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SOME FIXED POINT RESULTS OF RATIONAL TYPE CONTRACTIONS IN RATIONAL PARAMETRIC $(b,\theta)-$METRIC SPACES

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Abstract. In this paper, we discuss the existence of fixed points of Rational-type contraction mappings in rational parametric $(b,\theta)-$metric spaces. As an application, we study the existence and uniqueness of the solution of Fredholm integral equation.

Keywords: comparison function; α-admissible; rational-type contraction; extended b-metric space; parametric metric space; rational parametric $(b,\theta)-$metric space; Fredholm integral equation.

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1. Introduction

In this paper, we are concerned with the idea of two basic concepts of convergence of sequences and continuity of functions. Bakhtin [1] introduced the b-metric space in 1989 as a generalization of metric space (see also [11]). Kamran et al. [2] introduced the concept of an extended b-metric space and presented the Banach principle of contraction mapping. In 2014, Hussain et al. [3] introduced the concept of parametric space. Again, in 2015, Hussain et al.

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[4] introduced the concept of parametric b-metric space as a generalization of parametric metric space. Mahendra and Khan [5] introduced the concept of parametric (b,θ)-metric space as an extended form of parametric b-metric space and proved some fixed point results dealing with general contractive conditions involving rational-type expressions.

Throughout the chapter, we shall use $\mathbb{N}$ to denote the set of natural numbers and $\mathbb{R}$ to denote the set of real numbers.

Definition 1.1. [1] The pair (X,d_b) is called a b-metric space if $d_b : X \times X \to [0,\infty)$ is a b-metric on X satisfying the following conditions, for all $u,v,w \in X$

(B_1) $d_b(u,v) = 0$ iff $u = v$;

(B_2) $d_b(u,v) = d_b(v,u)$

(B_3) $d_b(u,v) \leq b[d_b(u,w) + d_b(w,v)],$ where $b \geq 1$ is a real number.

If $b = 1$, it becomes a metric space. Hence, every b-metric space is a metric space, but the converse is not true.

Example 1.1. Let (X,d) be a metric space, where $X \neq \emptyset$ and $d_b(u,v) = [d(u,v)]^r$, for all $u,v \in X$. Then (X,d_b) is a b-metric space with $b = 2^{r-1}$, where $r > 1$ is a real number, but d_b is not a metric on X.

Definition 1.2. [2] The pair (X,d_θ) is called an extended b-metric space, if $X \neq \emptyset$ is an arbitrary set, $\theta : X \times X \to [1,\infty)$, and $d_b : X \times X \to [0,\infty)$ is an extended b-metric on X satisfying the following conditions, for all $u,v,w \in X$,

(E_1) $d_\theta(u,v) = 0$ iff $u = v$;

(E_2) $d_\theta(u,v) = d_\theta(v,u)$;

(E_3) $d_\theta(u,v) \leq \theta(u,v)[d_\theta(u,w) + d_\theta(w,v)]$.

If $\theta(u,v) = b > 1$, then the extended b-metric space becomes a b-metric space. Therefore, every metric space is a b-metric space and every b-metric space is an extended b-metric space, but the converse need not be true in general.

Example 1.2. Consider $X=\mathbb{R}$ and define $d_\theta : X\times X \to [0,\infty)$ as $d_\theta(u,v)=|u|+|v|\neq 0, u=v$, where $\theta(u,v)=1+|u|+|v|$, for all $u,v\in X$. Then (X,d_θ) is an extended b-metric space. However, for $u,v\in\mathbb{R}\backslash\{0\}, u\neq v$, we have

$$\frac{d_\theta(u,v)}{d_\theta(u,0)+d_\theta(0,v)} \leq 1+|u|+|v| = \theta(u,v).$$

If $\sup_{u,v\in X}\theta(u,v)=\infty$, it is impossible to find a finite $b=\theta(u,v)\geq 1$ satisfying (B_3). Therefore, (X,d_θ) is not a b-metric space. But every finite extended b-metric space is a b-metric space.

Definition 1.3. [3] The pair (X,P) is called a parametric metric space if $X\neq\emptyset$ is a set, and $P: X^2\times(0,\infty)\to[0,\infty)$ is a parametric metric satisfying the following conditions, for all $u,v,w\in X$ and for all $\sigma>0$,

$$(P_1) P(u,v,\sigma)=0 \text{ iff } u=v;$$

$$(P_2) P(u,v,\sigma)=P(v,u,\sigma);$$

$$(P_3) P(u,v,\sigma)\leq P(u,w,\sigma)+P(w,v,\sigma).$$

Example 1.3. [3] Suppose $X\neq\emptyset$ is a set containing all continuous functions $u:(0,\infty)\to\mathbb{R}$ and define $P: X^2\times(0,\infty)\to(0,\infty)$ by $P(u,v,\sigma)=\sigma|u(t)-v(t)|, \forall\sigma>0$. Then (X,P) is a parametric metric space.

Example 1.4. [3] Suppose $X=[0,\infty)$ and define $P: X^2\times(0,\infty)\to(0,\infty)$ by $P(u,v,\sigma)=\sigma\max\{u,v\}, u\neq v$ and $P(u,v,\sigma)=0, u=v, \forall\sigma>0$. Then (X,P) is a parametric metric space.

Definition 1.4. [4] The pair (X,P_b) is called a parametric b-metric space if $X\neq\emptyset$ is a set, $b\geq 1$ is a real number and $P_b: X^2\times(0,\infty)\to[0,\infty)$ is a parametric b-metric satisfying the following conditions, for all $u,v\in X$ and for all $\sigma>0$,

$$(P_b1)\ P_b(u,v,\sigma)=0 \text{ iff } u=v;$$

$$(P_b2)\ P_b(u,v,\sigma)=P_b(v,u,\sigma);$$

$$(P_b3)\ P_b(u,v,\sigma)\leq b[P_b(u,w,\sigma)+P_b(w,v,\sigma)]$$

If $b=1$, then it becomes a parametric metric space, and hence every parametric metric space is a parametric b-metric space, but the converse need not be true. Note that a parametric b-metric space, for $b>1$, may not be continuous.

Example 1.5. [5] Let $X = \mathbb{N} \cup \{\infty\}$ and let $P : X^2 \times (0,\infty) \to \mathbb{R}$ be defined by

$$P(m,n,\sigma) = \begin{cases} 0, & \text{if } m = n \\ \sigma|\frac{1}{m} - \frac{1}{n}|, & \text{if } m,n \text{ are even or } mn = \infty \text{ ,} \\ 3\sigma, & \text{if } m \text{ and } n \text{ are odd and } m \neq n, \\ 2\sigma, & \text{otherwise.} \end{cases}$$

But for all $m,n,p \in X$, we get $P(m,n,\sigma) \leq \frac{3}{2}(P(m,p,\sigma) + P(p,n,\sigma))$. Thus, (X,P) is a parametric b-metric space with $b = \frac{3}{2}$. We need to show that P is not a continuous function. Suppose $u_n = 2n$ and $v_n = 1$, then we have $u_n \to \infty, v_n \to 1$. Also,

$$P(2n,\infty,\sigma) = \frac{\sigma}{2n} \to 0,$$

and

$$P(u_n, v_n, \sigma) = 0 \to 0.$$

On the other hand,

$$P(u_n, v_n, \sigma) = P(u_n, 1, \sigma) = 2\sigma,$$

and $P(\infty, 1, \sigma) = 1$. Hence, $\lim_{n\to\infty} P(u_n, v_n, \infty) \neq P(u, v, \sigma)$. So P is not continuous.

Example 1.6. [3] Suppose $X = [0,\infty)$ and define $P_b(u,v,\sigma) = \sigma|u-v|^p$, for all $u,v \in X$, and for all $\sigma > 0$. Then (X,P_b) is a parametric b-metric space with constant $b = 2^p$, where $p \geq 1$.

Example 1.7. Suppose $X = [0,\infty)$ and define $P(u,v,\sigma) = \sigma \max\{u,v\}$ $u \neq v$ and $P(u,v,\sigma) = 0$, $u = v$, for all $\sigma > 0$. Then (X,P) is a parametric metric space.

Motivated by the works of Kamran et al. [2], Mahendra and Khan [5] introduced the concept of parametric (b,θ)-metric space.

Definition 1.5. [5] The pair (X,P_θ) is called a parametric (b,θ)-metric space if $X \neq \emptyset$ is a set, $\theta : X^2 \times (0,\infty) \to [1,\infty)$ and $P_\theta : X^2 \times (0,\infty) \to [0,\infty)$ is a parametric (b,θ)-metric satisfying the following conditions, for all $u,v,w \in X$ and for all $\sigma > 0$

$(P_\theta 1)$ $P_\theta(u,v,\sigma)=0$ iff $u=v$;

$(P_\theta 2)$ $P_\theta(u,v,\sigma)=P_\theta(v,u,\sigma)$;

$(P_\theta 3)$ $P_\theta(u,v,\sigma)\leq \theta(u,v,\sigma)[P_\theta(u,w,\sigma)+P_\theta(v,w,\sigma)]$.

If $\theta(u,v,\sigma)=b\geq 1$, then P_θ becomes P_b. Note that every parametric metric space and every parametric b-metric space is a P_θ metric space. Note that P_θ with $b>1$ is not a continuous function, so it is P_θ.

Example 1.8. Suppose $X=\mathbb{R}$ and let $\theta : X^2\times(0,\infty)\to[1,\infty)$ be defined by $\theta(u,v,\sigma)=1+\sigma(|u|+|v|), \forall u,v\in X$ and $\forall\sigma>0$. Let $P_\theta : X^2\times(0,\infty)\to[0,\infty)$ be given by $P_\theta(u,v,\sigma)=\sigma(|u|^p+|v|^p), u\neq v$ and $P_\theta(u,v,\sigma)=0, u=v\ \forall\sigma>0$, where $p\geq 1$. Then (X,P_θ) is a parametric (b,θ)-metric space.

Example 1.9. Let $\theta : X^2\times(0,\infty)\to[1,\infty)$, where $X=[0,1]$, be a function defined by $\theta(u,v,\sigma)=2[\frac{1+\sigma(u+v)}{u+v}], u+v>0$, and $\theta(0,0,\sigma)=1, \forall\sigma>0$. Define $P_\theta : X^2\times(0,\infty)\to[0,\infty)$ as

$$\begin{aligned}
P_\theta(u,v,\sigma) &= \frac{\sigma}{uv}, \forall u,v\in(0,1], u\neq v;\\
P_\theta(u,v,\sigma) &= 0, u=v;\\
P_\theta(u,0,\sigma) &= P_\theta(0,u,\sigma)\\
&= \frac{\sigma}{u}, u\in(0,1], \sigma>0.
\end{aligned}$$

Clearly, $(P_\theta 1)$ and $(P_\theta 2)$ are hold. For $(P_\theta 3)$, we have the following cases:

(i) For $u,v,w\in(0,1], \forall\sigma>0$, we have

$$\begin{aligned}
&P_\theta(u,v,\sigma)\leq\theta(u,v,\sigma)[P_\theta(u,w,\sigma)+P_\theta(w,v,\sigma)]\\
&\Leftrightarrow \frac{\sigma}{uv}\leq 2\frac{[1+\sigma(u+v)]}{(u+v)}\frac{\sigma(u+v)}{uv\sigma}.\\
&\Leftrightarrow \sigma\leq 2[1+\sigma(u+v)].
\end{aligned}$$

(ii) For $u,v\in(0,1]$ and $w=0, \forall\sigma>0$, we have

$$\begin{aligned}P_\theta(u,v,\sigma) &\leq \theta(u,v,\sigma)[P_\theta(u,0,\sigma)+P_\theta(0,v,\sigma)]\\ \Leftrightarrow \frac{\sigma}{uv} &\leq 2\frac{[1+\sigma(u+v)]}{u+v}\left(\frac{\sigma}{u}+\frac{\sigma}{v}\right)\\ \Leftrightarrow 1 &\leq 2[1+\sigma(u+v)].\end{aligned}$$

(iii) For $u,w\in(0,1]$ and $v=0,\forall\sigma>0$, we have

$$\begin{aligned}P_\theta(u,0,\sigma) &\leq \theta(u,0,\sigma)[P_\theta(u,w,\sigma)+P_\theta(w,0,\sigma)]\\ \Leftrightarrow \frac{\sigma}{u} &\leq 2\frac{(1+\sigma u)}{u}\left(\frac{\sigma}{uw}+\frac{\sigma}{w}\right)\\ \Leftrightarrow uw &\leq 2(1+\sigma u)(1+u).\end{aligned}$$

It shows that $(P_\theta 3)$ is satisfied. Thus, (X,P_θ) is a parametric (b,θ)-metric space.

Definition 1.6. [5] Let (X,P_θ) be a parametric (b,θ)-metric space and $\{u_n\}$ be a sequence in X, then

(i) the sequence $\{u_n\}$ is said to be convergent to $u\in X$ and symbolically, we write $u_n\to u$ as $n\to\infty$ iff $\forall\sigma>0, P_\theta(u_n,u,\sigma)\to 0$ as $n\to\infty$;

(ii) the sequence $\{u_n\}$ is said to be a Cauchy in X iff $\forall\sigma>0, P_\theta(u_m,u_n,\sigma)\to 0$ as $m,n\to\infty$;

(iii) (X,P_θ) is said to be complete iff every Cauchy sequence $\{u_n\}$ in X is convergent.

Definition 1.7. [5] Let (X,P_θ) be a parametric (b,θ)-metric space and $T:X\to X$ be a mapping. Then we say that T is continuous at $u\in X$ if for any sequence $\{u_n\}$ in X such that $u_n\to u$ as $n\to\infty$, we have $Tu_n\to Tu$ as $n\to\infty$.

Example 1.10. Let $X=[0,1]$ and define $P_\theta(u,v,\sigma)=\sigma(|u|^p+|v|^p), u\neq v$, and $P_\theta(u,v,\sigma)=0, u=v$, where $\theta(u,v,\sigma)=1+\sigma(|u|+|v|),\forall u,v\in X$ and for all $\sigma>0$. Let $T:X\to X$ be a mapping defined by $Tu=\frac{u}{5},\forall u\in X$. For any $u_0\in X$, we define a sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that $u_n=T^n u_0=(\frac{1}{5})^n u_0$. Clearly, $u_n\to 0$ as $n\to\infty$ and $Tu_n=(\frac{1}{5})^{n+1}u_0\to T0=0$ as $n\to\infty$, i.e., $\lim_{n\to\infty}P_\theta(Tu_n,T0,\sigma)=0$, whenever $\lim_{n\to\infty}P_\theta(u_n,0,\sigma)=0$. Therefore, T is continuous at 0.

Definition 1.8. [8] A mapping $T:X\to X$ is said to be α-admissible if $u,v\in X,\alpha(u,v)\geq 1\Rightarrow \alpha(Tu,Tv)\geq 1$.

Example 1.11. Let $X = \mathbb{R}$. Define $f : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by

$$f(u) = \begin{cases} ln|u|, & \text{if } u \neq 0 \\ 2, & \text{otherwise.} \end{cases}$$

and

$$\alpha(u,v) = \begin{cases} 2, & \text{if } u \geq v \\ 0, & \text{otherwise} \end{cases}$$

Case(i) $u \neq 0$ and $u \geq v$

$$\alpha(f(u), f(v)) = \alpha(ln|u|, ln|v|)$$
$$= 2 \geq 1.$$

Case(ii) If $u = 0$ and $u \geq v$

$$\alpha(f(u), f(v)) = \alpha(2,2)$$
$$= 2 \geq 1.$$

Therefore, in all possible cases, f is α-admissible.

Definition 1.9. [9] A mapping $T : X \to X$ is said to be α-orbital admissible if $u \in X, \alpha(u, Tu) \geq 1 \Rightarrow \alpha(Tu, T^2u) \geq 1$.

Example 1.12. Let $X = \{0,1,2\}$ with usual metric $d(u,v) = |u - v|$. Define $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$ as $T0 = 0, T1 = 2, T2 = 1$ and $\alpha(u,v) = 1$, if $(u,v) \in \{(0,1),(0,2)\}$ and $\alpha(u,v) = 0$, otherwise. Note that T is α-admissible as $\alpha(0,1) = \alpha(0,2) = 1, \alpha(T0,T1) = \alpha(T0,T2) = 1$. But there does not exist $\omega \in X$ such that $\alpha(u,v) = \alpha(T^n\omega, T^{n+1}\omega) = 1, n \in \mathbb{N} \cup \{0\}$. So, T is not an α-orbital admissible mapping.

Definition 1.10. [5] A mapping $T : X \to X$ is said to be parametric α-admissible if $u,v \in X, \alpha(u,v,\sigma) \geq 1 \Rightarrow \alpha(Tu,Tv,\sigma) \geq 1, \forall \sigma > 0$. In addition, we say that T is a parametric α^*-admissible if for all $u,v \in Fix(T) \neq \emptyset$, $\alpha(u,v,\sigma) \geq 1, \forall \sigma > 0$.

Example 1.13. [5] Let $X = [0,\infty)$ and $T : X \to X$ be a mapping defined by $Tu = \frac{u^2}{2}, \forall u \in X$. Define $\alpha : X^2 \times (0,\infty) \to \mathbb{R}$ as $\alpha(u,v,\sigma) = 1 + \sigma(u+v), \forall u,v \in [0,2]$ and $\alpha(u,v,\sigma) = 0$ otherwise, for all $\sigma > 0$. Note that $Fix(T) = \{0,2\}$. Then T is a parametric α-admissible and parametric α^*-admissible.

Definition 1.11. [10] A continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance if it is non-decreasing and $\phi(r) = 0$ iff $r = 0$, and Φ denotes the set of all altering distance functions.

Example 1.14. [5] Let $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$, where $i = 1, 2$ be defined by

(i) $\phi_1(u) = e^u + mu - 1$;

(ii) $\phi_2(u) = mu^2 + ln(nu + 1)$, where $m, n > 0$.

Then $\phi_1(u)$ and $\phi_2(u)$ are altering distance functions.

Definition 1.12. [14] A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a comparison function if it is monotonically increasing and $\lim_{n\to\infty} \psi^n(r) = 0$ for all $r > 0$. The symbol Ψ denotes the set of all the comparison functions.

Lemma 1.1. [14] Let $\psi \in \Psi$. Then, $\psi(r) < r$ for all $r > 0$ and $\psi(0) = 0$.

Lemma 1.2. [5] Let (X, P_θ) be a parametric (b, θ)-metric space and $\{u_n\}$ be any sequence in X. If $\psi \in \Psi$ satisfies

$$\lim_{m,n\to\infty} \frac{\theta(u_n, u_m, \sigma)\psi^n(P_\theta(u_0, u_1, \sigma))}{\psi^{n-1}(P_\theta(u_0, u_1, \sigma))} < 1$$

and

$$0 < P_\theta(u_n, u_{n-1}, \sigma) \leq \psi(P_\theta(u_{n-1}, u_n, \sigma))$$

for any $m > n \geq 1, m, n \in \mathbb{N}$ and for all $\sigma > 0$, then the sequence $\{u_n\}$ is a Cauchy sequence in X.

Lemma 1.3. [5] Let (X, P_θ) be a parametric (b, θ)-metric space and $\{u_n\}$ be any sequence in X such that

$$\begin{aligned} 0 &< P_\theta(u_n, u_{n+1}, \sigma) \\ &\leq kP_\theta(u_{n-1}, u_n, \sigma) \\ \text{and } \lim_{n,m\to\infty} \theta(u_n, u_m, \sigma) &< \frac{1}{k}, \end{aligned}$$

where $k \in [0, 1)$, for any $m > n \geq 1$ and for all $\sigma > 0$, then $\{u_n\}$ is a Cauchy sequence in X.

2. Fixed Points of Rational Type Contractions

Let (X,P_θ) be a parametric (b,θ)-metric space with the continuous functional P_θ, and $T:X\to X$ be a mapping. For $u,v\in X$, we denote

$$\mathscr{U}(u,v,\sigma) = \max\{P_\theta(u,v,\sigma), \tfrac{P_\theta(v,Tv,\sigma)P_\theta(u,Tu,\sigma)}{P_\theta(u,v,\sigma)}, \tfrac{P_\theta(u,Tu,\sigma)[1+P_\theta(v,Tv,\sigma)]}{1+P_\theta(u,v,\sigma)}, \tfrac{P_\theta(v,Tv,\sigma)[1+P_\theta(u,Tu,\sigma)]}{1+P_\theta(u,v,\sigma)}\},$$

$$\mathscr{V}(u,v,\sigma) = \max\{P_\theta(u,v,\sigma), \frac{P_\theta(u,Tu,\sigma)P_\theta(u,Tv,\sigma)+P_\theta(v,Tv,\sigma)P_\theta(v,Tu,\sigma)}{\max\{P_\theta(u,Tv,\sigma),P_\theta(v,Tu,\sigma)\}}, \frac{P_\theta(u,Tu,\sigma)P_\theta(v,Tv,\sigma)+P_\theta(u,Tv,\sigma)P_\theta(v,Tu,\sigma)}{\max\{P_\theta(v,Tv,\sigma),P_\theta(v,Tu,\sigma)\}}\}.$$

We establish theorems on the existence and uniqueness of fixed points for a class of parametric α-admissible mappings in rational-type contractions in the setting of a parametric (b,θ)-metric space and extend our result to a parametric (b,θ)-metric space endowed with partial order.

Theorem 2.1. Let (X,P_θ) be a complete parametric (b,θ)-metric space and $T:X\to X$ is a continuous mapping on X. Assume that there exist $\alpha: X^2\times(0,\infty)\to\mathbb{R}$, $\phi\in\Phi$ and $\psi\in\Psi$ such that $\phi(r)\geq r>\psi(r)$, for $r>0$ satisfying

$$\alpha(u,v,\sigma)\phi(P_\theta(Tu,Tv,\sigma))\leq\psi(\mathscr{U}(u,v,\sigma)) \tag{1}$$

for all $u,v\in X$ and for all $\sigma>0$. If

(i) T is a parametric α-orbitally admissible;

(ii) there exists $u_0\in X$ such that $\alpha(u_0,Tu_0,\sigma)\geq 1$, for all $\sigma>0$;

(iii) $\lim\limits_{n,m\to\infty}\frac{\theta(u_n,u_m,\sigma)\psi^n(P_\theta(u_0,u_1,\sigma))}{\psi^{n-1}(P_\theta(u_0,u_1,\sigma))}<1$, where $u_n=T^nu_0, m>n\geq 1$, for all $\sigma>0$.

Then there exists $w\in X$ such that $Tw=w$, i.e., $Fix(T)\neq\emptyset$.

Proof. According to the condition (ii), there exists $u_0\in X$ such that $\alpha(u_0,Tu_0,\sigma)\geq 1$, for all $\sigma>0$. Let us consider a sequence $\{u_n\}$ in X such that $u_n=T^nu_0$, for all $n\in\mathbb{N}$. If $u_{k-1}=u_k=Tu_{k-1}$, and hence u_{k-1} is a fixed point of T.

Without loss of generality, we suppose that $u_{n-1}\neq u_n$, for all $u\in\mathbb{N}$, then $P_\theta(u_n,u_{n+1},\sigma)>0$, for all $\sigma>0$. Using condition (i), T is a parametric α-orbitally admissible, $\alpha(u_0,u_1,\sigma)=\alpha(u_0,Tu_0,\sigma)\geq 1$ implies

$\alpha(u_1,u_2,\sigma)=\alpha(Tu_0,T^2u_0,\sigma)\geq 1$ for all $\sigma>0$. Similarly, $\alpha(u_1,u_2,\sigma)=\alpha(Tu_0,T^2u_0,\sigma)\geq 1$, $\alpha(u_2,u_3,\sigma)=\alpha(T^2u_0,T^3u_0,\sigma)\geq 1$, and so on. Then, we obtain that $\alpha(u_{n-1},u_n,\sigma)\geq 1$, where $n\in\mathbb{N}$, and for all $\sigma>0$. By the inequality, we get

$$\begin{aligned}\phi(P_\theta(u_n,u_{n+1},\sigma)) &= \phi(P_\theta(Tu_{n-1},Tu_n,\sigma))\\ &\leq \alpha(u_{n-1},u_n,\sigma)\phi(P_\theta(Tu_{n-1},Tu_n,\sigma))\\ &\leq \psi(\mathscr{U}(u_{n-1},u_n,\sigma)),\end{aligned}$$

where

$$\begin{aligned}\mathscr{U}(u_{n-1},u_n,\sigma) &= \max\{P_\theta(u_{n-1},u_n,\sigma),\frac{P_\theta(u_n,Tu_n,\sigma)P_\theta(u_{n-1},Tu_{n-1},\sigma)}{P_\theta(u_{n-1},u_n,\sigma)},\\ &\quad \frac{P_\theta(u_{n-1},Tu_{n-1},\sigma)[1+P_\theta(u_n,Tu_n,\sigma)]}{1+P_\theta(u_{n-1},u_n,\sigma)},\\ &\quad \frac{P_\theta(u_n,Tu_n,\sigma)[1+P_\theta(u_{n-1},Tu_{n-1},\sigma)]}{1+P_\theta(u_{n-1},u_n,\sigma)}\}\\ &= \max\{P_\theta(u_{n-1},u_n,\sigma),\frac{P_\theta(u_n,u_{n+1},\sigma)P_\theta(u_{n-1},u_n,\sigma)}{P_\theta(u_{n-1},u_n,\sigma)},\\ &\quad \frac{P_\theta(u_{n-1},u_n,\sigma)[1+P_\theta(u_n,u_{n+1},\sigma)]}{1+P_\theta(u_{n-1},u_n,\sigma)},\\ &\quad \frac{P_\theta(u_n,u_{n+1},\sigma)[1+P_\theta(u_{n-1},u_n,\sigma)]}{1+P_\theta(u_{n-1},u_n,\sigma)}\}\\ &= \max\{P_\theta(u_{n-1},u_n,\sigma),P_\theta(u_n,u_{n+1},\sigma),\\ &\quad \frac{P_\theta(u_{n-1},u_n,\sigma)[1+P_\theta(u_n,u_{n+1},\sigma)]}{1+P_\theta(u_{n-1},u_n,\sigma)},P_\theta(u_n,u_{n+1},\sigma)\}\\ &= \max\{P_\theta(u_{n-1},u_n,\sigma),P_\theta(u_n,u_{n+1},\sigma),\\ &\quad \frac{P_\theta(u_{n-1},u_n,\sigma)[1+P_\theta(u_n,u_{n+1},\sigma)]}{1+P_\theta(u_{n-1},u_n,\sigma)}\}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\phi(P_\theta(u_n,u_{n+1},\sigma)) &\leq \psi(\max\{P_\theta(u_{n-1},u_n,\sigma),P_\theta(u_n,u_{n+1},\sigma),\\ &\quad \frac{P_\theta(u_{n-1},u_n,\sigma)[1+P_\theta(u_n,u_{n+1},\sigma)]}{1+P_\theta(u_{n-1},u_n,\sigma)}\})\end{aligned}\tag{2}$$

We prove that

$$0<P_\theta(u_n,u_{n+1},\sigma)\leq\psi(P_\theta(u_{n-1},u_n,\sigma)),\forall n\in\mathbb{N}.\tag{3}$$

We finish the proof via three cases.

(i) If $\mathscr{U}(u_{n-1},u_n,\sigma) = P_\theta(u_{n-1},u_n,\sigma)$ then by (1), it follows that
$0 < P_\theta(u_n,u_{n+1},\sigma) \leq \psi(P_\theta(u_{n-1},u_n,\sigma))$.

(ii) If $\mathscr{U}(u_{n-1},u_n,\sigma) = P_\theta(u_n,u_{n+1},\sigma)$, then by (1), we have
$0 < P_\theta(u_n,u_{n+1},\sigma) \leq \psi(P_\theta(u_n,u_{n+1},\sigma)) < P_\theta(u_n,u_{n+1},\sigma)$, which is a contradiction.

(iii) If $\mathscr{U}(u_{n-1},u_n,\sigma) = \frac{P_\theta(u_{n-1},u_n,\sigma)[1+P_\theta(u_n,u_{n+1},\sigma)]}{1+P_\theta(u_{n-1},u_n,\sigma)}$, then by (2), it is clear that

$$\max\{P_\theta(u_{n-1},u_n,\sigma), P_\theta(u_n,u_{n+1},\sigma)\} \leq \frac{P_\theta(u_{n-1},u_n,\sigma)[1+P_\theta(u_n,u_{n+1},\sigma)]}{1+P_\theta(u_{n-1},u_n,\sigma)} \tag{4}$$

In this case, we discuss it with two sub-cases,

(i) If $\max\{P_\theta(u_{n-1},u_n,\sigma), P_\theta(u_n,u_{n+1},\sigma)\} = P_\theta(u_{n-1},u_n,\sigma)$, then

$$P_\theta(u_{n-1},u_n,\sigma) > P_\theta(u_n,u_{n+1},\sigma). \tag{5}$$

By (4), we have $P_\theta(u_{n-1},u_n,\sigma) \leq \frac{P_\theta(u_{n-1},u_n,\sigma)[1+P_\theta(u_n,u_{n+1},\sigma)]}{1+P_\theta(u_{n-1},u_n,\sigma)}$,
i.e., $P_\theta(u_{n-1},u_n,\sigma) \leq P_\theta(u_n,u_{n+1},\sigma)$. This contradicts (5).

(ii) If $\max\{P_\theta(u_{n-1},u_n,\sigma), P_\theta(u_n,u_{n+1},\sigma)\} = P_\theta(u_n,u_{n+1},\sigma)$, then

$$P_\theta(u_n,u_{n+1},\sigma) > P_\theta(u_{n-1},u_n,\sigma). \tag{6}$$

By (4), we get $P_\theta(u_n,u_{n+1},\sigma) \leq \frac{P_\theta(u_{n-1},u_n,\sigma)[1+P_\theta(u_n,u_{n+1},\sigma)]}{1+P_\theta(u_{n-1},u_n,\sigma)}$. This shows that $P_\theta(u_n,u_{n+1}\sigma) \leq P_\theta(u_{n-1},u_n,\sigma)$. This contradicts (6).

If $P_\theta(u_{n-1},u_n,\sigma) < P_\theta(u_n,u_{n+1},\sigma)$, then from (2), we get

$$\phi(P_\theta(u_n,u_{n+1},\sigma)) \leq \psi(P_\theta(u_n,u_{n+1},\sigma)) < \phi(P_\theta(u_n,u_{n+1},\sigma))$$

This is a contradiction, and hence $P_\theta(u_n,u_{n+1},\sigma) \leq P_\theta(u_{n-1},u_n,\sigma)$, for all $n \in \mathbb{N}$. Therefore from (1) with $\phi(r) \geq r > \psi(r), r > 0$, we obtain $0 < \phi(P_\theta(u_n,u_{n+1},\sigma)) \leq \psi(P_\theta(u_{n-1},u_n,\sigma))$, for all $\sigma > 0$. Also from (iii), we obtain $\lim_{n,m\to\infty} \frac{\theta(u_n,u_m,\sigma)\psi^n(P_\theta(u_0,u_1,\sigma))}{\psi^{n-1}(P_\theta(u_0,u_1,\sigma))} < 1$ for all $\sigma > 0$, where $m > n \geq 1$. By Lemma 1.2, the sequence $\{u_n\}$ is a Cauchy sequence in X. Since (X,P_θ) is complete, there exists $w \in X$ such that $u_n \to w$ as $n \to \infty$, i.e., $\lim_{n,m\to\infty} P_\theta(u_n,w,\sigma) = 0$, for all $\sigma > 0$. Suppose that T is continuous on X, then $Tu_n \to Tw$ as $n \to \infty$, but $Tu_n = u_{n+1} \to w$ as $n \to \infty$. Therefore $Tw = w$. □

Example 2.1. Let$X = [0,\infty)$ and $P_\theta : X^2 \times (0,\infty)$ be a parametric (b,θ)-metric defined as $P_\theta(u,v,\sigma) = \sigma|u-v|^2$, where $\theta(u,v,\sigma) = 2+\sigma(u+v)$, for all $u,v \in X$ and $\sigma > 0$. Let $T : X \to X$ is a continuous mapping defined by $Tu = \frac{1}{3}u, u \in [0,1]$ and $Tu = 2u - \frac{4}{3}, u > 1$. Define $\alpha : X^2 \times (0,\infty) \to \mathbb{R}$ as $\alpha(u,v,\sigma) = 1$, where $u,v \in [0,1]$ and $\alpha(u,v,\sigma) = 0$, otherwise, for all $\sigma > 0$. So T is parametric α-admissible. Also, setting $\psi(r) = kr$ and $\phi(r) = r$, where $k = \frac{1}{9}$, then $\phi(r) \geq r > \psi(r)$, for $r > 0$. Actually, for all $u,v \in X$ and for all $\sigma > 0$, we get

$$\begin{aligned}\alpha(u,v,\sigma)\phi(P_\theta(Tu,Tv,\sigma)) &= \sigma|Tu-Tv|^2\\ &= \frac{1}{9}\sigma|u-v|^2 = kP_\theta(u,v,\sigma) \leq \psi(\mathscr{U}(u,v,\sigma)).\end{aligned}$$

As T is parametric α-orbitally admissible, sequence $\{u_n\}$ in X such that

$$\alpha(u_n,u_{n+1},\sigma) = \alpha(T^n u_0, T^{n+1}u_0,\sigma) \geq 1, \text{ for all } \sigma > 0.$$

Since $\alpha(u_n,u_{n+1},\sigma) \geq 1, \forall n \in \mathbb{N}\cup\{0\}$, so $u_n \in [0,1], \forall n \in \mathbb{N}\cup\{0\}$. In fact, $u_n = T^n u_0 = (\frac{1}{3})^n u_0 \to 0$ as $n \to \infty$ and $\lim\limits_{n,m\to\infty} \theta(T^n u_0, T^m u_0,\sigma) = 2 < \frac{1}{k}$.
Hence, T satisfies all the conditions of Theorem (2.1), and hence $Fix(T) \neq \emptyset$.

Theorem 2.2. Let (X,P_θ) be a complete parametric (b,θ)-metric space and $T : X \to X$ be a mapping on X. Assume that there exists $\alpha : X^2 \times (0,\infty) \to \mathbb{R}, \phi \in \Phi$ and $\psi \in \Psi$ such that $\phi(r) \geq r > \psi(r),\ r > 0$ satisfying $\alpha(u,v,\sigma)\phi(P_\theta(Tu,Tv,\sigma)) \leq \psi(\mathscr{U}(u,v,\sigma)), \forall u,v \in X$ and $\forall \sigma > 0$. If

(i) T is a parametric α-orbitally admissible;
(ii) there exists $u_0 \in X$ such that $\alpha(u_0,Tu_0,\sigma) \geq 1, \forall \sigma > 0$;
(iii) $\lim\limits_{n,m\to\infty} \frac{\theta(u_n,u_m,\sigma)\psi^n(P_\theta(u_0,u_1,\sigma))}{\psi^{n-1}(P_\theta(u_0,u_1,\sigma))} < 1$, where $u_n = T^n u_0, m > n \geq 1, \forall \sigma > 0$;
(iv) $\{u_n\}$ is a sequence in X such that $\alpha(u_n,u_{n+1},\sigma) \geq 1$ and $u_n \to \xi \in X$ as $n \to \infty$, then there exists a sub-sequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\alpha(u_{n_k},\xi,\sigma) \geq 1, \forall \sigma > 0$, where $n_k \geq n_0 \geq 1$.

Then there exists $\xi \in X$ such that $T\xi = \xi$, i.e., $Fix(T) \neq \emptyset$.

Proof. As in Theorem (2.1), one can show that the sequence $\{u_n\}$ is a Cauchy sequence in X. Since (X,P_θ) is complete, there exists $\xi \in X$ such that $u_n \to \xi$ as $n \to \infty$. From (iv), we obtain

$\alpha(u_{n_k},\xi,\sigma)\geq 1, u_k\geq u_0\geq 1, \forall\sigma>0$. Taking $u=u_{n_k}$ and $v=\xi, \forall\sigma>0$, we obtain

$$\begin{aligned}\phi(P_\theta(u_{n_k+1},T_\xi,\sigma)) &= \phi(P_\theta(Tu_{n_k},T\xi,\sigma))\\ &\leq \alpha(u_{n_k},\xi,\sigma)\phi(P_\theta(Tu_{n_k},T\xi,\sigma))\\ &\leq \psi(\mathscr{U}(u_{n_k},\xi,\sigma))\\ &< \phi(\mathscr{U}(u_{n_k},\xi,\sigma)).\end{aligned}$$

Now,

$$\begin{aligned}\mathscr{U}(u_{n_k},\xi,\sigma) &= \max\{P_\theta(u_{n_k},\xi,\sigma), \frac{P_\theta(u_{n_k},Tu_{n_k},\sigma)P_\theta(\xi,T\xi,\sigma)}{P_\theta(u_{n_k},\xi,\sigma)},\\ &\quad \frac{P_\theta(u_{n_k},Tu_{n_k},\sigma)[1+P_\theta(\xi,T\xi,\sigma)]}{1+P_\theta(u_{n_k},\xi,\sigma)},\\ &\quad \frac{P_\theta(\xi,T\xi,\sigma)[1+P_\theta(u_{n_k},Tu_{n_k},\sigma)]}{1+P_\theta(u_{n_k},\xi,\sigma)}\}\\ &= \max\{P_\theta(u_{n_k},\xi,\sigma), \frac{P_\theta(u_{n_k},u_{n_k+1},\sigma)P_\theta(\xi,T\xi,\sigma)}{P_\theta(u_{n_k},\xi,\sigma)},\\ &\quad \frac{P_\theta(u_{n_k},u_{n_k+1},\sigma)[1+P_\theta(\xi,T\xi,\sigma)]}{1+P_\theta(u_{n_k},\xi,\sigma)},\\ &\quad \frac{P_\theta(\xi,T\xi,\sigma)[1+P_\theta(u_{n_k},u_{n_k+1},\sigma)]}{1+P_\theta(u_{n_k},\xi,\sigma)}\}.\end{aligned}$$

Suppose $k\to\infty$ and continuity of ϕ, we obtain

$$\begin{aligned}\phi(P_\theta(\xi,T\xi,\sigma)) &< \phi(\lim_{n_k\to\infty}\mathscr{U}(u_{n_k},\xi,\sigma))\\ &= \phi(P_\theta(\xi,T\xi,\sigma))\end{aligned}$$

which is a contradiction. Then, we can say that $P_\theta(\xi,T\xi,\sigma)=0$, and hence $T\xi=\xi$, i.e., $Fix(T)\neq\emptyset$. □

Theorem 2.3. In addition to all the conditions of Theorems (2.1) and (2.2), suppose that T is parametric α^*-orbitally admissible. Then, T possesses a unique fixed point $\xi\in X$.

Proof. Following Theorem (2.1)(respectively Theorem (2.2)),Tpossesses a fixed point in X.Thus,$Fix(T)\neq\emptyset$.Assume that $T$is α^*-orbitally admissible,then $\alpha(\xi,\xi^*,\sigma)=\alpha(T\xi,T\xi^*,\sigma)\geq 1$,for all $\xi,\xi^*\in Fix(T)$ and for all$\sigma>0$, if $\xi\neq\xi^*$ for all $\sigma>0$, we obtain

$$\begin{aligned}\phi(P_\theta(\xi,\xi^*,\sigma)) &= \phi(P_\theta(T\xi,T\xi^*,\sigma))\\ &\le \alpha(\xi,\xi^*,\sigma)\phi(P_\theta(T\xi,T\xi^*,\sigma))\\ &\le \psi(P_\theta(\xi,\xi^*,\sigma))\\ &< \phi(P_\theta(\xi,\xi^*,\sigma)),\ \text{which is a contradiction.}\end{aligned}$$

Therefore, T possesses a unique fixed point in X. □

Corollary 2.1. Let T be a continuous self-mapping on a complete parametric (b,θ)-metric space such that $P_\theta(Tu,Tv,\sigma) \le k\mathscr{U}(u,v,\sigma), \forall u,v \in X$ and $\forall \sigma > 0, u \neq v$, where $k \in [0,1)$. That is, T is a rational-type contraction. In addition, suppose that $\forall u_0 \in X$, $\lim_{n,m\to\infty} \theta(u_n,u_m,\sigma) < \frac{1}{k}$, where $u_n = T^n u_0$ and $0 \le k < 1, \forall \sigma > 0$. Then T has a unique fixed point.

Theorem 2.4. Let (X,P_θ) be a complete parametric (b,θ)-metric space and $T : X \to X$ be a mapping on X. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$, $\phi \in \Phi$ and $\psi \in \Psi$ such that $\phi(r) \ge r > \psi(r)$, for $r > 0$ satisfying

$$\alpha(u,v,\sigma)\phi(P_\theta(Tu,Tv,\sigma)) \le \psi(\mathscr{V}(u,v,\sigma)),\ \forall u,v \in X \text{ and } \forall \sigma > 0. \tag{7}$$

If

(i) T is parametric α-orbitally admissible;
(ii) there exists $u_0 \in X$ and $\alpha(u_0,Tu_0,\sigma) \ge 1$;
(iii) $\lim_{n\to\infty} \frac{\psi^n(P_\theta(u_0,u_1,\sigma))}{\psi^{n-1}(P_\theta(u_0,u_1,\sigma))} < 1$, where $u_n = T^n u_0$;
(iv) T is continuous, the sequence $\{u_n\}$ is in X such that $\alpha(u_n,u_{n+1},\sigma) \ge 1$ and $u_n \to \xi \in X$ as $n \to \infty$, then there exists a sub-sequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\alpha(u_{n_k},\xi,\sigma) \ge 1$, where $u_k \ge n_0 \ge 1$.

Then there exists $\xi \in X$ such that $T\xi = \xi$, i.e., $Fix(T) \neq \emptyset$.

Proof. By (ii), we define a sequence $\{u_n\}$ in X such that $u_{n+1} = Tu_n = T^{n+1}u_0, \forall n \in \mathbb{N} \cup \{0\}$. Since T is parametric α-orbitally admissible, then $\alpha(u_0,u_1,\sigma) = \alpha(u_0,Tu_0,\sigma) \ge 1$ implies $\alpha(u_1,u_2,\sigma) = \alpha(Tu_0,T^2u_0,\sigma) \ge 1$. Repeating this process, we obtain that $\alpha(u_n,u_{n+1},\sigma) \ge 1, \forall n \in \mathbb{N} \cup \{0\}$.

From equation (7), we get

$$\phi(P_\theta(u_{n+1},u_{n+2},\sigma)) \leq \alpha(u_n,u_{n+1},\sigma)\phi(P_\theta(u_n,u_{n+1},\sigma))$$
$$\leq \psi(\mathscr{V}(u_n,u_{n+1},\sigma)),$$

where

$$\begin{aligned}\mathscr{V}(u_n,u_{n+1},\sigma) &= \max\{P_\theta(u_n,u_{n+1},\sigma),\\ &\quad \frac{P_\theta(u_n,Tu_n,\sigma)P_\theta(u_n,Tu_{n+1},\sigma)+P_\theta(u_{n+1},Tu_{n+1},\sigma)P_\theta(u_{n+1},Tu_n,\sigma)}{\max\{P_\theta(u_n,Tu_{n+1},\sigma),P_\theta(u_{n+1},Tu_n,\sigma)\}},\\ &\quad \frac{P_\theta(u_n,Tu_n,\sigma)P_\theta(u_{n+1},Tu_{n+1},\sigma)+P_\theta(u_n,Tu_{n+1},\sigma)P_\theta(u_{n+1},Tu_n,\sigma)}{\max\{P_\theta(u_{n+1},Tu_{n+1},\sigma),P_\theta(u_{n+1},Tu_n,\sigma)\}}\}\\ &= \max\{P_\theta(u_n,u_{n+1},\sigma),\\ &\quad \frac{P_\theta(u_n,u_{n+1},\sigma)P_\theta(u_n,u_{n+2},\sigma)+P_\theta(u_{n+1},u_{n+2},\sigma)P_\theta(u_{n+1},u_{n+1},\sigma)}{\max\{P_\theta(u_n,u_{n+2},\sigma),P_\theta(u_{n+1},u_{n+1},\sigma)\}},\\ &\quad \frac{P_\theta(u_n,u_{n+1},\sigma)P_\theta(u_{n+1},u_{n+2},\sigma)+P_\theta(u_n,u_{n+2},\sigma)P_\theta(u_{n+1},u_{n+1},\sigma)}{\max\{P_\theta(u_{n+1},u_{n+2},\sigma),P_\theta(u_{n+1},u_{n+1},\sigma)\}}\}\\ &= P_\theta(u_n,u_{n+1},\sigma).\end{aligned}$$

Consequently,

$$\phi(P_\theta(u_n,u_{n+1},\sigma)) \leq \psi(P_\theta(u_n,u_{n+1},\sigma)) \tag{8}$$

$\implies \phi(P_\theta(u_n,u_{n+1},\sigma)) \leq \psi(P_\theta(u_n,u_{n+1},\sigma)) < \phi(P_\theta(u_n,u_{n+1},\sigma))$. This is a contradiction and $P_\theta(u_n,u_{n+1},\sigma) \leq P_\theta(u_{n-1},u_n,\sigma)$.

From (8), it follows that

$$\phi(P_\theta(u_n,u_{n+1},\sigma)) \leq \psi(P_\theta(u_{n-1},u_n,\sigma)) \tag{9}$$

since $\phi(r) \geq r > \psi(r)$ for $r > 0$, we get

$$0 < P_\theta(u_n,u_{n+1},\sigma) \leq \phi(P_\theta(u_n,u_{n+1},\sigma)) \leq \psi(P_\theta(u_{n-1},u_n,\sigma)) \tag{10}$$

Consequently, we get

$$\begin{aligned}0 &< P_\theta(u_n,u_{n+1},\sigma)\\ &\leq \psi(P_\theta(u_{n-1},u_n,\sigma))\\ &\ \ \vdots\end{aligned}$$

$$\leq \psi^n(P_\theta(u_0,u_1,\sigma)) \tag{11}$$

From (11), using the triangular inequality, taking with $r = P_\theta(u_0,u_1,\sigma)$, for $p \geq 1$ and $n \in \mathbb{N}$, we get

(12)

$$\begin{aligned} P_\theta(u_n,u_{n+p},\sigma) &\leq P_\theta(u_n,u_{n+1},\sigma) + P_\theta(u_{n+1},u_{n+2},\sigma) + \cdots + P_\theta(u_{n+p-1},u_{n+p},\sigma) \\ &\leq \psi^n(P_\theta(u_0,u_1,\sigma)) + \psi^{n+1}(P_\theta(u_0,u_1,\sigma)) + \cdots + \psi^{n+p-1}(P_\theta(u_0,u_1,\sigma)) \\ &= \sum_{i=n}^{n+p-1} \psi^i(r) \\ &= \sum_{i=1}^{n+p-1} \psi^i(r) - \sum_{i=1}^{n-1} \psi^i(r) \end{aligned}$$

From (iii), we obtain

$$\lim_{n\to\infty} \frac{\psi^n(P_\theta(u_0,u_1,\sigma))}{\psi^{n-1}(P_\theta(u_0,u_1,\sigma))} = \lim_{n\to\infty} \frac{\psi^n(r)}{\psi^{n-1}(r)} < 1.$$

By the Ratio test, $\sum_{i=1}^{\infty} \psi^i(r)$ is convergent. Let $\mathscr{S} = \sum_{i=1}^{\infty} \psi^i(r)$ and $\mathscr{S}_n = \sum_{i=1}^{n} \psi^i(r)$, according by partial sum, the equation (12) becomes $P_\theta(u_n,u_{n+p},\sigma) \leq [\mathscr{S}_{n+p-1} - \mathscr{S}_{n-1}], \forall n \in \mathbb{N}$ and $p \geq 1$. As $n \to \infty$, we get $\lim_{n\to\infty} P_\theta(u_n,u_{n+1},\sigma) = 0$. Thus, the sequence $\{u_k\}$ is a Cauchy sequence in X. Since (X,P_θ) is complete, so the sequence $\{u_n\}$ converges to $\xi \in X$ so that $T\xi = \xi$, i.e., $Fix(T) \neq \emptyset$. □

Theorem 2.5. In addition to Theorem 2.4, suppose that T is α^*-orbitally admissible. Then T has a unique fixed point $\gamma \in X$.

Proof. By Theorem 2.4, T possesses a fixed point in X, i.e., $Fix(T) \neq \emptyset$. For uniqueness, suppose $\gamma,\gamma^* \in Fix(T)$ such that $\gamma \neq \gamma^*, \alpha^*$-orbital admissibility of T, we have $\alpha^*(\gamma,\gamma^*,\sigma) \geq 1$, $\forall\, \sigma > 0$.

Theorem 2.5 divides the proof into two cases.

Case (i): Let us assume that $\max\{P_\theta(\gamma,T\gamma^*,\sigma), P_\theta(\gamma^*,T\gamma,\sigma)\} \neq 0$ and $\max\{P_\theta(\gamma^*,T\gamma^*,\sigma), P_\theta(\gamma^*,T\gamma,\sigma)\} \neq 0$.

From (7), we get

$$\begin{aligned} P_\theta(\gamma,\gamma^*,\sigma) &= P_\theta(T\gamma,T\gamma^*,\sigma) \\ &\leq \alpha(\gamma,\gamma^*,\sigma)P_\theta(T\gamma,T\gamma^*,\sigma) \\ &\leq \psi(\mathscr{V}(\gamma,\gamma^*,\sigma)) \end{aligned}$$

where,

$$\begin{aligned} \mathscr{V}(\gamma,\gamma^*,\sigma) &= \max\{P_\theta(\gamma,\gamma^*,\sigma), \\ &\quad \frac{P_\theta(\gamma,T\gamma,\sigma)P_\theta(\gamma,T\gamma^*,\sigma)+P_\theta(\gamma^*,T\gamma^*,\sigma)P_\theta(\gamma^*,T\gamma,\sigma)}{\max\{P_\theta(\gamma,T\gamma^*,\sigma),P_\theta(\gamma^*,T\gamma,\sigma)\}}, \\ &\quad \frac{P_\theta(\gamma,T\gamma,\sigma)P_\theta(\gamma^*,T\gamma^*,\sigma)+P_\theta(\gamma,T\gamma^*,\sigma)P_\theta(\gamma^*,T\gamma,\sigma)}{\max\{P_\theta(\gamma^*,T\gamma^*,\sigma),P_\theta(\gamma^*,T\gamma,\sigma)\}}\} \\ &= P_\theta(\gamma,\gamma^*,\sigma). \end{aligned}$$

Therefore $P_\theta(\gamma,\gamma^*,\sigma) \leq \psi(P_\theta(\gamma,\gamma^*,\sigma)) < P_\theta(\gamma,\gamma^*,\sigma)$, which is a contradiction.

Case (ii): Assume that $\max\{P_\theta(\gamma,T\gamma^*,\sigma),P_\theta(\gamma^*,T\gamma,\sigma)\} = 0$ or $\max\{P_\theta(\gamma^*,T\gamma^*,\sigma),P_\theta(\gamma^*,T\gamma,\sigma)\} = 0$. Consequently, $\gamma = T\gamma^* = T\gamma = \gamma^*$.

Thus, T possesses a unique fixed point in X. □

Suppose $(X,\preceq)$ is a partially ordered set. Then a mapping $T: X \to X$ is a monotonic non-decreasing mapping if $u,v \in X,\ u \preceq v$ implies $Tu \preceq Tv$.

Definition 2.1. Let $(X,P_\theta,\preceq)$ be a parametric (b,θ)-metric space endowed with a partial order $\preceq$. If for every monotonic non-decreasing sequence $\{u_n\} \subset X$, which converges to $u \in X$, there exists a sub-sequence $\{u_{n_k}\}$ of $\{u_n\}$ satisfying $u_{n_k} \preceq u$, then $(X,P_\theta,\preceq)$ is said to be regular.

Theorem 2.6. Suppose $(X,\preceq)$ is a partially ordered set, and such that there exists a complete parametric (b,θ)-metric space. Let $T: X \to X$ be a monotonic non-decreasing self-mapping w.r.t. $\preceq$ and there exist $\phi \in \Phi$ and $\psi \in \Psi, \phi(r) \geq r > \psi(r)$, for $r > 0$ satisfying

$$\phi(P_\theta(Tu,Tv,\sigma)) \leq \psi(\mathscr{U}(u,v,\sigma)), \forall u,v \in X \text{ with } u \preceq v \text{ and } \forall \sigma > 0.$$

If

(i) there exists $u_0 \in X$ such that $u_0 \preceq Tu_0$;

(ii)

$$\lim_{n,m\to\infty} \frac{\theta(u_n,u_m,\sigma)\psi^n(P_\theta(u_0,u_1,\sigma))}{\psi^{n-1}(P_\theta(u_0,u_1,\sigma))} < 1,$$

(iii) T is continuous, or $\{u_n\}$ is a non-decreasing sequence in X such that $u_n \to \xi$ as $n \to \infty$, then there exists a sub-sequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \preceq \xi$, where $n_k > n_0$.

Then $Fix(T) \neq \emptyset$. Furthermore, if every pair of elements $\xi, \xi^* \in Fix(T)$ is comparable, then T has a unique fixed point.

Proof. Let a mapping $\alpha : X^2 \times (0,\infty) \to [0,\infty)$ be defined as $\alpha(u,v,\sigma) = 1,\ u \preceq v$ or $v \preceq u$ and $\alpha(u,v,\sigma) = 0$, otherwise, $\forall \sigma > 0$. Then, $\alpha(u,v,\sigma)\ \phi(P_\theta(Tu,Tv,\sigma)) \leq \phi(\mathscr{U}(u,v,\sigma))$, $\forall u,v \in X$ with $u \preceq v$ and $\forall \sigma > 0$, but T is non-decreasing mapping w.r.t. $\preceq$, so T is parametric α-admissible, in fact if $u,v \in X$ such that $\alpha(u,v,\sigma) \geq 1$, $\forall \sigma > 0$, then $u \preceq v$, or $v \preceq u$. Because, T is monotonic non-decreasing mapping w.r.t. $\preceq$, we get $Tu \preceq Tv$, or $Tv \preceq Tu$, and which gives $\alpha(Tu,Tv,\sigma) \geq 1, \forall \sigma > 0$. From (iii), if T is continuous, then all the hypotheses of Theorem 2.4 are satisfied. And in the second part of (iii), suppose that $\{u_n\}$ is a monotonic non-decreasing sequence in X such that $u_n \to \xi$ as $n \to \infty$, then there is a sub-sequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \preceq \xi, n_k \geq n_0$. Therefore, $\alpha(u_{n_k}, \xi, \sigma) \geq 1, \forall \sigma > 0$.

Hence all the hypotheses of Theorem 2.4 are satisfied. Consequently, T possesses a fixed point in X, i.e., $Fix(T) \neq \emptyset$

Next, assume that every pair of elements $\xi, \xi^* \in Fix(T)$ are comparable, then $\xi \leq \xi^*$, or $\xi^* \leq \xi$, which is also $\alpha(\xi, \xi^*, \sigma) \geq 1, \forall \sigma > 0$. So, T is a parametric α^*-admissible. Thus, all the hypotheses of Theorem 2.5 are satisfied, and hence T has a unique fixed point. □

3. Application

Let $X = \mathscr{C}(a,b)$ be the set of all real-valued continuous functions on $[a,b]$. Define two mappings $\theta : X^2 \times (0,\infty) \to [1,\infty)$ and $P_\theta : X^2 \times (0,\infty) \to [0,\infty)$ by $\theta(x,y,\sigma) = 2^{p-1} + \sigma(|x(t)| + |y(t)|$ and $P_\theta(x,y,\sigma) = \sup_{t\in[a,b]} \sigma|x(t) - y(t)|^p$ respectively, where $p > 1$ is a constant. Then, (X, P_θ) is a complete parametric (b,θ)-metric space.

Define a Fredholm integral equation by

$$x(t) = \eta(t) + \lambda \int_a^b I(t,s,x(s))ds,$$

where $t \in [a,b], |\lambda| > 0$ and $I : [a,b] \times [a,b] \times X \to \mathbb{R}$ and $\eta : [a,b] \to \mathbb{R}$ are continuous functions. Let $T : X \to X$ be an integral operator defined by

$$Tx(t) = \eta(t) + \lambda \int_a^b I(t,s,x(s))ds. \tag{13}$$

Theorem 3.1. Let $T : X \to X$ be an integral operator defined in (13). Suppose that the following conditions hold:

(i) for any $x_0 \in X$, $\lim_{n,m\to\infty} \theta(T^n x_0, T^m x_0, \sigma) < \frac{1}{k}$, where $k = \frac{1}{2^p}$,

(ii) for any $x, y \in X$, $x \neq y$, it satisfies

$$|I(t,s,x(s) - I(t,s,y(s)| \leq \xi(t,s)|x(s) - y(s)|, \tag{14}$$

where $(t,s) \in [a,b] \times [a,b]$ and $\xi : [a,b] \times [a,b] \to \mathbb{R}$ is a continuous function satisfying

$$\sup_{t\in[a,b]} \int_a^b \xi^p(t,s)ds < \frac{1}{2^p|\lambda|^p(b-a)^{p-1}}. \tag{15}$$

Then, the integral operator T has a unique solution in X.

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}$ in X by $x_n = T^n x_0, n \geq 1$. From (13), we obtain

$$x_{n+1} = Tx_n(t) = \eta(t) + \lambda \int_a^b I(t,s,x_n(s))ds.$$

Let $q > 1$ be a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Making full use of (14) and Holder's inequality, we speculate that

$$\begin{aligned}
|Tx(t) - Ty(t)|^p &= \left|\lambda \int_a^b \{I(t,s,x(s)) - I(t,s,y(s))\}\, ds\right|^p \\
&\leq \left(\int_a^b |\lambda|\, |I(t,s,x(s)) - I(t,s,y(s))|\, ds\right)^p \\
&\leq \left(\int_a^b |\lambda|^q ds\right)^{\frac{p}{q}} \left(\left(\int_a^b |I(t,s,x(s)) - I(t,s,y(s))|^p ds\right)^{\frac{1}{p}}\right)^p \\
&= |\lambda|^p (b-a)^{p-1} \left(\int_a^b |I(t,s,x(s)) - I(t,s,y(s))|^p ds\right) \\
&\leq |\lambda|^p (b-a)^{p-1} \int_a^b \xi^p(t,s)|x(s) - y(s)|^p ds \qquad (16)
\end{aligned}$$

Making the most of (16) and (15), we deduce that

$$P_\theta(Tx,Ty,\sigma) = \sup_{t\in[a,b]} \sigma|Tx(t) - Ty(t)|^p$$

$$\begin{aligned}
&\leq |\lambda|^p (b-a)^{p-1} \sup_{t\in[a,b]} \left[\int_a^b \xi^p(t,s)|x(s)-y(s)|^p ds\right] \\
&\leq |\lambda|^p (b-a)^{p-1} \sup_{t\in[a,b]} |x(s)-y(s)|^p \left(\sup_{t\in[a,b]} \int_a^b \xi^p(t,s) ds\right) \\
&\leq \frac{1}{2^p}\mathscr{U}(x,y,\sigma).
\end{aligned}$$

Setting $k = \frac{1}{2^p}$, we obtain that

$$P_\theta(Tx,Ty,\sigma) \leq k\mathscr{U}(x,y,\sigma).$$

Thus, all the conditions of Corollary 2.1 are satisfied, and hence T possesses a unique fixed point in X. □

Conflict of Interests

The author declares that there is no conflict of interests.

References

[1] I.A. Bakhtin, The Contraction Mapping Principle in Almost Metric Spaces, Funct. Anal. Gos. Ped. Inst. Unianowsk 30 (1989), 26–37.

[2] T. Kamran, M. Samreen, Q. UL Ain, A Generalization of B-Metric Space and Some Fixed Point Theorems, Mathematics 5 (2017), 19. https://doi.org/10.3390/math5020019.

[3] N. Hussain, S. Khaleghizadeh, P. Salimi, A.A.N. Abdou, A New Approach to Fixed Point Results in Triangular Intuitionistic Fuzzy Metric Spaces, Abstr. Appl. Anal. 2014 (2014), 690139. https://doi.org/10.1155/2014/690139.

[4] N. Hussain, P. Salimi, V. Parvaneh, Fixed Point Results for Various Contractions in Parametric and Fuzzy b-Metric Spaces, J. Nonlinear Sci. Appl. 08 (2015), 719–739. https://doi.org/10.22436/jnsa.008.05.24.

[5] Y.M. Singh, M.S. Khan, On parametric (b,θ)-Metric Space and Some Fixed Point Theorems, in: P. Debnath, N. Konwar, S. Radenović (eds.) Metric Fixed Point Theory, Forum for Interdisciplinary Mathematics, Springer, Singapore, 2021, pp. 101–115. https://doi.org/10.1007/978-981-16-4896-0_7.

[6] H. Huang, Y.M. Singh, M.S. Khan, S. Radenović, Rational Type Contractions in Extended b-Metric Spaces, Symmetry 13 (2021), 614. https://doi.org/10.3390/sym13040614.

[7] A. Ran, M. Reurings, A Fixed Point Theorem in Partially Ordered Sets and Some Applications to Matrix Equations, Proc. Am. Math. Soc. 132 (2003), 1435–1443. https://doi.org/10.1090/s0002-9939-03-07220-4.

[8] B. Samet, C. Vetro, P. Vetro, Fixed Point Theorems for $\alpha - \psi$-Contractive Type Mappings, Nonlinear Anal.: Theory Methods Appl. 75 (2012), 2154–2165. https://doi.org/10.1016/j.na.2011.10.014.

[9] O. Popescu, Some New Fixed Point Theorems for α-Geraghty Contraction Type Maps in Metric Spaces, Fixed Point Theory Appl. 2014 (2014), 190. https://doi.org/10.1186/1687-1812-2014-190.

[10] M. Khan, M. Swaleh, S. Sessa, Fixed Point Theorems by Altering Distances Between the Points, Bull. Aust. Math. Soc. 30 (1984), 1–9. https://doi.org/10.1017/s0004972700001659.

[11] S. Czerwik, Contraction Mapping in b-Metric Spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5–11.

[12] A. Aghajani, M. Abbas, J. Roshan, Common Fixed Point of Generalized Weak Contractive Mappings in Partially Ordered b-Metric Spaces, Math. Slovaca 64 (2014), 941–960. https://doi.org/10.2478/s12175-014-0250-6.

[13] H.H. Alsulami, S. Gülyaz, E. Karapınar, İ.M. Erhan, Fixed Point Theorems for a Class of α-Admissible Contractions and Applications to Boundary Value Problem, Abstr. Appl. Anal. 2014 (2014), 187031. https://doi.org/10.1155/2014/187031.

[14] V. Berinde, On the Approximation of Fixed Points of Weak Contractive Mappings, Carpathian J. Math. 19 (2003), 7–22.

[15] E. Karapınar, P. Kumam, P. Salimi, On α-ψ-Meir-Keeler Contractive Mappings, Fixed Point Theory Appl. 2013 (2013), 94. https://doi.org/10.1186/1687-1812-2013-94.