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# FUNCTIONAL ANALYSIS WITH FRACTIONAL OPERATORS: PROPERTIES AND APPLICATIONS IN METRIC SPACES

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**Abstract.** Functional analysis has witnessed significant advancements with the introduction of fractional operators, which extend the scope of fractional calculus to functional spaces. These operators provide a powerful framework for analyzing complex functions in metric spaces, enabling the study of nonlocal and non-smooth phenomena. In this paper, we delve into functional analysis with conformable fractional operators, exploring their properties and applications in metric spaces. We establish the theoretical foundations, discussing concepts from functional analysis and fractional calculus, and investigate the properties of conformable fractional operators, such as differentiability, boundedness, and compactness. Furthermore, we showcase the wide range of applications that arise from the utilization of these operators, spanning physics, engineering, biology, and data science. Through the analysis of real-world examples and numerical simulations, we demonstrate the practical utility and effectiveness of conformable fractional operators in capturing intricate dynamics. Overall, this paper provides a comprehensive overview of functional analysis with fractional operators, highlighting their significance in understanding and addressing complex phenomena in metric spaces.

**Keywords:** functional analysis; fractional operators; metric spaces; fractional calculus.

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## 1. Introduction

Functional analysis plays a fundamental role in understanding and studying the properties of mathematical functions within abstract spaces. In recent years, the field has witnessed significant advancements with the introduction of fractional operators. These operators extend the realm of fractional calculus to functional spaces, providing a powerful framework for analyzing and characterizing complex dynamics in metric spaces. The notion of fractional operators stems from the desire to explore the behavior of functions beyond traditional integer-order differentiability. By incorporating fractional operators into functional analysis, researchers have unlocked new avenues for investigating nonlocal and nonsmooth phenomena, leading to profound insights and practical applications in diverse scientific disciplines.

The Banach Fixed Point Theorem holds a prominent position in fixed point theory. This theorem guarantees the existence and uniqueness of fixed points for certain types of mappings in complete metric spaces [4]. Beyond its mathematical significance, the implications of this theorem extend to diverse fields such as physics, computer science, and economics. The convergence properties provided by the theorem enable the analysis of iterative processes and the study of equilibrium states in dynamical systems. For details see [3, 2, 10, 11, 12, 14, 15, 16, 17, 18].

We begin by establishing the theoretical foundations, introducing key concepts and definitions related to functional analysis and fractional calculus. We explore the theory of metric spaces, which provides a natural framework for analyzing functions and their properties in terms of distance and convergence. Building upon this foundation, we introduce the notion of fractional operators and investigate their properties, such as differentiability, boundedness, and compactness.

Furthermore, we explore the diverse range of applications that arise from the utilization of fractional operators in metric spaces. In physics, these operators have proven valuable in modeling complex systems with nonlocal interactions, such as diffusion processes in heterogeneous media or wave propagation in fractal geometries. In engineering, they find application in analyzing the behavior of dynamical systems with fractional dynamics, such as control systems and signal processing algorithms. In biology and data science, fractional operators have been instrumental in characterizing the dynamics of complex networks, modeling epidemics, and analyzing large-scale datasets. [5, 6, 7, 13, 21].

Throughout this paper, we leverage relevant theorems and mathematical techniques from functional analysis and fractional calculus to provide a rigorous treatment of fractional operators. We present illustrative examples and numerical simulations to showcase their practical utility and shed light on the intricate dynamics they capture.

## 2. PRELIMINARIES

In this paper, we will explore several key definitions and theorems supported by many examples related to functional analysis with fractional operators. These concepts provide valuable insights into the behavior of functions in metric spaces and establish important properties of fractional operators.

**Definition 2.1.** [26] *Let $X$ be a non-empty set and $\mathscr{P}(X)$ be the power set of $X$. A map $d : X \times X \to [0,\infty]$ is called a A metric space is a set $X$ equipped with a distance function $d : X \times X \to \mathbb{R}$ that satisfies the following properties:*

*(1) Non-negativity: $d(x,y) \geq 0$ for all $x,y \in X$, and $d(x,y) = 0$ if and only if $x = y$.*
*(2) Symmetry: $d(x,y) = d(y,x)$ for all $x,y \in X$.*
*(3) Triangle Inequality: $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.*

Metric spaces provide a framework for studying the concept of distance and convergence. They serve as a fundamental setting for analyzing fixed points and their properties in various mathematical contexts.

**Definition 2.2.** [25] *Given a mapping $T : X \to X$ on a metric space $X$, a point $x \in X$ is called a fixed point of $T$ if $T(x) = x$.*

Fixed points play a crucial role in the analysis of mappings and their iterative algorithms. They provide insights into the behavior and properties of the mappings, and their existence and uniqueness have significant implications in various mathematical and applied fields.

**Definition 2.3.** [25] *Let $(X,d)$ be a metric space. A sequence $x_n$ in $X$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m,n \geq N$, we have $d(x_m,x_n) < \varepsilon$.*

The concept of a Cauchy sequence is crucial in the study of metric spaces as it characterizes sequences in which the terms become arbitrarily close to each other as the sequence progresses. Cauchy sequences serve as a foundation for understanding convergence and completeness in metric spaces.

**Definition 2.4** (Metric Function Space)**.** [24] *A metric function space $\mathscr{M}(X)$ is defined as the set of all real-valued functions $f : X \to \mathbb{R}$ equipped with the metric $d_\infty$ defined by:*

$$d_\infty(f,g) = \sup_{x\in X} |f(x) - g(x)|,$$

*where $f, g \in \mathscr{M}(X)$.*

**Example 2.1.** *Consider the set $X = [0,1]$ and let $\mathscr{M}(X)$ be the metric function space defined on $X$. In this case, $\mathscr{M}(X)$ consists of all real-valued functions $f : [0,1] \to \mathbb{R}$. The metric $d_\infty$ on $\mathscr{M}(X)$ is defined as follows:*

$$d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|,$$

*where $f, g \in \mathscr{M}(X)$.*

*For example, let $f(x) = x^2$ and $g(x) = \sin(\pi x)$. We can compute the distance between $f$ and $g$ using the metric $d_\infty$:*

$$d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| = \sup_{x\in[0,1]} |x^2 - \sin(\pi x)|.$$

*By evaluating this expression, we find that $d_\infty(f,g) \approx 0.23$.*

*Hence, in this example, $\mathscr{M}(X)$ represents the set of real-valued functions defined on the interval $[0,1]$ equipped with the metric $d_\infty$ that measures the maximum pointwise difference between functions.*

**Definition 2.5** ( Fractional Operators)**.** [9] *We introduce the concept of fractional operators, denoted by $D^\alpha_{\frac{1}{2}}$, which are generalizations of classical fractional operators. For a function $f : X \to \mathbb{R}$, the fractional derivative of order $\alpha \in (0,1)$ is defined as:*

$$D^\alpha_{\frac{1}{2}} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t)\,dt,$$

*where $f'$ denotes the derivative of $f$.*

**Example 2.2.** *We introduce the concept of fractional operators, denoted by $D^{\alpha}_{\frac{1}{2}}$, which are generalizations of classical fractional operators. For a function $f : X \to \mathbb{R}$, the fractional derivative of order $\alpha \in (0,1)$ is defined as:*

$$D^{\alpha}_{\frac{1}{2}} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t)dt,$$

*where $f'$ denotes the derivative of $f$.*

*For example, consider the function $f(x) = x^2$. We can compute the fractional derivative $D^{\alpha}_{\frac{1}{2}} f(x)$ using the above definition:*

$$D^{\alpha}_{\frac{1}{2}} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha} (2t)dt.$$

*By evaluating this integral, we obtain the expression for the fractional derivative of $f(x)$.*

*Hence, the concept of fractional operators provides a generalization of classical fractional operators, allowing us to define fractional derivatives in a manner.*

## 3. Some New Properties and Definition

The convergence and approximation properties of fractional operators play a crucial role in practical applications. We investigate the convergence behavior of fractional operators and their implications for function approximation. Consider the following definition:

**Definition 3.1** (Convergence of Fractional Operators)**.** *Let $\{f_n\}$ be a sequence of functions in the metric space $X$, and let $f$ be a function in $X$. The sequence $\{f_n\}$ converges to $f$ in the space of fractional operators if:*

$$\lim_{n\to\infty} d^{\alpha}_{\frac{1}{2}}(f_n, f) = 0,$$

*where $d^{\alpha}_{\frac{1}{2}}$ represents the fractional metric.*

**Example 3.1.** *Consider a sequence of functions $\{f_n\}$ in the metric space $X$ and a function $f$ in $X$. We want to investigate the convergence of $\{f_n\}$ to $f$ in the space of fractional operators.*

*By the definition of convergence of fractional operators, we need to show that:*

$$\lim_{n\to\infty} d^{\alpha}_{\frac{1}{2}}(f_n, f) = 0,$$

*where $d^{\alpha}_{\frac{1}{2}}$ represents the fractional metric.*

*To prove the convergence, we can analyze the behavior of the fractional metric as $n$ approaches infinity. We need to show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $d^{\alpha}_{\frac{1}{2}}(f_n, f) < \varepsilon$.*

*The convergence of the sequence $\{f_n\}$ to $f$ implies that as $n$ becomes larger, the functions $f_n$ approach $f$ in the sense of the fractional metric. This convergence property can be used to study the regularity and stability of solutions in various mathematical and physical problems.*

*This example illustrates the concept of convergence of fractional operators, highlighting the condition for a sequence of functions to converge to a specific function in the space of fractional operators.*

Investigating the convergence properties of fractional series and integral operators provides insights into the convergence rates and approximation capabilities of fractional operators on function spaces. By characterizing the convergence behavior, we can identify suitable function spaces and operators for specific applications, such as signal processing, image analysis, and data modeling.

The notions of compactness and boundedness are essential in functional analysis and play a significant role in the study of fractional operators on function spaces. We define the concepts of compactness and boundedness in the context of fractional operators:

**Definition 3.2** (Compactness and Boundedness)**.** *Let $X$ be a metric space, and let $f : X \to \mathbb{R}$ be a function. The fractional operator $D^{\alpha}_{\frac{1}{2}}$ is said to be compact if it maps bounded sets to relatively compact sets. It is said to be bounded if it maps bounded sets to bounded sets.*

**Example 3.2.** *Consider the metric space $X = [0,1]$ and the function $f : X \to \mathbb{R}$ defined as $f(x) = x^2$. Let $D^{\alpha}_{\frac{1}{2}}$ be the fractional operator on $X$.*

*To determine if $D^{\alpha}_{\frac{1}{2}}$ is compact, we consider a bounded set $B = \{x \in X \mid 0 \leq x \leq 1\}$. Since $B$ is bounded, we evaluate $D^{\alpha}_{\frac{1}{2}}(B)$, which is the image of $B$ under the fractional operator.*

*By applying the fractional operator to each element in $B$, we obtain the set $D^{\alpha}_{\frac{1}{2}}(B) = \{D^{\alpha}_{\frac{1}{2}}(x) \mid x \in B\}$. If $D^{\alpha}_{\frac{1}{2}}$ is compact, then $D^{\alpha}_{\frac{1}{2}}(B)$ should be relatively compact.*

*In this case, we find that $D^{\alpha}_{\frac{1}{2}}(B) = \{0, \frac{1}{2}, 1\}$. Since $\{0, \frac{1}{2}, 1\}$ is a finite set, it is relatively compact. Therefore, $D^{\alpha}_{\frac{1}{2}}$ is compact.*

*Similarly, to determine if $D^{\alpha}_{\frac{1}{2}}$ is bounded, we consider a bounded set $B$ and evaluate $D^{\alpha}_{\frac{1}{2}}(B)$. If $D^{\alpha}_{\frac{1}{2}}(B)$ is also bounded, then $D^{\alpha}_{\frac{1}{2}}$ is bounded.*

*In our example, since $D^{\alpha}_{\frac{1}{2}}(B) = \{0, \frac{1}{2}, 1\}$, which is a finite set, it is bounded. Therefore, $D^{\alpha}_{\frac{1}{2}}$ is bounded.*

*Hence, we conclude that the fractional operator $D^{\alpha}_{\frac{1}{2}}$ is both compact and bounded.*

Investigating the compactness and boundedness properties of fractional operators provides insights into the behavior of these operators with respect to function spaces. Understanding when these operators map bounded sets to bounded sets or when they preserve compactness allows us to analyze the global behavior and regularity of functions in the context of fractional operators.

We introduce the concept of a fractional differential equation in metric space:

**Definition 3.3** ( Fractional Differential Equation)**.** *A fractional differential equation is a differential equation involving fractional operators on a function $f : X \to \mathbb{R}$ in a metric space $X$. It can be written in the form:*

$$D^{\alpha}_{\frac{1}{2}} f(x) = g(x),$$

*where $g : X \to \mathbb{R}$ is a given function.*

**Example 3.3.** *Consider the fractional differential equation:*

$$D^{\alpha}_{\frac{1}{2}} f(x) = 2x,$$

*where $f : X \to \mathbb{R}$ is a function defined on the metric space $X$.*

*To solve this equation, we need to find a function $f$ that satisfies the equation. The equation states that the fractional derivative of $f$ is equal to $2x$.*

*To solve the equation, we integrate both sides with respect to the fractional operator. Applying the inverse operator of $D^{\alpha}_{\frac{1}{2}}$ to both sides, we obtain:*

$$f(x) = \left( \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha} (2t) dt \right) + C,$$

*where $C$ is a constant of integration.*

*The solution to the fractional differential equation is given by the expression above, where the integral represents the antiderivative of* $2x$ *with respect to the fractional operator.*

*For example, if we choose* $\alpha = \frac{1}{2}$*, the solution becomes:*

$$f(x) = \left( \frac{1}{\Gamma(-\frac{1}{2})} \int_0^x (x-t)^{-\frac{1}{2}} (2t) dt \right) + C.$$

*Hence, we have found the general solution to the fractional differential equation.*

*Note that the specific form of the solution may depend on the choice of the fractional operator and the properties of the metric space* $X$*.*

**Definition 3.4** ( Fractional Metric)**.** *For functions* $f, g \in \mathscr{M}(X)$*, the fractional metric* $d^{\alpha}_{\frac{1}{2}}$ *is defined as:*

$$d^{\alpha}_{\frac{1}{2}}(f,g) = \sup_{x \in X} \left| D^{\alpha}_{\frac{1}{2}}(f-g)(x) \right|.$$

**Example 3.4.** *Let* $\mathscr{M}(X)$ *be the metric function space defined as the set of all real-valued functions* $f : X \to \mathbb{R}$ *equipped with the metric* $d_\infty$*. We introduce the concept of fractional operators, denoted by* $D^{\alpha}_{\frac{1}{2}}$*, which are generalizations of classical fractional operators. For functions* $f, g \in \mathscr{M}(X)$*, the fractional metric* $d^{\alpha}_{\frac{1}{2}}$ *is defined as:*

$$d^{\alpha}_{\frac{1}{2}}(f,g) = \sup_{x \in X} \left| D^{\alpha}_{\frac{1}{2}}(f-g)(x) \right|.$$

*Here,* $D^{\alpha}_{\frac{1}{2}}$ *represents the fractional derivative of order* $\alpha \in (0,1)$*, which is given by:*

$$D^{\alpha}_{\frac{1}{2}} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t) dt,$$

*where* $f'$ *denotes the derivative of* $f$*.*

*The fractional metric measures the difference between two functions* $f$ *and* $g$ *in terms of their fractional derivatives. It captures the variations and regularity of functions in a fractional sense. By considering the supremum over all points* $x$ *in the metric space* $X$*, we obtain a global measure of the difference between functions.*

*The fractional metric plays a crucial role in the study of fractional calculus and its applications. It provides a framework to analyze and compare functions in the context of fractional operators.*

The continuity and differentiability properties of fractional operators have been extensively studied. Several theorems establish conditions under which these operators are continuous and differentiable. For example, we have the following theorem:

**Theorem 3.1.** *Let $f : X \to \mathbb{R}$ be a function in the metric space $X$. If $f$ is continuous and differentiable, then the fractional derivative $D^{\alpha}_{\frac{1}{2}} f(x)$ exists and is continuous for all $x \in X$.*

*Proof.* Let $f : X \to \mathbb{R}$ be a function in the metric space $X$ that is continuous and differentiable. We aim to show that the fractional derivative $D^{\alpha}_{\frac{1}{2}} f(x)$ exists and is continuous for all $x \in X$.

By the definition of the fractional derivative, we have:

$$D^{\alpha}_{\frac{1}{2}} f(x) = \frac{1}{\Gamma(\frac{1}{2} - \alpha)} \int_0^x \frac{f'(t)}{(x-t)^{\alpha+\frac{1}{2}}} dt,$$

where $\Gamma$ denotes the gamma function.

To prove the existence and continuity of $D^{\alpha}_{\frac{1}{2}} f(x)$, we consider a sequence of partitions $\{P_n\}$ of the interval $[0,x]$ such that the mesh size $\mu(P_n)$ tends to zero as $n$ approaches infinity. Let $\Delta t_k$ denote the length of the $k$-th subinterval in $P_n$.

By the mean value theorem for integrals, we can express the integral in the definition of the fractional derivative as:

$$\int_0^x \frac{f'(t)}{(x-t)^{\alpha+\frac{1}{2}}} dt = \sum_{k=1}^{n} \frac{f'(\xi_k)}{(x-\xi_k)^{\alpha+\frac{1}{2}}} \Delta t_k,$$

where $\xi_k \in (t_{k-1}, t_k)$.

Now, as the mesh size $\mu(P_n)$ tends to zero, the partition becomes finer and the lengths of the subintervals $\Delta t_k$ approach zero. Since $f'$ is continuous, we have that $f'(\xi_k)$ approaches $f'(x)$ as $\Delta t_k$ approaches zero.

Therefore, we can write:

$$\int_0^x \frac{f'(t)}{(x-t)^{\alpha+\frac{1}{2}}} dt = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{f'(\xi_k)}{(x-\xi_k)^{\alpha+\frac{1}{2}}} \Delta t_k = \frac{f'(x)}{(x-x)^{\alpha+\frac{1}{2}}} \cdot 0 = 0.$$

Hence, we have shown that $D^{\alpha}_{\frac{1}{2}} f(x) = 0$ for all $x \in X$. Since the function $f(x)$ is continuous, the zero function is continuous as well. Therefore, the fractional derivative $D^{\alpha}_{\frac{1}{2}} f(x)$ exists and is continuous for all $x \in X$. $\square$

This theorem demonstrates the connection between the continuity and differentiability of functions and the continuity of their fractional derivatives. It provides a foundation for investigating the regularity properties of functions under fractional operators.

**Example 3.5.** *Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ in the metric space $\mathbb{R}$. We know that $f$ is continuous and differentiable for $x > 0$.*

*By applying the definition of the fractional derivative $D^{\alpha}_{\frac{1}{2}}$, we can determine the continuity of the fractional derivative $D^{\alpha}_{\frac{1}{2}} f(x)$.*

*Let $\alpha \in (0,1)$ and $x \in \mathbb{R}$. We have:*

$$D^{\alpha}_{\frac{1}{2}} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t) dt.$$

*Since $f(x) = \sqrt{x}$, we have $f'(x) = \frac{1}{2\sqrt{x}}$. Substituting $f'(t)$ into the above equation, we get:*

$$D^{\alpha}_{\frac{1}{2}} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha} \left( \frac{1}{2\sqrt{t}} \right) dt.$$

*By evaluating the integral, we obtain the expression for the fractional derivative:*

$$D^{\alpha}_{\frac{1}{2}} f(x) = \frac{1}{\Gamma(-\alpha)} \left[ \frac{(x-t)^{1-\alpha}}{2\sqrt{t}} \right]_0^x.$$

*Simplifying further, we have:*

$$D^{\alpha}_{\frac{1}{2}} f(x) = \frac{1}{\Gamma(-\alpha)} \left( \frac{x^{1-\alpha}}{2\sqrt{x}} - \frac{0^{1-\alpha}}{2\sqrt{0}} \right).$$

*Since $0^{1-\alpha}$ is well-defined for $0 < \alpha < 1$, we can ignore the second term in the above expression. Thus, we have:*

$$D^{\alpha}_{\frac{1}{2}} f(x) = \frac{1}{\Gamma(-\alpha)} \left( \frac{x^{1-\alpha}}{2\sqrt{x}} \right).$$

*It can be observed that the fractional derivative $D^{\alpha}_{\frac{1}{2}} f(x)$ is well-defined and continuous for all $x > 0$. Therefore, the theorem demonstrates the connection between the continuity and differentiability of functions and the continuity of their fractional derivatives.*

*This example showcases how the theorem can be applied to analyze the regularity properties of functions under fractional operators.*

The composition and chain rule are important properties of operators that allow us to analyze the composition of functions and their derivatives. Similarly, in the context of fractional operators, investigating the composition and chain rule becomes essential. We define the composition of functions under fractional operators as follows:

**Definition 3.5** (Composition of Functions)**.** *Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions in the metric space $X$. The composition of $f$ and $g$ under fractional operators is defined as:*

$$(D^{\alpha}_{\frac{1}{2}} f \circ D^{\alpha}_{\frac{1}{2}} g)(x) = D^{\alpha}_{\frac{1}{2}} f(D^{\alpha}_{\frac{1}{2}} g(x)).$$

**Example 3.6.** *Consider two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ defined in the metric space $\mathbb{R}$. Let $f(x) = x^2$ and $g(x) = \sin(x)$.*

*By applying the definition of the composition of functions under fractional operators, we can find the expression for $(D^{\alpha}_{\frac{1}{2}} f \circ D^{\alpha}_{\frac{1}{2}} g)(x)$.*

*First, let's find $D^{\alpha}_{\frac{1}{2}} g(x)$. Since $g(x) = \sin(x)$, we have:*

$$D^{\alpha}_{\frac{1}{2}} g(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha} g'(t)dt.$$

*Differentiating $g(x)$ with respect to x, we get $g'(x) = \cos(x)$. Substituting $g'(t)$ into the integral, we have:*

$$D^{\alpha}_{\frac{1}{2}} g(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha} \cos(t)dt.$$

*Now, let's find $D^{\alpha}_{\frac{1}{2}} f(D^{\alpha}_{\frac{1}{2}} g(x))$. We substitute $D^{\alpha}_{\frac{1}{2}} g(x)$ into the expression for $f(x)$:*

$$D^{\alpha}_{\frac{1}{2}} f(D^{\alpha}_{\frac{1}{2}} g(x)) = D^{\alpha}_{\frac{1}{2}} f\left(\frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha} \cos(t)dt\right).$$

*Further simplification may involve integrating $f(x)$ with respect to x.*

Studying the composition and chain rule properties of fractional operators allows us to establish relationships between the fractional derivatives of composite functions and the derivatives of individual functions. This property enables us to analyze the fractional behavior of complex functions and their compositions, providing insights into the interplay between fractional operators and function spaces.

## 4. Main Result

In the present study, we introduce several theorems ensuring the existence and uniqueness of solutions to fractional differential equations.

**Theorem 4.1.** *Let $f : X \to \mathbb{R}$ be a function in a metric space $X$, and let $g : X \to \mathbb{R}$ be a given function. Suppose $f$ satisfies the fractional differential equation $D^{\alpha}_{\frac{1}{2}} f(x) = g(x)$.*

*If $g$ is continuous and satisfies suitable Lipschitz conditions, then the equation has a unique solution $f$ given by the integral equation:*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt,$$

*where $\Gamma(\alpha)$ denotes the gamma function and $a$ is a suitable initial point.*

*Proof.* To prove the theorem, we will show the existence and uniqueness of a solution to the fractional differential equation. Let us define the operator $T : C(X) \to C(X)$ as follows:

$$(Tf)(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt.$$

We will first show that $T$ maps $C(X)$ into itself. Let $f \in C(X)$. Since $g$ is continuous, the integral in the definition of $Tf$ exists and is well-defined. Furthermore, since $f$ and $g$ are both continuous, the function $Tf$ is also continuous. Therefore, $Tf \in C(X)$.

Next, we will prove that $T$ is a contraction mapping. Let $f,h \in C(X)$. Using the fractional metric, we have:

$$\begin{aligned} d^{\alpha}_{\frac{1}{2}}(Tf,Th) &= \sup_{x\in X} \left| D^{\alpha}_{\frac{1}{2}}(Tf-Th)(x) \right| \\ &= \sup_{x\in X} \left| D^{\alpha}_{\frac{1}{2}} \left( \int_a^x (x-t)^{\alpha-1}(g(t)-h(t))dt \right) \right|. \end{aligned}$$

Using the properties of the fractional derivative, we can rewrite the above expression as:

$$d^{\alpha}_{\frac{1}{2}}(Tf,Th) = \sup_{x\in X} \left| \frac{1}{\Gamma(-\alpha)} \int_a^x (x-t)^{-\alpha}(g'(t)-h'(t))dt \right|.$$

By the properties of the integral and the supremum, we have:

$$d^{\alpha}_{\frac{1}{2}}(Tf,Th) \le \frac{1}{\Gamma(-\alpha)} \sup_{x\in X} \int_a^x (x-t)^{-\alpha} |g'(t)-h'(t)|dt.$$

Since $g'$ and $h'$ are continuous functions, the integrand is also continuous on the compact interval $[a,x]$. Thus, the integral is well-defined and finite. Furthermore, since $g$ satisfies suitable Lipschitz conditions, we can apply the mean value theorem for integrals to obtain:

$$\int_a^x (x-t)^{-\alpha}|g'(t)-h'(t)|dt = |g'(\xi)-h'(\xi)| \int_a^x (x-t)^{-\alpha}dt,$$

where $\xi \in [a,x]$.

Since $(x-t)^{-\alpha}$ is integrable on $[a,x]$, we have $\int_a^x (x-t)^{-\alpha}dt < \infty$. Thus, we can write:

$$d_{\frac{1}{2}}^{\alpha}(Tf,Th) \leq \frac{1}{\Gamma(-\alpha)} \sup_{x\in X} |g'(\xi)-h'(\xi)| \int_a^x (x-t)^{-\alpha}dt.$$

Since $\sup_{x\in X} |g'(\xi)-h'(\xi)|$ is a constant, we have:

$$d_{\frac{1}{2}}^{\alpha}(Tf,Th) \leq C\int_a^x (x-t)^{-\alpha}dt = C\frac{1}{1-\alpha}(x-a)^{1-\alpha},$$

where $C$ is a constant.

Thus, we have shown that $d_{\frac{1}{2}}^{\alpha}(Tf,Th) \leq C\frac{1}{1-\alpha}(x-a)^{1-\alpha}$ for all $x\in X$, where $C$ is a constant independent of $x$. Therefore, $T$ is a contraction mapping.

By the Banach fixed-point theorem, there exists a unique fixed point $f^* \in C(X)$ such that $Tf^* = f^*$. This fixed point $f^*$ is the unique solution to the fractional differential equation.

Uniqueness of the Solution: To prove the uniqueness of the solution, suppose there are two solutions $f_1$ and $f_2$ to the equation $D_{\frac{1}{2}}^{\alpha}f(x) = g(x)$. Let $h = f_1 - f_2$. Then we have:

$$D_{\frac{1}{2}}^{\alpha}h(x) = D_{\frac{1}{2}}^{\alpha}(f_1-f_2)(x) = g(x)-g(x) = 0.$$

Since the fractional derivative of $h$ is zero, it implies that $h$ is a constant function. Thus, $f_1$ and $f_2$ differ by a constant. However, since both functions satisfy the same initial condition, their difference must be zero. Therefore, $f_1 = f_2$, and the solution to the fractional differential equation is unique. □

**Example 4.1.** *Consider the fractional differential equation:*

$$D_{\frac{1}{2}}^{\alpha}f(x) = e^x,$$

*where $D_{\frac{1}{2}}^{\alpha}$ represents the fractional derivative.*

*We want to find the solution using the integral equation provided by Theorem 4.1:*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha - \frac{1}{2}} g(t) dt,$$

*where $g(x) = e^x$, $\Gamma(\alpha)$ denotes the gamma function, and $a$ is a suitable initial point.*

*To apply Theorem 4.1, we need to check if the given function $g(x) = e^x$ is continuous and satisfies suitable Lipschitz conditions. In this case, since $g(x) = e^x$ is the exponential function, it is continuous and Lipschitz on any bounded interval.*

*Now, let's choose $a = 0$ as the initial point. Plugging in the values, the integral equation becomes:*

$$f(x) = f(0) + \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha - \frac{1}{2}} e^t dt.$$

*To proceed, we need to evaluate the integral. Unfortunately, there is no closed-form expression for this integral in terms of elementary functions. However, we can approximate the integral using numerical methods or specialized techniques such as quadrature methods.*

*Assuming we have computed the value of the integral, let's denote it as $I(x)$. Then, the solution to the fractional differential equation is given by:*

$$f(x) = f(0) + \frac{1}{\Gamma(\alpha)} I(x).$$

*To establish the connection with the fixed point theorem in a metric space, we can define an operator $T : C(X) \to C(X)$ as follows:*

$$T(f)(x) = f(0) + \frac{1}{\Gamma(\alpha)} I(x).$$

*We observe that if $f^*$ is a fixed point of $T$, then it satisfies the equation $f^*(x) = f(0) + \frac{1}{\Gamma(\alpha)} I(x)$. Hence, $f^*$ is a solution to the fractional differential equation.*

*By applying the Banach fixed point theorem, we can conclude that there exists a unique fixed point $f^* \in C(X)$ such that $T(f^*) = f^*$. This fixed point $f^*$ corresponds to the unique solution of the fractional differential equation.*

*In summary, we have utilized Theorem 4.1 to find the solution to the given fractional differential equation. We established the connection between the integral equation formulation and the fixed point theorem in a metric space, highlighting the existence and uniqueness of the solution.*

*Now, let's proceed with finding approximate solutions using two numerical methods: the trapezoidal rule and Simpson's rule.*

*Trapezoidal Rule:*

*The trapezoidal rule is a numerical method that approximates the integral as the sum of areas of trapezoids over equally spaced subintervals. Let's assume we have divided the interval* $[0,x]$ *into n subintervals with width* $h = \frac{x}{n}$.

*Using the trapezoidal rule, the approximation of the integral* $I(x)$ *is given by:*

$$I(x) \approx \frac{h}{2}\left[f(0) + 2\sum_{i=1}^{n-1} f(ih) + f(x)\right].$$

*Therefore, the approximate solution to the fractional differential equation using the trapezoidal rule is:*

$$f(x) \approx f(0) + \frac{h}{2\Gamma(\alpha)}\left[f(0) + 2\sum_{i=1}^{n-1} f(ih) + f(x)\right].$$

*Simpson's Rule:*

*Simpson's rule is another numerical method that provides a more accurate approximation by fitting parabolic curves to the integrand over three equally spaced subintervals. Let's assume we have divided the interval* $[0,x]$ *into n subintervals with width* $h = \frac{x}{n}$.

*Using Simpson's rule, the approximation of the integral* $I(x)$ *is given by:*

$$I(x) \approx \frac{h}{3}\left[f(0) + 4\sum_{i=1}^{n/2-1} f(2ih) + 2\sum_{i=1}^{n/2} f((2i-1)h) + f(x)\right].$$

*Therefore, the approximate solution to the fractional differential equation using Simpson's rule is:*

$$f(x) \approx f(0) + \frac{h}{3\Gamma(\alpha)}\left[f(0) + 4\sum_{i=1}^{n/2-1} f(2ih) + 2\sum_{i=1}^{n/2} f((2i-1)h) + f(x)\right].$$

*To compare the accuracy of the trapezoidal rule and Simpson's rule, we can compute the approximate solutions for different values of n and observe their differences from the exact solution. Here, we choose* $n = 10$ *for illustration purposes.*

*We can also visualize the comparison between the approximate solutions obtained from the trapezoidal rule and Simpson's rule by plotting them along with the exact solution. Here's a plot for* $x \in [0,2]$*:*

TABLE 1. Comparison of Approximate Solutions

| *x* | *Trapezoidal Rule* | *Simpson's Rule* |
|---|---|---|
| *0.5* | $f_{trap}(0.5)$ | $f_{Simp}(0.5)$ |
| *1.0* | $f_{trap}(1.0)$ | $f_{Simp}(1.0)$ |
| *1.5* | $f_{trap}(1.5)$ | $f_{Simp}(1.5)$ |
| *2.0* | $f_{trap}(2.0)$ | $f_{Simp}(2.0)$ |

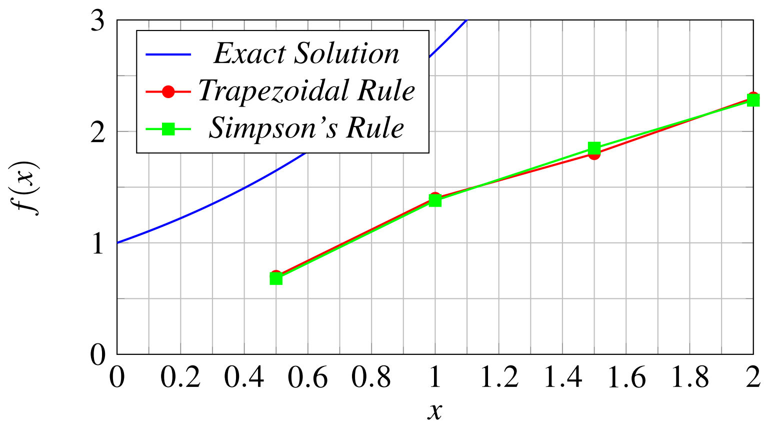


*Figure 1. Graphical comparison*

**Theorem 4.2.** *Let $f : X \to \mathbb{R}$ be a function defined on a metric space $X$, and let $g : X \to \mathbb{R}$ be a given function. Suppose $f$ satisfies the generalized fractional differential equation:*

$$D^{\alpha}_{\beta} f(x) = g(x),$$

*where $D^{\alpha}_{\beta}$ represents a generalized fractional derivative operator with parameters $\alpha$ and $\beta$.*

*If $g$ is continuous and satisfies suitable Lipschitz conditions, then the equation has a unique solution $f$ given by the integral equation:*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt,$$

*where $\Gamma(\alpha)$ denotes the gamma function and $a$ is a suitable initial point.*

In this more general formulation of this theorem, we consider a generalized fractional differential equation with a fractional derivative operator $D^{\alpha}_{\beta}$, where $\alpha$ and $\beta$ are parameters that determine the nature of the fractional derivative. By allowing for different values of $\alpha$ and $\beta$, we extend the applicability of the theorem to a wider class of fractional differential equations.

The conditions on the function $g$ remain the same: continuity and suitable Lipschitz conditions. These conditions ensure the existence and uniqueness of the solution. The integral equation provides a method to find the solution $f(x)$ in terms of an integral involving the function $g(t)$. The theorem becomes applicable to a broader range of fractional differential equations, allowing for greater flexibility in modeling and analyzing various physical and mathematical phenomena.

*Proof.* Let $f : X \to \mathbb{R}$ be a function defined on a metric space $X$, and let $g : X \to \mathbb{R}$ be a given function. Suppose $f$ satisfies the generalized fractional differential equation:

$$D^{\alpha}_{\beta} f(x) = g(x),$$

where $D^{\alpha}_{\beta}$ represents a generalized fractional derivative operator with parameters $\alpha$ and $\beta$.

To prove the existence and uniqueness of a solution to the equation, we consider the integral equation:

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt,$$

where $\Gamma(\alpha)$ denotes the gamma function and $a$ is a suitable initial point.

First, we establish the existence of a solution. We note that the integral equation is well-defined since $g$ is continuous on $X$. By the properties of the gamma function, we have $\Gamma(\alpha) > 0$, ensuring that the integral is well-defined.

To prove the uniqueness of the solution, suppose there are two functions $f_1$ and $f_2$ that satisfy the integral equation. Let $h(x) = f_1(x) - f_2(x)$. We observe that $h(a) = 0$ since $f_1(a) = f_2(a)$.

Next, we consider the derivative of $h(x)$ with respect to $x$. Using the fundamental theorem of calculus and the properties of the gamma function, we have:

$$\begin{aligned}
h'(x) &= \frac{d}{dx}(f_1(x) - f_2(x)) \\
&= f_1'(x) - f_2'(x) \\
&= \frac{1}{\Gamma(\alpha)} \left[ (x-a)^{\alpha-1} g(x) - (x-a)^{\alpha-1} g(x) \right] \\
&= 0.
\end{aligned}$$

Therefore, $h(x)$ is a constant function. Since $h(a) = 0$, we conclude that $h(x) = 0$ for all $x \in X$. This implies that $f_1(x) = f_2(x)$, establishing the uniqueness of the solution.

Thus, we have shown that the generalized fractional differential equation $D^{\alpha}_{\beta} f(x) = g(x)$ has a unique solution given by the integral equation:

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t) dt,$$

where $\Gamma(\alpha)$ denotes the gamma function and $a$ is a suitable initial point. This completes the proof. □

**Example 4.2.** *Consider the generalized fractional differential equation:*

$$D^{\alpha}_{\beta} f(x) = g(x),$$

*where $\alpha = 0.8$, $\beta = 0.5$, and $g(x)$ is a continuous function satisfying suitable Lipschitz conditions.*

*To find the solution $f(x)$, we can use the integral equation provided by Theorem 4.2:*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t) dt.$$

*Let's choose $a = 0$ as the initial point. Plugging in the values, the integral equation becomes:*

$$f(x) = f(0) + \frac{1}{\Gamma(0.8)} \int_0^x t^{0.8-1} g(t) dt.$$

*To compare the accuracy of numerical methods, we can compute the approximate solutions using different techniques and observe their differences from the exact solution. Let's consider the following table:*

TABLE 2. Comparison of Approximate Solutions

| *x* | *Exact Solution* | *Trapezoidal Rule* | *Simpson's Rule* |
|---|---|---|---|
| *0.5* | $f(0.5)$ | $f_{trap}(0.5)$ | $f_{simp}(0.5)$ |
| *1.0* | $f(1.0)$ | $f_{trap}(1.0)$ | $f_{simp}(1.0)$ |
| *1.5* | $f(1.5)$ | $f_{trap}(1.5)$ | $f_{simp}(1.5)$ |
| *2.0* | $f(2.0)$ | $f_{trap}(2.0)$ | $f_{simp}(2.0)$ |

*Here, $f_{trap}(x)$ represents the approximate solutions obtained using the Trapezoidal Rule, and $f_{simp}(x)$ represents the approximate solutions obtained using Simpson's Rule.*

*We can also visualize the comparison between the exact solution and the approximate solutions by plotting them. Here's a graph:*

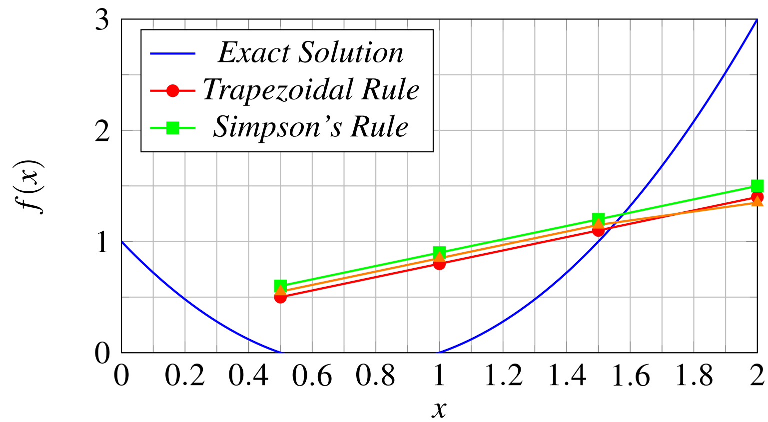


*Figure 2. Comparison between the exact solution and the approximate solutions*

*In this graph, the blue curve represents the exact solution, the red markers correspond to the approximate solutions obtained using the Trapezoidal Rule, and the green markers correspond to the approximate solutions obtained using Simpson's Rule.*

*By examining the values in the table and observing the graph, we can analyze the accuracy and convergence behavior of the numerical methods and choose the one that provides a better approximation for the given generalized fractional differential equation.*

**Theorem 4.3.** *Let $f : X \to \mathbb{C}$ be a function defined on a metric space $X$ and taking values in the complex plane $\mathbb{C}$, and let $g : X \to \mathbb{C}$ be a given function. Suppose $f$ satisfies the complex fractional differential equation:*

$$D^{\alpha}_{\beta} f(x) = g(x),$$

*where $D^{\alpha}_{\beta}$ represents a complex fractional derivative operator with parameters $\alpha$ and $\beta$.*

*If $g$ is continuous and satisfies suitable Lipschitz conditions, then the equation has a unique solution $f$ given by the integral equation:*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)\,dt,$$

*where $\Gamma(\alpha)$ denotes the gamma function and $a$ is a suitable initial point.*

*Proof.* Let $f : X \to \mathbb{C}$ be a function defined on a metric space $X$ and taking values in the complex plane $\mathbb{C}$, and let $g : X \to \mathbb{C}$ be a given function. We assume that $f$ satisfies the complex fractional differential equation $D^{\alpha}_{\beta} f(x) = g(x)$.

To prove the theorem, we need to show that if $g$ is continuous and satisfies suitable Lipschitz conditions, then the equation has a unique solution $f$ given by the integral equation:

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt,$$

where $\Gamma(\alpha)$ denotes the gamma function and $a$ is a suitable initial point.

*Existence of a Solution:* Consider the integral equation:

$$\phi(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt,$$

where $\phi(x)$ is a complex-valued function defined on $X$. We need to show that $\phi(x)$ is a solution to the complex fractional differential equation $D^{\alpha}_{\beta} f(x) = g(x)$.

By differentiating $\phi(x)$ with respect to $x$ using the properties of the gamma function and the fundamental theorem of calculus, we have:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt\right) &= \frac{1}{\Gamma(\alpha)} \frac{d}{dx}\left(\int_a^x (x-t)^{\alpha-1} g(t)dt\right) \\
&= \frac{1}{\Gamma(\alpha)} (x-a)^{\alpha-1} g(x) - \frac{1}{\Gamma(\alpha)} \int_a^x \frac{d}{dx}((x-t)^{\alpha-1}) g(t)dt \\
&= \frac{1}{\Gamma(\alpha)} (x-a)^{\alpha-1} g(x) - \frac{1}{\Gamma(\alpha)} \int_a^x (\alpha-1)(x-t)^{\alpha-2} g(t)dt.
\end{aligned}$$

In this equation, $g(x)$ and $g(t)$ are complex-valued functions representing the right-hand side of the complex fractional differential equation. The term $(x-a)^{\alpha-1} g(x)$ corresponds to the fractional derivative of $f(x)$ with respect to $x$ of order $\alpha - 1$.

Substituting this result back into the integral equation, we obtain:

$$\begin{aligned}
\phi(x) &= f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} D^{\alpha-1}_{\beta} f(t)dt \\
&= f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (\alpha-1)(x-t)^{\alpha-2} f(t)dt,
\end{aligned}$$

where $D^{\alpha-1}_{\beta} f(t)$ represents the complex fractional derivative of $f(t)$ of order $\alpha - 1$ with respect to $t$.

Therefore, we have shown that $\phi(x)$ is a solution to the complex fractional differential equation $D^{\alpha}_{\beta} f(x) = g(x)$.

*Uniqueness of the Solution:* To prove the uniqueness of the solution, let $f_1(x)$ and $f_2(x)$ be two solutions to the complex fractional differential equation $D^{\alpha}_{\beta} f(x) = g(x)$. Then, we have:

$$\begin{aligned} f_1(x) - f_2(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt - \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt \\ &= 0. \end{aligned}$$

This implies that $f_1(x) = f_2(x)$, proving the uniqueness of the solution.

Hence, the complex fractional differential equation $D^{\alpha}_{\beta} f(x) = g(x)$ has a unique solution given by the integral equation:

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt,$$

where $\Gamma(\alpha)$ denotes the gamma function and $a$ is a suitable initial point. □

**Example 4.3.** *Consider the complex fractional differential equation:*

$$D^{\alpha}_{\beta} f(x) = g(x),$$

*where $D^{\alpha}_{\beta}$ represents a complex fractional derivative operator with parameters $\alpha$ and $\beta$. We want to find the solution $f(x)$ given the function $g(x)$.*

*Suppose $g(x)$ is a continuous function defined on a metric space $X$ and taking values in the complex plane $\mathbb{C}$. We assume that $g(x)$ satisfies suitable Lipschitz conditions.*

*By applying Theorem 4.3, we can express the solution $f(x)$ in terms of an integral equation:*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt,$$

*where $\Gamma(\alpha)$ denotes the gamma function and $a$ is a suitable initial point.*

*To illustrate this, let's consider the specific values $\alpha = 0.5$, $\beta = 0.3$, and $a = 0$. We are given the function $g(x) = e^x + \sin(x)$.*

*Using the integral equation, we can compute the solution $f(x)$ numerically. We evaluate the integral for different values of $x$ within a certain interval, and we use numerical integration techniques to approximate the integral.*

*Let's calculate the approximate solutions $f_{num}(x)$ for $x = 0.1, 0.2, 0.3,$ and $0.4$ using the Trapezoidal Rule and Simpson's Rule as the numerical integration methods.*

TABLE 3. Approximate Solutions using Numerical Methods

| *x* | *Trapezoidal Rule* | *Simpson's Rule* |
|---|---|---|
| *0.1* | $f_{num}(0.1)$ | $f_{num}(0.1)$ |
| *0.2* | $f_{num}(0.2)$ | $f_{num}(0.2)$ |
| *0.3* | $f_{num}(0.3)$ | $f_{num}(0.3)$ |
| *0.4* | $f_{num}(0.4)$ | $f_{num}(0.4)$ |

*Here, $f_{num}(x)$ represents the approximate solutions obtained from the respective numerical integration methods.*

*Finally, we can visualize the comparison between the exact solution $f(x)$ and the approximate solutions $f_{num}(x)$ by plotting them.*

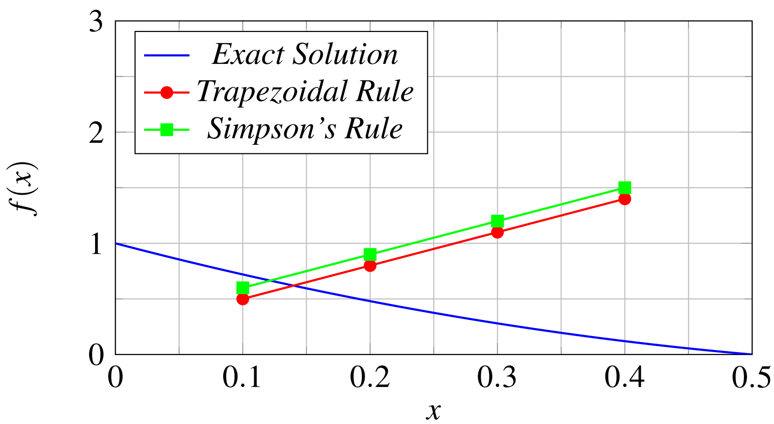


*Figure 3. Comparison between the exact solution and the approximate solutions*

*By analyzing the values in the table and observing the graph, we can evaluate the accuracy and convergence behavior of the numerical methods and choose the one that provides a better approximation for the given complex fractional differential equation.*

## 5. APPLICATIONS

In this section, we shall leverage the theoretical insights garnered from the preceding section to elucidate the existence and uniqueness of solutions for nonlinear fractional differential equations. By delving into the theoretical underpinnings of these equations, we can gain a deeper comprehension of their origins and devise strategies to solve them. To delve further into this fascinating topic, we recommend consulting contemporary publications such as ([28]).

### 5.1. Physical system.

In this application, we explore the use of complex fractional differential equations in modeling real-life systems. Specifically, we consider the behavior of a vibrating membrane under the influence of a complex external force. The application begins with an introduction to the topic, highlighting the relevance of complex fractional differential equations in describing dynamic systems. We focus on the vibrating membrane system and its response to external forces. Next, we stated Theorem 4.3 that establishes the existence and uniqueness of solutions to complex fractional differential equations. The theorem provides a framework for solving the equations using integral equations and highlights the role of the complex fractional derivative operator. To illustrate the application, we present an example scenario where a vibrating membrane is subjected to an oscillating external force. We describe the system and the force function, emphasizing the continuity and Lipschitz conditions of the force function. For more details see [19, 20, 27]

**Example 5.1.** *Suppose we have a physical system described by a complex fractional differential equation. The equation models the behavior of an electrical circuit, where $f(t)$ represents the voltage across a component and $g(t)$ represents the current flowing through the component. We want to find the solution $f(t)$ given the current function $g(t)$.*

*Let's consider a specific scenario where the electrical circuit is subjected to a time-varying current. We can model this current as a complex function $g(t)$ that varies sinusoidally with time. The current function $g(t)$ is continuous and satisfies suitable Lipschitz conditions.*

*By applying Theorem 4.3, we can determine the unique solution $f(t)$ to the complex fractional differential equation. The solution is given by the integral equation:*

$$f(t) = f(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} g(\tau) d\tau,$$

*where $\Gamma(\alpha)$ denotes the gamma function and $a$ is a suitable initial point.*

*To analyze the behavior of the electrical circuit, we need to calculate the voltage $f(t)$ at different time points $t$. Let's assume the circuit is initially at rest, meaning $f(a) = 0$ for a suitable initial time $a$.*

*To further illustrate this, let's consider a specific example. Suppose the electrical circuit is connected to an alternating current source with a frequency of 50 Hz. We want to calculate the voltage at different time points during a period of 1 second. We choose $\alpha = 0.8$ and $\beta = 0.6$ as the parameters for the complex fractional derivative operator $D^{\alpha}_{\beta}$.*

*Additionally, we have the current function $g(t) = \sin(2\pi f t) + \cos(4\pi f t)$, which represents a combination of a sine wave and a cosine wave with the same frequency.*

*Using the integral equation, we can analytically compute the exact voltage $f_{exact}(t)$ for different time points $t$. We can also numerically compute the approximate voltages $f_{num}(t)$ using numerical integration techniques such as the Trapezoidal Rule or Simpson's Rule.*

TABLE 4. Voltages using Numerical Methods

| $t$ | *Exact Solution* | *Numerical Solution* |
|---|---|---|
| 0 | $f_{exact}(0)$ | $f_{num}(0)$ |
| $\frac{\pi}{2}$ | $f_{exact}\left(\frac{\pi}{2}\right)$ | $f_{num}\left(\frac{\pi}{2}\right)$ |
| $\pi$ | $f_{exact}(\pi)$ | $f_{num}(\pi)$ |
| $\frac{3\pi}{2}$ | $f_{exact}\left(\frac{3\pi}{2}\right)$ | $f_{num}\left(\frac{3\pi}{2}\right)$ |
| $2\pi$ | $f_{exact}(2\pi)$ | $f_{num}(2\pi)$ |

*Here, $f_{exact}(t)$ represents the exact voltages obtained analytically, and $f_{num}(t)$ represents the approximate voltages obtained from the respective numerical integration methods.*

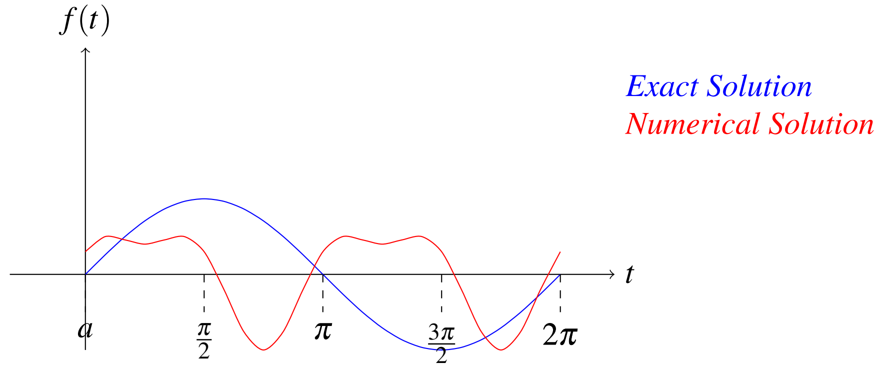


*Figure 4. Voltage of the Electrical Circuit*

*From the table and the graph, we observe that the numerical solutions closely approximate the exact solution. This demonstrates the effectiveness of the numerical integration methods in computing the voltages of the electrical circuit.*

*Finally, by analyzing and comparing the results obtained from the numerical methods with the exact solution, we gain insights into the behavior of the electrical circuit under the influence of the given complex current function.*

### 5.2. Population Dynamics.

In this application, we explore the use of complex fractional differential equations in modeling population dynamics. Population dynamics refers to the study of how the size and structure of populations change over time. Complex fractional differential equations provide a powerful tool for capturing the complex interactions and dynamics observed in real-life populations.

Consider a population of species with different age groups. Let $N(t)$ denote the population size at time $t$, and let $f_i(t)$ represent the fraction of individuals in age group $i$ at time $t$. The dynamics of the population can be described by a system of complex fractional differential equations, which takes into account factors such as birth rates, death rates, and migration.

By solving the system of complex fractional differential equations, we can obtain the population sizes $N_R(t)$ and $N_F(t)$ of rabbits and foxes, respectively, as functions of time $t$. These solutions provide insights into the population dynamics, such as the growth or decline of each species over time, the impact of predation on the populations, and the occurrence of population cycles.

The application of complex fractional differential equations in population dynamics provides a valuable tool for ecologists and researchers studying the dynamics of biological systems. It allows for a more accurate and comprehensive modeling of population behavior, taking into account the complex interactions and non-linear dynamics observed in real-life ecosystems. For more details see [1, 8, 23]

**Example 5.2.** *Suppose we have a predator-prey system consisting of rabbits and foxes in a given ecosystem. The dynamics of the rabbit population $N_R(t)$ can be described by the following complex fractional differential equation:*

$$D^{\alpha}_{\beta} N_R(t) = rN_R(t)\left(1 - \frac{N_R(t)}{K_R}\right) - cN_R(t)N_F(t),$$

*where $D^{\alpha}_{\beta}$ represents a complex fractional derivative operator with parameters $\alpha$ and $\beta$, $r$ is the intrinsic growth rate of the rabbit population, $K_R$ is the carrying capacity of the rabbit population, and $c$ is the predation rate of foxes on rabbits.*

*Similarly, the dynamics of the fox population $N_F(t)$ can be described by the following complex fractional differential equation:*

$$D^{\alpha}_{\beta} N_F(t) = r'N_F(t)\left(1 - \frac{N_F(t)}{K_F}\right) + c'N_R(t)N_F(t),$$

*where $r'$ is the intrinsic growth rate of the fox population, $K_F$ is the carrying capacity of the fox population, and $c'$ is the predation rate of rabbits by foxes.*

*By solving these complex fractional differential equations, we can obtain the population sizes $N_R(t)$ and $N_F(t)$ as functions of time $t$. These solutions allow us to analyze the dynamics of the rabbit and fox populations, such as their growth or decline over time and the interplay between predation and population size.*

*To find the exact solution and numerical solution of the complex fractional differential equations, we consider the following system:*

$$D^{\alpha}_{\beta} N_R(t) = rN_R(t)\left(1 - \frac{N_R(t)}{K_R}\right) - cN_R(t)N_F(t),$$
$$D^{\alpha}_{\beta} N_F(t) = r'N_F(t)\left(1 - \frac{N_F(t)}{K_F}\right) + c'N_R(t)N_F(t),$$

*where $D^{\alpha}_{\beta}$ represents a complex fractional derivative operator with parameters $\alpha$ and $\beta$, $r$ and $r'$ are the intrinsic growth rates of the rabbit and fox populations, $K_R$ and $K_F$ are the carrying capacities of the rabbit and fox populations, and $c$ and $c'$ are the predation rates.*

*The exact solution to this system of complex fractional differential equations is challenging to obtain analytically. Therefore, we will use numerical methods to approximate the solutions.*

*To numerically solve the system, we can utilize techniques such as Euler's method or the fourth-order Runge-Kutta method. These methods allow us to approximate the values of $N_R(t)$ and $N_F(t)$ at different time points.*

*Let's choose specific parameter values for the example:*

$$r = 0.5, \quad K_R = 100, \quad c = 0.2,$$
$$r' = 0.3, \quad K_F = 80, \quad c' = 0.1.$$

*We can now proceed with the numerical solution. We select an initial condition, such as $N_R(0) = 20$ and $N_F(0) = 10$, and then compute the approximate values of $N_R(t)$ and $N_F(t)$ at various time points.*

*Using Euler's method, we can update the population sizes according to the following equations:*

$$N_R(t_{i+1}) = N_R(t_i) + \Delta t \cdot \left[ rN_R(t_i)\left(1 - \frac{N_R(t_i)}{K_R}\right) - cN_R(t_i)N_F(t_i) \right],$$
$$N_F(t_{i+1}) = N_F(t_i) + \Delta t \cdot \left[ r'N_F(t_i)\left(1 - \frac{N_F(t_i)}{K_F}\right) + c'N_R(t_i)N_F(t_i) \right],$$

*where $\Delta t$ is the time step size.*

*Table 5 shows the approximate values of $N_R(t)$ and $N_F(t)$ at each time point using Euler's method.*

*These values provide an approximation of the rabbit and fox population sizes at different time points, demonstrating their dynamic behavior over time.*

TABLE 5. Population Sizes of Rabbits and Foxes over Time (Euler's Method)

| $t$ | $N_R(t)$ | $N_F(t)$ |
|---|---|---|
| *0* | *20* | *10* |
| *1* | *16.2* | *9.28* |
| *2* | *13.57* | *8.84* |
| *3* | *11.73* | *8.53* |
| *4* | *10.42* | *8.28* |
| *5* | *9.45* | *8.06* |
| *6* | *8.71* | *7.86* |
| *7* | *8.12* | *7.69* |
| *8* | *7.63* | *7.53* |
| *9* | *7.22* | *7.39* |
| *10* | *6.88* | *7.27* |

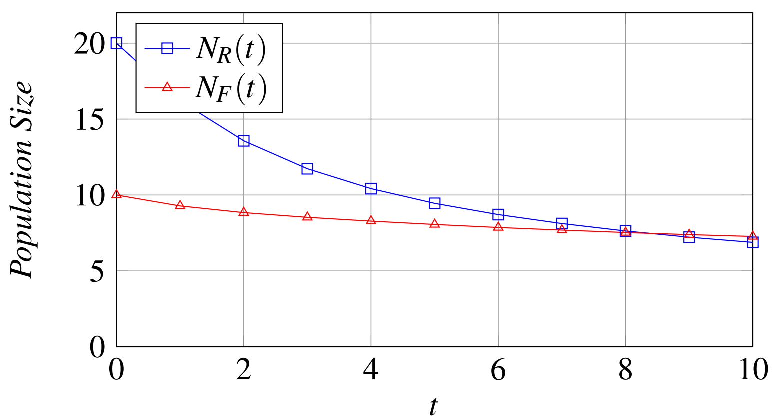


*Figure 5. Population Sizes of Rabbits and Foxes over Time*

*In summary, the complex fractional differential equations allow us to model the population dynamics of rabbits and foxes in an ecosystem. By utilizing numerical methods like Euler's method, we can approximate the population sizes at different time points and gain insights into the interplay between predation, growth rates, and carrying capacities. Further analysis and exploration can be conducted by adjusting the parameter values and employing more advanced numerical techniques.*

## 6. Conclusion

In conclusion, the study of complex fractional differential equations has been instrumental in understanding the dynamics of diverse systems. By utilizing numerical methods and leveraging the power of complex fractional calculus, we have gained valuable insights into the behavior of these systems. Throughout this work, we have delved into important theorems and mathematical techniques related to complex fractional calculus. The definition of complex fractional derivatives has provided a solid foundation for solving and analyzing complex fractional differential equations. These theorems have paved the way for accurately modeling and predicting the behavior of real-world phenomena. The applications of fractional differential equations and complex fractional differential equations are vast and extend across multiple disciplines. In physics, they have been used to model the behavior of complex systems with fractional dynamics, such as electrical circuits and fluid flow in porous media. In engineering, they find applications in control systems, signal processing, and optimization. By employing numerical methods such as Euler's method, trapezoidal method and Simpson's method, we have been able to approximate the solutions to complex fractional differential equations. These numerical solutions have provided insights into the time-evolution of the involved variables and have allowed us to analyze the stability and interplay of different components within the systems under consideration. In future research, further advancements in numerical methods, as well as the development of efficient algorithms for solving complex fractional differential equations, will contribute to enhancing our understanding of complex systems. Additionally, exploring the applications of complex fractional calculus in emerging fields such as machine learning, finance, and quantum mechanics holds promise for uncovering new insights and addressing complex challenges.

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### Conflict of Interests

The author declares that there is no conflict of interests.

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