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NOTES ON HARDY-HILBERT'S INTEGRAL INEQUALITY

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Abstract : New kinds of Hardy-Hilbert's integral inequalities are presented.

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1. Introduction

If $f, g \geq 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ 0 < \int_0^\infty f^p(x)\,dx < \infty$ *and* $0 < \int_0^\infty g^q(x)\,dx < \infty,$ then

$$(1.1)\qquad \int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dxdy \leq \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty g^q(x)\,dx\right)^{1/q},$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Inequality (1.1) is called

Hardy-Hilbert's integral inequality (see [1]) is important in analysis and applications (cf. Mitrinovic et al. [2]). Hardy- et al.[1] gave an inequality similar to (1.1) as :

$$(1.2)\qquad \int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{\max\{x,y\}}\,dx\,dy \ \leq\ pq\left(\int_0^\infty f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty g^q(t)\,dt\right)^{1/q},$$

where the constant factor pq is the best possible.

Other mathematicians present generalizations or new kinds of (1.2) as follows :

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Theorem 1.1 [3]. *If* $\lambda > 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \ f, g \geq 0$ *such that*

$$0 < \int_0^\infty t^{p-1-\lambda} f^p(t)\,dt < \infty,\ \ 0 < \int_0^\infty t^{q-1-\lambda} g^q(t)\,dt < \infty\ ,$$

then one has

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{\max\{x^\lambda, y^\lambda\}}\,dx\,dy \ \leq\ \frac{pq}{\lambda}\left(\int_0^\infty t^{p-1-\lambda} f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty t^{q-1-\lambda} g^q(t)\,dt\right)^{1/q}, \tag{1.3}$$

where the constant factor $\frac{pq}{\lambda}$ *is the best possible.*

Theorem 1.2 [4]. *Suppose* $f, g \geq 0,\ 0 < \int_0^\infty f^2(x)\,dx < \infty,\ 0 < \int_0^\infty g^2(x)\,dx < \infty$.

Then

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x + y + \max\{x, y\}}\,dx\,dy \leq c\left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(x)\,dx\right)^{1/2}, \tag{1.4}$$

where $c = \sqrt{2}\left(\pi - 2\arctan\sqrt{2}\right) \approx 1.7408.$

The Beta function denoted by $B(p,q)$, is defined, by

$$B(p,q) \ = \int_0^1 t^{p-1}(1-t)^{q-1}dt = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,dt\ , \quad p > 0,\ q > 0. \tag{1.5}$$

2. Main Results

Lemma 2.1. *Let* $p > 0,\ q > 0.$ *Then*

$$B(p,q) = \int_1^\infty \frac{x^{-p-q}}{(x-1)^{1-p}}\ dx. \tag{2.1}$$

Proof . The proof follows by putting $x = 1/t$ in (1.5).

The following is our main result

Theorem 2.2. *Assume that* f, g, $h \geq 0$, *h is homogeneous of degree* $\lambda > 0,\ \gamma > 0,$ $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\quad a, b > 0.$ *Then*

(2.2) $$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{h(x, y) \left| a \min\{x, y\} + b \max\{x, y\} \right|^\gamma} dxdy$$

$$\leq \left(C \int_0^\infty t^{(1-\lambda)(p-1)-\gamma} f^p(t)\, dt \right)^{1/p} \left(K \int_0^\infty t^{(1-\lambda)(q-1)-\gamma} g^q(t)\, dt \right)^{1/q},$$

provided the integrals on the R.H.S do exist, where

$$C = b^{-\gamma} C_1 + a^{-\gamma} C_2, \quad K = a^{-\gamma} K_1 + b^{-\gamma} K_2,$$

$$C_1 = \int_0^1 \frac{t^{\lambda-1}}{h(1, t) \left| 1 + at/b \right|^\gamma} dt, \quad C_2 = \int_1^\infty \frac{t^{\lambda-1}}{h(1, t) \left| 1 + bt/a \right|^\gamma} dt, \quad K_1 = \int_0^1 \frac{t^{\lambda-1}}{h(t, 1) \left| 1 + at/b \right|^\gamma} dt,$$

and $$K_2 = \int_1^\infty \frac{t^{\lambda-1}}{h(t, 1) \left| 1 + bt/a \right|^\gamma} dt.$$

Proof.

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{h(x, y) \left| a \min\{x, y\} + b \max\{x, y\} \right|^\gamma} dxdy$$

$$= \int_0^\infty \int_0^\infty \frac{f(x)\, y^{(\lambda-1)\frac{1}{p}}}{x^{(\lambda-1)\frac{1}{q}} h^{\frac{1}{p}}(x, y) \left| a \min\{x, y\} + b \max\{x, y\} \right|^{\frac{\gamma}{p}}}$$

$$\times \frac{f(x)\, x^{(\lambda-1)\frac{1}{q}}}{y^{(\lambda-1)\frac{1}{p}} h^{\frac{1}{q}}(x, y) \left| a \min\{x, y\} + b \max\{x, y\} \right|^{\frac{\gamma}{q}}} dxdy$$

$$\leq \left(\int_0^\infty \int_0^\infty \frac{f^p(x)\, y^{\lambda-1}}{x^{(\lambda-1)\frac{p}{q}} h(x,y) \left| a \min\{x,y\} + b \max\{x,y\} \right|^\gamma} dx\, dy \right)^{1/p} \times$$

$$\left(\int_0^\infty \int_0^\infty \frac{g^q(y)\, x^{\lambda-1}}{y^{(\lambda-1)\frac{q}{p}} h(x,y) \left| a \min\{x,y\} + b \max\{x,y\} \right|^\gamma} dx\, dy \right)^{1/q}$$

$$= M^{1/p} N^{1/q}.$$

We first consider

$$M = \int_0^\infty \int_0^\infty \frac{f^p(x)\, y^{\lambda-1}}{x^{(\lambda-1)\frac{p}{q}} h(x,y) \left| a \min\{x,y\} + b \max\{x,y\} \right|^\gamma} dx\, dy$$

$$= \int_0^\infty x^{(1-\lambda)(p-1)} f^p(x) \, (M_1 + M_2) dx,$$

where, for $x > 0$,

$$M_1 = \int_0^x \frac{y^{\lambda-1}}{h(x,y) \left| a \min\{x,y\} + b \max\{x,y\} \right|^\gamma} dy,$$

$$M_2 = \int_x^\infty \frac{y^{\lambda-1}}{h(x,y) \left| a \min\{x,y\} + b \max\{x,y\} \right|^\gamma} dy.$$

Now,

$$M_1 = \int_0^x \frac{y^{\lambda-1}}{h(x,y) |ay + bx|^\gamma} dy$$

$$= \int_0^x \frac{y^{\lambda-1}}{h(x, xx^{-1}y) |ay + bx|^\gamma} dy$$

$$= \int_0^x \frac{y^{\lambda-1}}{x^\lambda h(1, y/x) |ay + bx|^\gamma} dy$$

$$=b^{-\lambda}\, x^{-\lambda} \int_0^1 \frac{u^{\lambda-1}}{h(1,u)\left|1+au/b\right|^{\gamma}}\, du \qquad (u = y/x).$$

$$= b^{-\gamma} C_1 x^{-\gamma}.$$

Also, via similar steps,

$$M_2 = a^{-\gamma} x^{-\gamma} \int_1^{\infty} \frac{u^{\lambda-1}}{h(1,u)\left|1+bu/a\right|^{\gamma}}\, du$$

$$= a^{-\gamma} C_2 x^{-\gamma},$$

and hence

$$M = C \int_0^{\infty} x^{(1-\lambda)(p-1)-\gamma} f^{\,p}(x)\, dx.$$

Similarly,

$$N = K \int_0^{\infty} y^{(1-\lambda)(q-1)-\gamma} g^{\,q}(x)\, dy.$$

Collecting the above estimates, we obtain

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)\, g(y)}{h(x,y)\left|a \min\{x,y\} + b \max\{x,y\}\right|^{\gamma}}\, dxdy$$

$$\leq \left(C\int_0^{\infty} t^{(1-\lambda)(p-1)-\gamma} f^{\,p}(t)\, dt \right)^{1/p} \left(K\int_0^{\infty} t^{(1-\lambda)(q-1)-\gamma} g^{\,q}(t)\, dt \right)^{1/q}.$$

The proof is complete.

3. Applications

Corollary 3.1. *Assume that* f, g, $h \geq 0$, *h is homogeneous of degree* λ, $0 < \lambda < 1/2$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

(3.1) $$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{|x-y|^\lambda\,|\min\{x,y\}+\max\{x,y\}|^\lambda}\,dxdy$$

$$\leq C\left(\int_0^\infty t^{(1-\lambda)(p-1)-\gamma} f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty t^{(1-\lambda)(q-1)-\gamma} g^q(t)\,dt\right)^{1/q},$$

provided the integrals on the R.H.S do exist, where

$$C = 2B(\lambda, 1-2\lambda).$$

Proof. The proof follows from theorem 2.2 via lemma 2.1 by putting

$$a = b = 1,\ \gamma = \lambda.$$

Corollary 3.2. *Assume that* f, g, $h \geq 0$, *h is homogeneous of degree* λ, $0 < \lambda < 1/2$, $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$. *Then*

(3.2) $$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{(x+y)^{2\lambda}}\,dxdy$$

$$\leq C\left(\int_0^\infty t^{(1-\lambda)(p-1)-\gamma} f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty t^{(1-\lambda)(q-1)-\gamma} g^q(t)\,dt\right)^{1/q},$$

provided the integrals on the R.H.S do exist, where

$$C = B(\lambda, \lambda).$$

Proof. The proof follows from theorem 2.2 by putting

$$a = b = 1,\ \gamma = \lambda.$$

Corollary 3.3. *Assume that* f, g, $h \geq 0$, *h is homogeneous of degree* λ, $1 < \lambda < 2$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

(3.3) $$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{\left|x^\lambda - y^\lambda\right|\left|\min\{x,y\} + \max\{x,y\}\right|^\lambda}\,dxdy$$

$$\leq C\left(\int_0^\infty t^{(1-\lambda)(p-1)-\gamma} f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty t^{(1-\lambda)(q-1)-\gamma} g^q(t)\,dt\right)^{1/q},$$

provided the integrals on the R.H.S do exist, where

$$C < \tfrac{1}{2}B(\lambda/2, 1-\lambda/2) + \tfrac{1}{2}.B(\lambda/4, 1-\lambda/2).$$

Proof. The proof follows from theorem 2.2 by putting

$$h(x,y) = \left|x^\lambda - y^\lambda\right|,\ a = b = 1,\ \gamma = \lambda/2,$$

as follows :

Since $1 < \lambda < 2$, then $\dfrac{1}{1-t^\lambda} < \dfrac{1}{1-t} < \dfrac{1}{(1-t)^{\lambda/2}}$, and hence, we have

$$K_1 = C_1 = \int_0^1 \frac{t^{\lambda-1}}{(1-t^\lambda)(1+t)^{\lambda/2}}\,dt < \int_0^1 \frac{t^{\lambda-1}}{(1-t)^{\lambda/2}(1+t)^{\lambda/2}}\,dt = \int_0^1 \frac{t^{\lambda-1}}{(1-t^2)^{\lambda/2}}\,dt$$

$$= \tfrac{1}{2}\int_0^1 \frac{t^{\lambda/2-1}}{(1-t)^{\lambda/2}}\,d = \tfrac{1}{2}B(\lambda/2, 1-\lambda/2).$$

$$K_2 = C_2 = \int_1^\infty \frac{t^{\lambda-1}}{(t^\lambda - 1)(1+t)^{\lambda/2}}\,dt = \int_0^1 \frac{t^{\lambda/2-1}}{(1-t^\lambda)(1+t)^{\lambda/2}}\,dt < \int_0^1 \frac{t^{\lambda/2-1}}{(1-t)^{\lambda/2}(1+t)^{\lambda/2}}\,dt$$

$$= \int_0^1 \frac{t^{\lambda/2-1}}{(1-t^2)^{\lambda/2}}\,dt = \tfrac{1}{2}B(\lambda/4, 1-\lambda/2).$$

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