If $e_n = 1$ $(d_n = l_n = 0)$ for all $n \geq 1$ and $R_i = I$ (the identity mapping on E) for all $i \geq 1$ in (1.2), then the sequence $\{x_n\}$ defined by (1.2) can be written as follows:

$$\begin{cases} x_{n+1} = W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \\ y_n = W(x_n, S_n^n x_n, v_n; a_n, b_n, c_n), \ n = 1,2,... \end{cases} \tag{1.3}$$

which is the Ishikawa-type iterative sequence with errors considered in [5]. Further, if $e_n = b_n = 1$ $(d_n = l_n = 0,\ a_n = c_n = 0)$ for all $n \geq 1$ and $R_i = S_i = I$ for all $i \geq 1$, then (1.2) reduces to the following Mann-type iterative sequence with errors:

$$x_{n+1} = W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \ n = 1,2,.... \tag{1.4}$$

Lemma 1. [7]*Let the nonnegative sequences* $\{a_n\}$, $\{b_n\}$ *and* $\{\delta_n\}$ *satisfy that*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n,\ \ n = 1, 2, ..$$

and

$$\sum_{n=1}^{\infty} b_n < \infty,\ \ \sum_{n=1}^{\infty} \delta_n < \infty.$$

Then

(i) $\lim_{n\to\infty} a_n$ *exist;*

(ii) *if* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2. *Let* (X, d) *be a convex metric space and* E *be a nonempty subset of* X. $T_i, R_i, S_{\dot{I}} : E \to E$ *be uniformly quasi-Lipschitzian mappings with* $L_i > 0$, $L_i^{'} > 0$ *and* $L_i^{"} > 0$, *respectively,* $i = 1, 2, ...$ *such that* $\mathcal{F} = (\cap_{i=1}^{\infty} F(T_i)) \cap (\cap_{i=1}^{\infty} F(R_i)) \cap (\cap_{i=1}^{\infty} F(S_i)) \neq \emptyset$. *If* $\{L_i\}$, $\left\{L_i^{'}\right\}$ *and* $\left\{L_i^{"}\right\}$ *are bounded, then there exists a constant* $L \geq 0$ *such that*

$$d(T_i^n x, p) \leq Ld(x, p),\ \ d(R_i^n x, p) \leq Ld(x, p)\ \text{and}\ d(S_i^n x, p) \leq Ld(x, p)$$

for all $x \in E$, $p \in \mathcal{F}$ *and* $n = 1, 2, ...$

Proof. For each $n = 1, 2, ...$ and $i = 1, 2, ...$ we have

$$\begin{aligned} d(T_i^n x, p) &\leq L_i d(x, p) \leq Ld(x, p),\ \forall x \in E,\ p \in \mathcal{F}, \\ d(R_i^n x, p) &\leq L_i d(x, p) \leq Ld(x, p),\ \forall x \in E,\ p \in \mathcal{F}, \\ d(S_i^n x, p) &\leq L_i d(x, p) \leq Ld(x, p),\ \forall x \in E,\ p \in \mathcal{F}, \end{aligned}$$

where $L = \max\left\{\sup_{i\geq 1}\{L_i\}, \sup_{i\geq 1}\left\{L_i^{'}\right\}, \sup_{i\geq 1}\left\{L_i^{"}\right\}\right\}$.This completes proof. □

Lemma 3. *Let* E *be a nonempty closed convex subset of complete convex metric space* X. *Let* $T_i, R_i, S_{\dot{I}} : E \to E$ *be uniformly quasi-Lipschitzian mappings with* $L_i > 0$, $L_i^{'} > 0$ *and* $L_i^{"} > 0$, *respectively,*$i = 1, 2, ...$ *Suppose that* $\mathcal{F} = (\cap_{i=1}^{\infty} F(T_i)) \cap (\cap_{i=1}^{\infty} F(R_i)) \cap (\cap_{i=1}^{\infty} F(S_i))$ *is nonempty and bounded. Let* $\{x_n\}$ *be a sequence defined by (1.2), in which* $\{u_n\}$, $\{v_n\}$

and $\{w_n\}$ *are three bounded sequences. Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{l_n\}$ *be sequences in* $[0,1]$ *with*

$$\alpha_n+\beta_n+\gamma_n=a_n+b_n+c_n=d_n+e_n+l_n=1, \forall n\geq 1 \text{ and } \sum_{n=1}^{\infty}(\beta_n+\gamma_n)<\infty.$$

If $\{L_i\}$, $\left\{L_i^{'}\right\}$ *and* $\left\{L_i^{"}\right\}$ *are bounded, then*

(i) *for any* $p\in\mathcal{F}$ *and* $n=1,2,...$

$$d(x_{n+1},p)\leq\left[1+\beta_nL^2(1+L)\right]d(x_n,p)+M_0\eta_n \tag{1.5}$$

where $L=\max\left\{\sup_{i\geq 1}\{L_i\},\sup_{i\geq 1}\left\{L_i^{'}\right\},\sup_{i\geq 1}\left\{L_i^{"}\right\}\right\}$, $\eta_n=\beta_n+\gamma_n$.*and* $M_0=\sup_{p\in\mathcal{F},n\geq 1}\left\{d(u_n,p)+Ld(v_n,p)+L^2d(w_n,p)\right\}$,

(ii) *for any* $p\in\mathcal{F}$ *and* $n=1,2,...$

$$d(x_{n+m},p)\leq M_1d(x_n,p)+M_0M_1\sum_{k=n}^{n+m-1}\eta_k \tag{1.6}$$

where $M_1=e^{L^2(1+L)\sum\limits_{k=1}^{\infty}\beta_k}$.

Proof. (i) For any $p\in\mathcal{F}$, it follows from (1.1), (1.2),Lemma 1 and Lemma 2 that

$$\begin{aligned} d(x_{n+1},p) &= d(W(x_n,T_n^ny_n,u_n;\alpha_n,\beta_n,\gamma_n),p)\\ &\leq \alpha_nd(x_n,p)+\beta_nd(T_n^ny_n,p)+\gamma_nd(u_n,p)\\ &\leq \alpha_nd(x_n,p)+\beta_nLd(y_n,p)+\gamma_nd(u_n,p), \end{aligned} \tag{1.7}$$

$$\begin{aligned} d(y_n,p) &= d(W(R_n^nx_n,S_n^nz_n,v_n;a_n,b_n,c_n),p)\\ &\leq a_nd(R_n^nx_n,p)+b_nd(S_n^nz_n,p)+c_nd(v_n,p)\\ &\leq a_nLd(x_n,p)+b_nLd(z_n,p)+c_nd(v_n,p) \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} d(z_n,p) &= d(W(S_n^nx_n,R_n^nx_n,w_n;d_n,e_n,l_n),p)\\ &\leq d_nd(S_n^nx_n,p)+e_nd(R_n^nx_n,p)+l_nd(w_n,p)\\ &\leq d_nLd(x_n,p)+e_nLd(x_n,p)+l_nd(w_n,p), \end{aligned} \tag{1.9}$$

Substituting (1.9) into (1.8) and simplifying it, we have

$$
\begin{aligned}
d(y_n,p) &\leq a_nLd(x_n,p)+b_nL\left[d_nLd(x_n,p)+e_nLd(x_n,p)+l_nd(w_n,p)\right] \\
&\quad +c_nd(v_n,p) \\
&= a_nLd(x_n,p)+b_nL\left[(d_n+e_n)d(x_n,p)+l_nd(w_n,p)\right] \\
&\quad +c_nd(v_n,p) \\
&\leq a_nLd(x_n,p)+b_nL\left[Ld(x_n,p)+l_nd(w_n,p)\right]+c_nd(v_n,p) \\
&= (a_n+b_nL)Ld(x_n,p)+b_nl_nLd(w_n,p)+c_nd(v_n,p) \\
&\leq L(1+L)d(x_n,p)+Ld(w_n,p)+d(v_n,p) \qquad (1.10)
\end{aligned}
$$

By using (1.7) and (1.10) we obtain

$$
\begin{aligned}
d(x_{n+1},p) &\leq \alpha_nd(x_n,p)+\beta_nL\left[L(1+L)d(x_n,p)+Ld(w_n,p)+d(v_n,p)\right] \\
&\quad +\gamma_nd(u_n,p) \\
&\leq (\alpha_n+\beta_nL^2(1+L))d(x_n,p)+\beta_nL^2d(w_n,p)+\beta_nLd(v_n,p) \\
&\quad +\gamma_nd(u_n,p) \\
&\leq (1+\beta_nL^2(1+L))d(x_n,p) \\
&\quad +\left[d(u_n,p)+Ld(v_n,p)+L^2d(w_n,p)\right](\beta_n+\gamma_n) \\
&\leq (1+\beta_nL^2(1+L))d(x_n,p)+M_0\eta_n
\end{aligned}
$$

and so (1.5) holds.

(ii) It is well known that $1+x \leq e^x$ for all $x \geq 0$. Using this fact, for any $p \in \mathcal{F}$ and $m, n \geq 1$, it follws from (1.5) that

$$\begin{aligned} d(x_{n+m}, p) &\leq \left[1+\beta_{n+m-1}L^2(1+L)\right] d(x_{n+m-1}, p) + M_0\eta_{n+m-1} \\ &\leq e^{\beta_{n+m-1}L^2(1+L)} d(x_{n+m-1}, p) + M_0\eta_{n+m-1} \\ &\leq e^{\beta_{n+m-1}L^2(1+L)} \left[1+\beta_{n+m-2}L^2(1+L)d(x_{n+m-2}, p) + M_0\eta_{n+m-2}\right] \\ &\quad + M_0\eta_{n+m-1} \\ &\leq e^{(\beta_{n+m-1}+\beta_{n+m-2})L^2(1+L)} d(x_{n+m-2}, p) + M_0\left[\eta_{n+m-2}+\eta_{n+m-1}\right] \\ &\quad \vdots \\ &\leq M_1 d(x_n, p) + M_1 M_0 \sum_{k=n}^{n+m-1} \eta_k \end{aligned}$$

where $M_1 = e^{L^2(1+L)\sum_{k=1}^{\infty}\beta_k}$ and so (1.6) holds. This completes proof. $\square$

2. Main results

Theorem 1. *Let E be a nonempty closed convex subset of complete convex metric space X. Let $T_i, R_i, S_i : E \to E$ be uniformly quasi-Lipschitzian mappings with $L_i > 0$, $L_i^{'} > 0$ and $L_i^{"} > 0$, respectively, $i = 1, 2, \ldots$. Suppose that $\mathcal{F} = (\cap_{i=1}^{\infty} F(T_i)) \cap (\cap_{i=1}^{\infty} F(R_i)) \cap (\cap_{i=1}^{\infty} F(S_i))$ is nonempty and bounded. Let $\{x_n\}$ be a sequence defined by (1.2), in which $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{l_n\}$ be sequences in $[0,1]$ with*

$$\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + l_n = 1, \forall n \geq 1 \text{ and } \sum_{n=1}^{\infty}(\beta_n + \gamma_n) < \infty.$$

If $\{L_i\}$, $\{L_i^{'}\}$ and $\{L_i^{"}\}$ are bounded, then $\{x_n\}$ converges to a common fixed point of $p \in \mathcal{F}$ if and only if

$$\liminf_{n \to \infty} d(x_n, \mathcal{F}) = 0,$$

where $d(x, \mathcal{F}) = \inf\{d(x, p) : p \in \mathcal{F}\}$.

Proof. The necessity of the conditions is obvius. Thus, we only need to prove the sufficiency. By Lemma 3 (i), we have

$$d(x_{n+1}, F) \leq \left[1 + \beta_n L^2(1+L)\right] d(x_n, p) + M_0\eta_n,\ n \geq 1. \tag{2.1}$$

Since

$$\sum_{n=1}^{\infty} \eta_n = \sum_{n=1}^{\infty} (\beta_n + \gamma_n) < \infty,$$

it follows from (2.1) and Lemma 1 that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists. Now $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$ implies that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Next we prove that $\{x_n\}$ is a Cauchy sequence in E. For any $\varepsilon > 0$, there exists a positive integer N_0 such that, for all $n \geq N_0$,

$$d(x_n, \mathcal{F}) \leq \frac{\varepsilon}{4M_1},\ \sum_{n=N_0}^{\infty} \eta_n \leq \frac{\varepsilon}{4M_0M_1}.$$

Particularly, there exists a $p_1 \in \mathcal{F}$ and a positive integer $N_1 > N_0$ such that

$$d(x_{N_1}, p_1) \leq \frac{\varepsilon}{4M_1} \tag{2.2}$$

For any positive integers n, m with $n \geq N_1$, by (2.2) and Lemma 3 (ii), we have

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_1) + d(p_1, x_n) \\
&\leq M_1 d(x_{N_1}, p_1) + M_1M_0 \sum_{k=N_1}^{n+m-1} \eta_k + M_1 d(x_{N_1}, p_1) \\
&\quad + M_1M_0 \sum_{k=N_1}^{n-1} \eta_k \\
&\leq 2M_1 \frac{\varepsilon}{4M_1} + 2M_1M_0 \frac{\varepsilon}{4M_0M_1} \\
&= \varepsilon.
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in a nonempty closed convex subset E of a complete convex metric space X. Let $\lim_{n\to\infty} x_n = p^* \in E$.

Finally, we show that $p^* \in \mathcal{F}$. To this end, we only need to prove that $\mathcal{F}$ is closed because

$$d(p^*, \mathcal{F}) = \lim_{n\to\infty} d(x_n, \mathcal{F}) = 0.$$

Let $p_n \in \mathcal{F}$ be sequence such that $\lim_{n\to\infty} p_n = p'$. We show that $p' \in \mathcal{F}$. In fact, for any $i = 1, 2, \ldots.$

$$\begin{aligned} d(p', T_i p') &\leq d(p', p_n) + d(p_n, T_i p') \\ &= d(p', p_n) + d(T_i p_n, T_i p') \\ &\leq d(p', p_n) + Ld(p_n, p') \end{aligned}$$

and this implies that

$$d(p', T_i p') = 0,\ i = 1, 2, \ldots$$

Thus, $p' \in \mathcal{F}$ and so $\mathcal{F}$ is closed. This completes the proof. □

Taking $e_n = 1$, $\forall n \geq 1$ and $R_i = I$ $\forall i \geq 1$ in Theorem 1, then we have the following theorem.

Theorem 2. *Let E be a nonempty closed convex subset of complete convex metric space X. Let $T_i, S_i : E \to E$ be uniformly quasi-Lipschitzian mappings with $L_i > 0$ and L_i' > 0, respectively, $i = 1, 2, \ldots.$ Suppose that $\mathcal{F} = (\cap_{i=1}^{\infty} F(T_i)) \cap (\cap_{i=1}^{\infty} F(S_i))$ is nonempty and bounded. Let $\{x_n\}$ be a sequence defined by (1.3), in which $\{u_n\}$ and $\{v_n\}$ are two bounded sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences in $[0, 1]$ with*

$$\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = 1, \forall n \geq 1 \text{ and } \sum_{n=1}^{\infty} (\beta_n + \gamma_n) < \infty.$$

If $\{L_i\}$ and $\left\{L_i'\right\}$ are bounded, then $\{x_n\}$ converges to a common fixed point of $p \in \mathcal{F}$ if and only if

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0.$$

Taking $e_n = b_n = 1$, $\forall n \geq 1$ and $R_i = S_i = I$, $\forall i \geq 1$, in Theorem 1, then we have the following theorem.

Theorem 3. *Let E be a nonempty closed convex subset of complete convex metric space X. Let $T_i : E \to E$ be uniformly quasi-Lipschitzian mappings with $L_i > 0$, $i = 1, 2, \ldots.$ Suppose that $\mathcal{F} = \cap_{i=1}^{\infty} F(T_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence defined by*

(1.4), in which $\{u_n\}$ is bounded sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$ with

$$\alpha_n+\beta_n+\gamma_n=1, \forall n\geq 1 \text{ and } \sum_{n=1}^{\infty}(\beta_n+\gamma_n)<\infty.$$

If $\{L_i\}$ is bounded, then $\{x_n\}$ converges to a common fixed point of $p\in\mathcal{F}$ if and only if

$$\liminf_{n\to\infty} d(x_n,\mathcal{F})=0.$$

Similarly, we can obtain the following result.

Corollary 1. *Let E be a nonempty closed convex subset of complete convex metric space X. Let $T_i, R_i, S_i : E\to E$ be asymptotically quasi-nonexpansive mappings with $k_{n(i)}$, $k'_{n(i)}$ and $k''_{n(i)}$, respectively, $i=1,2,\ldots$. Suppose that $\mathcal{F}=(\cap_{i=1}^{\infty}F(T_i))\cap(\cap_{i=1}^{\infty}F(R_i))\cap(\cap_{i=1}^{\infty}F(S_i))$ is nonempty and bounded. Let $\{x_n\}$ be a sequence defined by (1.2), in which $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{l_n\}$ be sequences in $[0,1]$ with*

$$\alpha_n+\beta_n+\gamma_n=a_n+b_n+c_n=d_n+e_n+l_n=1, \forall n\geq 1 \text{ and } \sum_{n=1}^{\infty}(\beta_n+\gamma_n)<\infty.$$

If $\sup_{i\geq1}\left\{k_{n(i)}\right\}<\infty$, $\sup_{i\geq1}\left\{k'_{n(i)}\right\}<\infty$ and $\sup_{i\geq1}\left\{k''_{n(i)}\right\}<\infty$, then $\{x_n\}$ converges to a common fixed point of $p\in\mathcal{F}$ if and only if

$$\liminf_{n\to\infty} d(x_n,\mathcal{F})=0,$$

where $d(x,\mathcal{F})=\inf\{d(x,p):p\in\mathcal{F}\}$.

Proof. Since $T_i, R_i, S_i : E\to E$ are asymptotically quasi-nonexpansive mappings with $k_{n(i)}$, $k'_{n(i)}$ and $k''_{n(i)}$, respectively, we know that

$$\lim_{n\to\infty} k_{n(i)}=\lim_{n\to\infty} k'_{n(i)}=\lim_{n\to\infty} k''_{n(i)}=1$$

for $i=1,2,\ldots$. It follows from the $\sup_{i\geq1}\left\{k_{n(i)}\right\}<\infty$, $\sup_{i\geq1}\left\{k'_{n(i)}\right\}<\infty$ and $\sup_{i\geq1}\left\{k''_{n(i)}\right\}<\infty$ that

$$L_i=\sup_{n\geq1}\left\{k_{n(i)}\right\},\ L'_i=\sup_{n\geq1}\left\{k'_{n(i)}\right\} \text{ and } L''_i=\sup_{n\geq1}\left\{k''_{n(i)}\right\}$$

are bounded. Thus, $T_i, R_i, S_{\dot{I}} : E \to E$ are uniformly quasi-Lipschitzian mappings with $L_i > 0$, $L_i^{'} > 0$ and $L_i^{"} > 0$, respectively, $i = 1, 2, ...$ Now Theorem 1 shows that Theorem 1 is true. This completes the proof. □

Remark 1. *Theorems 1–2 generalize, improve, and unify some corresponding results in* [3]*,* [4] *and* [5]*.*

References

[1] W. Takahashi, A convexity in metric space and nonexpansive mappings, Kodai. Math. Sem. Rep. 22 (1970) 142–149.

[2] Y.X. Tian, Convergence of an Ishikawa type iterative scheme for asymptotically quasi-nonexpansive mappings, Comput. Math. Appl. 49 (2005) 1905–1912.

[3] C. Wang, L.W. Liu, Convergence theorems for fixed points of uniformly quasi-Lipschitzian mappings in convex metric spaces, Nonlinear Anal. TMA 70 (2009) 2067–2091.

[4] S.S. Chang, L. Yang, X.R. Wang, Stronger convergence theorems for an infinite family of uniformly quasi-Lipschitzian mappings in convex metric spaces, Appl. Math. Comp. 217 (2010) 277–282.

[5] Q.Y. Liu, Z.B. Liu, N.J. Huang, Approximating the common fixed points of two sequences of uniformly quasi-Lipschitzian mappings in convex metric spaces, Appl. Math. Comp. 216 (2010) 883-889.

[6] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by Ishikawa Iteration process, J.Math. Anal. Appl., 178, (1993) 301-308.

[7] 7 Q. Liu, "Iterative sequences for asymptotically quasi-nonexpansive mappings with error member," Journal of Mathematical Analysis and Applications, vol. 259, no. 1, pp. 18–24, 2001.

[8] H. Fukhar-ur-din, S.H. Khan, Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications, J. Math. Anal. Appl. 328 (2007) 821 829.

[9] I. Yıldırım, S.H. Khan, Convergence theorems for common fixed points of asymptotically quasi-nonexpansive mappings in convex metric spaces, Appl. Math. Comput. 218 (2012) 4860–4866.

[10] I. Yıldırım, S.H. Khan and M. Özdemır, An iteratıon process for common fıxed poınts of unıformly quasııpschıtzıan mappıngs in convex metrıc spaces, Journal of Nonlinear Analysis and Optimization: Theory & Application, (2013), Article in Press.

[11] G.S. Saluja, Three-step iteratıon process for a fınıte famıly of asymptotıcally quası-nonexpansıve mappıngs in the intermedıate sense in convex metrıc spaces, Bulletin of Mathematical Analysis and Applications, Vol. 3 Issue 1 (2011), Pages 89-101.

[12] A.R. Khan, M.A. Ahmed, Convergence of a general iterative scheme for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces and applications, Computers and Mathematics with Applications 59 (2010) 2990 2995.

[13] M. Abbas, Z. Kadelburg, D.R. Sahu, Fixed point theorems for Lipschitzian type mappings in CAT(0) spaces, Mathematical and Computer Modelling, 55 (2012) 1418–1427.