



Available online at <http://scik.org>

Adv. Inequal. Appl. 2014, 2014:3

ISSN: 2050-7461

## **n-EXPONENTIAL CONVEXITY FOR FAVARD'S AND BERWALD'S INEQUALITIES AND THEIR APPLICATIONS**

ASIF R. KHAN<sup>1,2,\*</sup>, NAVID LATIF<sup>3</sup>, AND JOSIP PEČARIĆ<sup>2,4</sup>

<sup>1</sup>Department of Mathematical Sciences, University of Karachi, Karachi, Pakistan

<sup>2</sup>Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan

<sup>3</sup>Department of Mathematics, GC University, Faisalabad, Pakistan

<sup>4</sup>Faculty Of Textile Technology, University Of Zagreb, Zagreb, Croatia

Copyright © 2014 A. Khan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

**Abstract.** In this paper, we give generalization of the results given in [5],[6],[7],[9] and [10] by using second order divided differences, also discuss  $n$ -exponential convexity of different positive linear functionals defined on special classes of functions. Some applications in terms of exponential convexity are also given with Cauchy type means in terms of weighted power means.

**Keywords:** Convex functions, Berwald's inequality, Favard's inequality, majorization,  $n$ -exponential convexity, log-convexity, Cauchy means.

**2000 AMS Subject Classification:** 26A51; 26D15

### **1. Introduction and Preliminaries**

Pečarić and Perić (2012) in [11] introduced the notion of  $n$ -exponentially convex function which is in fact generalization of exponentially convex function. In this paper, we use the same notion of  $n$ -exponential convexity and prove it for some important results extracted

---

\*Corresponding author

Received April 14, 2013

from [5],[6],[7],[9] and [10]. These results are basically extension of weighted Favard's and Berwald's inequalities and majorization type results whose details are as under.

Favard (1933) [2] proved the following result. Let  $f$  be a non-negative continuous concave function, not identically zero on  $[a, b] \subset \mathbb{R}$  and  $\phi$  be a convex function on  $[0, 2\tilde{f}] \subset \mathbb{R}$ , where

$$\tilde{f} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Then

$$\frac{1}{2\tilde{f}} \int_0^{2\tilde{f}} \phi(y) dy \geq \frac{1}{b-a} \int_a^b \phi(f(x)) dx.$$

Favard (1933) [2] also proved the following result. Let  $f$  be a non-negative concave function on  $[a, b] \subset \mathbb{R}$ . If  $q > 1$ , then

$$\frac{1}{b-a} \int_a^b f^q(x) dx \leq \frac{2^q}{q+1} \left( \frac{1}{b-a} \int_a^b f(x) dx \right)^q.$$

Some generalizations of the Favard's inequality and its reverse inequality are also given in [4, p.412-413]. Moreover, Berwald (1947) [1] proved the following generalization of Favard's inequality [4, p.413-414]. Let  $f$  be a non-negative continuous concave function, not identically zero on  $[a, b]$  and  $\psi$  be a strictly monotonic continuous function on  $[0, y_0]$ , where  $y_0$  is sufficiently large. If  $\alpha$  is the unique positive root of the equation

$$\frac{1}{\alpha} \int_0^\alpha \psi(y) dy = \frac{1}{b-a} \int_a^b \psi(f(x)) dx,$$

then for every function  $\phi : [0, y_0] \rightarrow \mathbb{R}$  which is convex with respect to  $\psi$  i.e.  $\phi \circ \psi^{-1}$  is convex, we have

$$\frac{1}{\alpha} \int_0^\alpha \phi(y) dy \geq \frac{1}{b-a} \int_a^b \phi(f(x)) dx.$$

Berwald (1947) [1] also proved the following result. If  $f$  is a non-negative concave function on  $[a, b]$ , then for  $0 < r < s$  we have

$$\left[ \frac{s+1}{b-a} \int_a^b f^s(x) dx \right]^{\frac{1}{s}} \leq \left[ \frac{r+1}{b-a} \int_a^b f^r(x) dx \right]^{\frac{1}{r}}.$$

The following two Theorems are generalizations of discrete weighted Favard's and Berwald's Inequalities proved by Latif et al. (2012) in [5].

**Theorem 1.1.**  $\mathbf{w}$ ,  $\mathbf{a}$  and  $\mathbf{b}$  be positive  $n$ -tuples and  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  be a convex function.

Let  $\mathbf{a}/\mathbf{b}$  be a decreasing  $n$ -tuple. If  $\mathbf{a}$  is an increasing  $n$ -tuple, then

$$(A1) \quad \Lambda_1 = \sum_{i=1}^n w_i \varphi\left(\frac{b_i}{\sum_{i=1}^n b_i w_i}\right) - \sum_{i=1}^n w_i \varphi\left(\frac{a_i}{\sum_{i=1}^n a_i w_i}\right) \geq 0.$$

If  $\mathbf{b}$  be a decreasing  $n$ -tuple then reverse inequality holds in (A1).

Let  $\mathbf{a}/\mathbf{b}$  be an increasing  $n$ -tuple. If  $\mathbf{b}$  is an increasing  $n$ -tuple, then

$$(A2) \quad \Lambda_2 = \sum_{i=1}^n w_i \varphi\left(\frac{a_i}{\sum_{i=1}^n a_i w_i}\right) - \sum_{i=1}^n w_i \varphi\left(\frac{b_i}{\sum_{i=1}^n b_i w_i}\right) \geq 0.$$

If  $\mathbf{a}$  be a decreasing  $n$ -tuple then reverse inequality holds in (A2).

If  $f$  is strictly convex function and  $\mathbf{a} \neq \mathbf{b}$ , then the strict inequality holds in (A1) and (A2) and their reverse cases.

**Theorem 1.2.** Let  $\mathbf{w}$ ,  $\mathbf{a}$  and  $\mathbf{b}$  be positive  $n$ -tuples. Suppose  $\psi, \varphi : [0, \infty) \rightarrow \mathbb{R}$  are such that  $\psi$  is a strictly increasing continuous function and  $\varphi$  is a convex function with respect to  $\psi$  i.e.  $\varphi \circ \psi^{-1}$  is convex.

Let  $z_1$  be such that

$$\sum_{i=1}^n w_i \psi(z_1 b_i) = \sum_{i=1}^n w_i \psi(a_i).$$

(1) Let  $\mathbf{a}/\mathbf{b}$  be a decreasing  $n$ -tuple. If  $\mathbf{a}$  is an increasing  $n$ -tuple, then

$$(A3) \quad \Lambda_3 = \sum_{i=1}^n w_i \varphi(z_1 b_i) - \sum_{i=1}^n w_i \varphi(a_i) \geq 0.$$

If  $\mathbf{b}$  is a decreasing  $n$ -tuple, then the reverse inequality holds in (A3).

(2) Let  $\mathbf{a}/\mathbf{b}$  be an increasing  $n$ -tuple. If  $\mathbf{b}$  is an increasing  $n$ -tuple, then

$$(A4) \quad \Lambda_4 = \sum_{i=1}^n w_i \varphi(a_i) - \sum_{i=1}^n w_i \varphi(z_1 b_i) \geq 0.$$

If  $\mathbf{a}$  is a decreasing  $n$ -tuple, then the reverse inequality holds in (A4).

If  $\varphi \circ \psi^{-1}$  is strictly convex function and  $\mathbf{a} \neq z_1 \mathbf{b}$ , then strict inequality holds in (A3) and (A4) and their reverse cases.

The following theorem is valid (see [9], p.32).

**Theorem 1.3.** Let  $\varphi$  be a convex function on an interval  $I \subseteq \mathbb{R}$ ,  $\mathbf{w}$  be a positive  $n$ -tuple and  $\mathbf{a}$ ,

$\mathbf{b} \in I^n$  satisfying

$$\sum_{i=1}^k w_i b_i \leq \sum_{i=1}^k w_i a_i, \quad k = 1, \dots, n-1,$$

and

$$\sum_{i=1}^n w_i b_i = \sum_{i=1}^n w_i a_i.$$

(1) If  $\mathbf{b}$  is decreasing  $n$ -tuple, then

$$(A5) \quad \Lambda_5 = \sum_{i=1}^n w_i \varphi(a_i) - \sum_{i=1}^n w_i \varphi(b_i) \geq 0.$$

(2) If  $\mathbf{a}$  is increasing  $n$ -tuple, then

$$(A6) \quad \Lambda_6 = \sum_{i=1}^n w_i \varphi(b_i) - \sum_{i=1}^n w_i \varphi(a_i) \geq 0.$$

If  $\varphi$  is strictly convex and  $\mathbf{a} \neq \mathbf{b}$ , then (A5) and (A6) are strict.

The following theorem is a slight extension of Theorem 1 in [7] which is proved by Pečarić and Abramovich (1997).

**Theorem 1.4.** Let  $\mathbf{w}$ ,  $\mathbf{a}$  and  $\mathbf{b}$  be positive  $n$ -tuples. Suppose  $\psi, \varphi : [0, \infty) \rightarrow \mathbb{R}$  are such that  $\psi$  is a strictly increasing function and  $\varphi$  is a convex function with respect to  $\psi$  i.e.,  $\varphi \circ \psi^{-1}$  is convex. Suppose also that

$$\sum_{i=1}^k w_i \psi(b_i) \leq \sum_{i=1}^k w_i \psi(a_i), \quad k = 1, \dots, n-1,$$

and

$$\sum_{i=1}^n w_i \psi(b_i) = \sum_{i=1}^n w_i \psi(a_i).$$

(1) If  $\mathbf{b}$  is a decreasing  $n$ -tuple, then

$$(A7) \quad \Lambda_7 = \sum_{i=1}^n w_i \varphi(a_i) - \sum_{i=1}^n w_i \varphi(b_i) \geq 0.$$

(2) If  $\mathbf{a}$  is an increasing  $n$ -tuple, then

$$(A8) \quad \Lambda_8 = \sum_{i=1}^n w_i \varphi(b_i) - \sum_{i=1}^n w_i \varphi(a_i) \geq 0.$$

If  $\varphi \circ \psi^{-1}$  is strictly convex and  $\mathbf{a} \neq \mathbf{b}$ , then (A7) and (A8) are strict.

The following theorem is an extension of Theorem 3 in [10] which is proved by Pečarić and Abramovich (1997).

**Theorem 1.5.** Let  $w$  be a weight function on  $[a, b]$  and let  $f$  and  $g$  be positive functions on  $[a, b]$ . Suppose  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  is a convex function.

(1) Let  $f/g$  be a decreasing function on  $[a, b]$ . If  $f$  is an increasing function on  $[a, b]$ , then

$$(A9) \quad \Lambda_9 = \int_a^b \varphi \left( \frac{g(t)}{\int_a^b g(t) w(t) dt} \right) w(t) dt - \int_a^b \varphi \left( \frac{f(t)}{\int_a^b f(t) w(t) dt} \right) w(t) dt \geq 0.$$

If  $g$  is a decreasing function on  $[a, b]$ , then the reverse inequality holds in (A9).

(2) Let  $f/g$  be an increasing function on  $[a, b]$ . If  $g$  is an increasing function on  $[a, b]$ , then

$$(A10) \quad \Lambda_{10} = \int_a^b \varphi \left( \frac{f(t)}{\int_a^b f(t) w(t) dt} \right) w(t) dt - \int_a^b \varphi \left( \frac{g(t)}{\int_a^b g(t) w(t) dt} \right) w(t) dt \geq 0.$$

If  $f$  is a decreasing function on  $[a, b]$ , then the reverse inequality holds in (A10).

If  $\varphi$  is strictly convex function and  $f \neq g$  (a.e.), then the strict inequality holds in (A9) and (A10) and their reverse cases.

The following theorem is a slight extension of Theorem 2 in [10] which is proved by Pečarić and Abramovich (1997).

**Theorem 1.6.** Let  $w$  be a weight function on  $[a, b]$  and let  $f$  and  $g$  be positive functions on  $[a, b]$ . Suppose  $\varphi, \psi : [0, \infty) \rightarrow \mathbb{R}$  are such that  $\psi$  is a strictly increasing function and  $\varphi$  is a convex function with respect to  $\psi$  i.e.,  $\varphi \circ \psi^{-1}$  is convex. Suppose also that

$$\int_a^x \psi(f(t)) w(t) dt \leq \int_a^x \psi(g(t)) w(t) dt, \quad x \in [a, b], \quad \text{and}$$

$$\int_a^b \psi(f(t)) w(t) dt = \int_a^b \psi(g(t)) w(t) dt.$$

(1) If  $f$  is a decreasing function on  $[a, b]$ , then

$$(A11) \quad \Lambda_{11} = \int_a^b \varphi(g(t)) w(t) dt - \int_a^b \varphi(f(t)) w(t) dt \geq 0.$$

(2) If  $g$  is an increasing function on  $[a, b]$ , then

$$(A12) \quad \Lambda_{12} = \int_a^b \varphi(f(t)) w(t) dt - \int_a^b \varphi(g(t)) w(t) dt \geq 0.$$

If  $\varphi \circ \psi^{-1}$  is strictly convex function and  $f \neq g$  (a. e.), then the strict inequality holds in (A11) and (A12).

The following theorem is a slight extension of Lemma 2 in [7] which is proved by Maligranda et al. (1995).

**Theorem 1.7.** Let  $w$  be a weight function on  $[a, b]$  and let  $f$  and  $g$  be positive functions on  $[a, b]$ . Suppose that  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  is a convex function and that

$$\int_a^x f(t) w(t) dt \leq \int_a^x g(t) w(t) dt, \quad x \in [a, b] \quad \text{and}$$

$$\int_a^b f(t) w(t) dt = \int_a^b g(t) w(t) dt.$$

(1) If  $f$  is a decreasing function on  $[a, b]$ , then

$$(A13) \quad \Lambda_{13} = \int_a^b \varphi(g(t)) w(t) dt - \int_a^b \varphi(f(t)) w(t) dt \geq 0.$$

(2) If  $g$  is an increasing function on  $[a, b]$ , then

$$(A14) \quad \Lambda_{14} = \int_a^b \varphi(f(t)) w(t) dt - \int_a^b \varphi(g(t)) w(t) dt \geq 0.$$

If  $\varphi$  is strictly convex function and  $f \neq g$  (a. e.), then (A13) and (A14) are strict.

**Theorem 1.8.** [7] Let  $w$  be a weight function on  $[a, b]$  and let  $f$  and  $g$  be positive functions on  $[a, b]$ . Suppose  $\varphi, \psi : [0, \infty) \rightarrow \mathbb{R}$  are such that  $\psi$  is a strictly increasing continuous function and  $\varphi$  is a convex function with respect to  $\psi$  i.e.  $\varphi \circ \psi^{-1}$  is convex.

Let  $z_1$  be such that

$$\int_a^b \psi(z_1 g(t)) w(t) dt = \int_a^b \psi(f(t)) w(t) dt.$$

(1) Let  $f/g$  be a decreasing function on  $[a, b]$ . If  $f$  is an increasing function on  $[a, b]$ , then

$$(A15) \quad \Lambda_{15} = \int_a^b \varphi(z_1 g(t)) w(t) dt - \int_a^b \varphi(f(t)) w(t) dt \geq 0.$$

If  $g$  is a decreasing function on  $[a, b]$ , then the reverse inequality holds in (A15).

(2) Let  $f/g$  be an increasing function on  $[a, b]$ . If  $g$  is an increasing function on  $[a, b]$ , then

$$(A16) \quad \Lambda_{16} = \int_a^b \varphi(f(t)) w(t) dt - \int_a^b \varphi(z_1 g(t)) w(t) dt \geq 0.$$

If  $f$  is a decreasing function on  $[a, b]$ , then the reverse inequality holds in (A16).

If  $\varphi \circ \psi^{-1}$  is strictly convex function and  $f \neq z_1 g$  (a.e.), then the strict inequality holds in (A15) and (A16) and their reverse cases.

This paper is divided into four sections. In “Introduction and Preliminaries” section, we have extracted some important results from [5],[6],[7],[9] and [10]. With the help of these results we have defined positive linear functionals  $\Lambda_1, \dots, \Lambda_{16}$  in  $A_1, \dots, A_{16}$  resp. In “ $n$ -Exponential Convexity” section, we give some definitions from [11] and we prove  $n$ -exponential convexity of the functionals by using different classes defined in  $D_1, D_2, D_3, \acute{D}_1, \acute{D}_2, \acute{D}_3$  etc. In third section, we discuss Cauchy means for a few linear functionals. The last section is of “Examples”, in this section we use different classes of functions and prove exponential convexity of linear functionals for these classes and also construct some means in terms of weighted power means.

## 2. n-Exponential Convexity

The following definitions and results are extracted from [11]. Throughout this section  $J$  is an interval of  $\mathbb{R}$  and  $n \in \mathbb{N}$ .

**Definition 2.1.** A function  $\phi : J \rightarrow \mathbb{R}$  is  $n$ -exponentially convex in the Jensen sense on the interval  $J$  if

$$\sum_{k,l=1}^n \alpha_k \alpha_l \phi \left( \frac{x_k + x_l}{2} \right) \geq 0$$

holds for  $\alpha_k \in \mathbb{R}$  and  $x_k \in J; k = 1, 2, \dots, n$ .

A function  $\phi : J \rightarrow \mathbb{R}$  is  $n$ -exponentially convex if it is  $n$ -exponentially convex in the Jensen sense and continuous on  $J$ .

**Remark 2.2.** From the definition it is clear that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also,  $n$ -exponentially convex functions in the Jensen sense are  $m$ -exponentially convex in the Jensen sense for every  $m \in \mathbb{N}, m \leq n$ .

**Proposition 2.3.** If  $\phi : J \rightarrow \mathbb{R}$  is an  $n$ -exponentially convex function in the Jensen sense on  $J$ , then the matrix  $\left[ \phi \left( \frac{x_k + x_l}{2} \right) \right]_{k,l=1}^m$  is a positive semi-definite matrix for all  $m \in \mathbb{N}, m \leq n$ . Particularly,

$$\det \left[ \phi \left( \frac{x_k + x_l}{2} \right) \right]_{k,l=1}^m \geq 0$$

for all  $m \in \mathbb{N}$ ,  $m \leq n$ .

**Definition 2.4.** A function  $\phi : J \rightarrow \mathbb{R}$  is exponentially convex in the Jensen sense on  $J$  if it is  $n$ -exponentially convex in the Jensen sense for all  $n \in \mathbb{N}$ . A function  $\phi : J \rightarrow \mathbb{R}$  is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

**Definition 2.5.** A function  $\phi : J \rightarrow \mathbb{R}^+$ , is said to be log-convex if  $\log \phi$  is convex, or equivalently if for all  $x, y \in J$  and all  $\lambda \in [0, 1]$ ,

$$\phi(\lambda x + (1 - \lambda)y) \leq \phi(x)^\lambda \phi(y)^{1-\lambda}.$$

**Proposition 2.6.** If  $\phi : J \rightarrow \mathbb{R}^+$  is exponentially convex function, then  $\phi$  is a log-convex function.

**Remark 2.7.** It is easy to show that  $\phi : J \rightarrow \mathbb{R}^+$  is log-convex in the Jensen sense if and only if

$$\alpha^2 \phi(x) + 2\alpha\beta\phi\left(\frac{x+y}{2}\right) + \beta^2 \phi(y) \geq 0$$

holds for every  $\alpha, \beta \in \mathbb{R}$  and  $x, y \in [a, b]$ . It follows that a function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense. Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

**Remark 2.8.** If convex functions in Jensen sense are continuous then these are convex and this is also true for log i.e., if log-convex functions in Jensen sense are continuous then these are log-convex.

**Definition 2.9.** [12, p.2] A function  $\psi$  is convex on an interval  $J \subseteq \mathbb{R}$ , if

$$(x_3 - x_2) \psi(x_1) + (x_1 - x_3) \psi(x_2) + (x_2 - x_1) \psi(x_3) \geq 0$$

holds for every  $x_1 < x_2 < x_3$ ;  $x_1, x_2, x_3 \in J$ .

Let  $f$  be a real-valued function defined on  $[a, b]$ , a second order divided difference of  $f$  at distinct points  $z_0, z_1, z_2 \in [a, b]$  is defined (as in [12, p.14]) recursively by

$$\begin{aligned} [z_i; f] &= f(z_i), \quad \text{for } i = 0, 1, 2; \\ [z_i, z_{i+1}; f] &= \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i}, \quad \text{for } i = 0, 1; \end{aligned}$$



and

$$[z_0, z_1, z_2; f] = \frac{[z_1, z_2; f] - [z_0, z_1; f]}{z_2 - z_0}.$$

The value  $[z_0, z_1, z_2; f]$  is independent of the order of the points  $z_0, z_1$ , and  $z_2$ . By taking limits this definition may be extended to include the cases in which any two or all three points coincide as follows:  $\forall z_0, z_1, z_2 \in [a, b]$

$$[z_0, z_0, z_2; f] = \lim_{z_1 \rightarrow z_0} [z_0, z_1, z_2; f] = \frac{f(z_2) - f(z_0) - f'(z_0)(z_2 - z_0)}{(z_2 - z_0)^2}, \quad z_2 \neq z_0$$

given that  $f'$  exist on  $[a, b]$  and

$$[z_0, z_0, z_0; f] = \lim_{z_i \rightarrow z_0} [z_0, z_1, z_2; f] = \frac{f''(z_0)}{2} \text{ for } i = 1, 2$$

provided that  $f''$  exist on  $[a, b]$ .

Let us define some classes to be used in the following theorem and let us denote Domain of  $f_t$  by  $D(f_t)$  where  $D(f_t)$  varies from functional to functional.

For an interval  $J \subseteq \mathbb{R}$ .

$D_1 = \{f_t : t \in J\}$  be a class of functions such that the function  $t \mapsto [z_0, z_1, z_2; f_t]$  is  $n$ -exponentially convex in the Jensen sense on  $J$  for every three mutually distinct points  $z_0, z_1, z_2 \in D(f_t)$ .

$D_2 = \{f_t : t \in J\}$  be a class of differentiable functions such that the function  $t \mapsto [z_0, z_0, z_2; f_t]$  is  $n$ -exponentially convex in the Jensen sense on  $J$  for any two distinct points  $z_0, z_2 \in D(f_t)$ .

$D_3 = \{f_t : t \in J\}$  be a class of twice differentiable functions such that the function  $t \mapsto [z_0, z_0, z_0; f_t]$  is  $n$ -exponentially convex in the Jensen sense on  $J$  for any point  $z_0 \in D(f_t)$ .

We use an idea from [3] to give an elegant method of producing an  $n$ -exponentially convex functions and exponentially convex functions applying the functionals  $\Lambda_k$ ,  $k = 1, \dots, 16$  on a given family with the same property.

**Theorem 2.10.** *Let  $\Lambda_k$  be linear functionals for  $k = 1, 2, 5, 6, 9, 10, 13, 14$  as defined in (A1), (A2), (A5), (A6), (A9),*

*Let  $J$  be an interval in  $\mathbb{R}$  and  $f_t \in D_j$ ;  $j = 1, 2, 3$ ;  $t \in J$ . Then the following statements are valid for  $\Lambda_k$ ;  $k = 1, 2, 5, 6, 9, 10, 13, 14$ :*

- (a) *The function  $t \mapsto \Lambda_k(f_t)$  is an  $n$ -exponentially convex function in the Jensen sense on  $J$ .*
- (b) *If the function  $t \mapsto \Lambda_k(f_t)$  is continuous on  $J$ , then the function  $t \mapsto \Lambda_k(f_t)$  is an  $n$ -exponentially convex function on  $J$ .*

**Proof.**

(a) Fix  $k = 1, 2, 5, 6, 9, 10, 13, 14$ .

Let us define the function

$$\Omega(z) = \sum_{l,m=1}^n b_l b_m f_{\frac{t_l+t_m}{2}}(z),$$

where  $\frac{t_l+t_m}{2} \in J$ ;  $b_l \in \mathbb{R}$ ;  $l = 1, 2, \dots, n$ .

Since the function  $t \rightarrow [z_0, z_1, z_2; f_t]$  is  $n$ -exponentially convex in the Jensen sense, we have

$$[z_0, z_1, z_2; \Omega] = \sum_{l,m=1}^n b_l b_m [z_0, z_1, z_2; f_{\frac{t_l+t_m}{2}}] \geq 0,$$

which implies that  $\Omega$  is convex function on  $D(f_t)$  and therefore we have  $\Lambda_k(\Omega) \geq 0$ .

Hence

$$\sum_{l,m=1}^n b_k b_l \Lambda_k(f_{\frac{t_l+t_m}{2}}) \geq 0.$$

We conclude that the function  $t \rightarrow \Lambda_k(f_t)$  is an  $n$ -exponentially convex function on  $J$  in Jensen sense.

(b) This part is easily followed by definition of  $n$ -exponentially convex function.

As a consequence of the above theorem we can give the following corollaries for different classes of functions.

$\bar{D}_1 = \{f_t : t \in J\}$  be a class of functions such that the function  $t \mapsto [z_0, z_1, z_2; f_t]$  is an exponentially convex in the Jensen sense on  $J$  for every three mutually distinct points  $z_0, z_1, z_2 \in D(f_t)$ .

$\bar{D}_2 = \{f_t : t \in J\}$  be a class of differentiable functions such that the function  $t \mapsto [z_0, z_0, z_2; f_t]$  is an exponentially convex in the Jensen sense on  $J$  for any two distinct points  $z_0, z_2 \in D(f_t)$ .

$\bar{D}_3 = \{f_t : t \in J\}$  be a class of twice differentiable functions such that the function  $t \mapsto [z_0, z_0, z_0; f_t]$  is an exponentially convex in the Jensen sense on  $J$  for any point  $z_0 \in D(f_t)$ .

**Corollary 2.11.** *Let  $\Lambda_k$  be linear functionals for  $k = 1, 2, 5, 6, 9, 10, 13, 14$  as defined in (A1), (A2), (A5), (A6), (A9) Let  $J$  be an interval in  $\mathbb{R}$  and  $f_t \in \bar{D}_j$ ;  $j = 1, 2, 3$ ;  $t \in J$ . Then the following statements are valid for  $\Lambda_k$ ;  $k = 1, 2, 5, 6, 9, 10, 13, 14$ :*

(a) *The function  $t \mapsto \Lambda_k(f_t)$  is an exponentially convex function in the Jensen sense on  $J$ .*

- (b) *If the function  $t \mapsto \Lambda_k(f_t)$  is continuous on  $J$ , then the function  $t \mapsto \Lambda_k(f_t)$  is an exponentially convex function on  $J$ .*

**Proof.** Proof follows directly from Theorem 2.10 and Proposition 2.3.

$\widehat{D}_1 = \{f_t : t \in J\}$  be a class of functions such that the function  $t \mapsto [z_0, z_1, z_2; f_t]$  is 2-exponentially convex in the Jensen sense on  $J$  for every three mutually distinct points  $z_0, z_1, z_2 \in D(f_t)$ .

$\widehat{D}_2 = \{f_t : t \in J\}$  be a class of differentiable functions such that the function  $t \mapsto [z_0, z_0, z_2; f_t]$  is 2-exponentially convex in the Jensen sense on  $J$  for any two distinct points  $z_0, z_2 \in D(f_t)$ .

$\widehat{D}_3 = \{f_t : t \in J\}$  be a class of twice differentiable functions such that the function  $t \mapsto [z_0, z_0, z_0; f_t]$  is 2-exponentially convex in the Jensen sense on  $J$  for any point  $z_0 \in D(f_t)$ .

**Corollary 2.12.** *Let  $\Lambda_k$  be linear functionals for  $k = 1, 2, 5, 6, 9, 10, 13, 14$  as defined in (A1), (A2), (A5), (A6), (A9) Let  $J$  be an interval in  $\mathbb{R}$  and  $f_t \in \widehat{D}_j$ ;  $j = 1, 2, 3$ ;  $t \in J$ . Then the following statements are valid for  $\Lambda_k$ ;  $k = 1, 2, 5, 6, 9, 10, 13, 14$ :*

- (a) *For  $t_1, \dots, t_m \in I$ , the matrix  $\left[ \Lambda_k \left( f_{\frac{t_i+t_j}{2}} \right) \right]_{i,j=1}^m$  is a positive semi-definite for all  $m \in \mathbb{N}, m \leq n$ . Particularly,*

$$\det \left[ \Lambda_k \left( f_{\frac{t_i+t_j}{2}} \right) \right]_{i,j=1}^m \geq 0$$

*for all  $m \in \mathbb{N}, m \leq n$ .*

- (b) *If the function  $t \mapsto \Lambda_k(f_t)$  is strictly positive continuous on  $J$ , then it is 2-exponentially convex on  $J$  and thus log convex function.*

- (c) *If the function  $t \mapsto \Lambda_k(f_t)$  is strictly positive and differentiable on  $J$ , then for every  $s, t, u, v \in J$ , such that  $s \leq u$  and  $t \leq v$ , we have*

$$(1) \quad \mu_{s,t}(\Lambda_k, \widehat{D}_j) \leq \mu_{u,v}(\Lambda_k, \widehat{D}_j)$$

*where*

$$(2) \quad \mu_{s,t}(\Lambda_k, \widehat{D}_j) = \begin{cases} \left( \frac{\Lambda_k(f_s)}{\Lambda_k(f_t)} \right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp \left( \frac{\frac{d}{ds} \Lambda_k(f_s)}{\Lambda_k(f_s)} \right), & s = t \end{cases}$$

*for  $f_s, f_t \in \widehat{D}_j$  for  $j = 1, 2, 3$ .*

**Proof.**

- (a) Direct consequence of Proposition 2.3.
- (b) It follows directly from Theorem 2.10 and Remark 2.7.
- (c) From the definition of convex function  $\phi$ , we have the following inequality [12, p.2]

$$(3) \quad \frac{\phi(s) - \phi(t)}{s - t} \leq \frac{\phi(u) - \phi(v)}{u - v},$$

$\forall s, t, u, v \in J$  such that  $s \leq u, t \leq v, s \neq t, u \neq v$ .

Since by (b),  $\Lambda_k(f_s)$  is log-convex, so set  $\phi(x) = \log \Lambda_k(f_x)$  in (3) we have

$$(4) \quad \frac{\log \Lambda_k(f_s) - \log \Lambda_k(f_t)}{s - t} \leq \frac{\log \Lambda_k(f_u) - \log \Lambda_k(f_v)}{u - v}$$

for  $s \leq u, t \leq v, s \neq t, u \neq v$ , which equivalent to (2). The cases for  $s = t$ , and / or  $u = v$  are easily followed from (4) by taking respective limits.

$\mathcal{D}_1 = \{f_t : t \in J\}$  be a class of functions such that the function

$t \mapsto [z_0, z_1, z_2; f_t \circ \psi^{-1}]$  is  $n$ -exponentially convex in the Jensen sense on  $J$  for every three mutually distinct points  $z_0, z_1, z_2 \in [0, \infty)$  where the function  $\psi$  is strictly increasing (and continuous also for functionals  $\Lambda_k, k = 3, 4, 11, 12$ ).

$\mathcal{D}_2 = \{f_t : t \in J\}$  be a class of differentiable functions such that the function

$t \mapsto [z_0, z_0, z_2; f_t \circ \psi^{-1}]$  is  $n$ -exponentially convex in the Jensen sense on  $J$  for any two distinct points  $z_0, z_2 \in [0, \infty)$  where the function  $\psi$  is strictly increasing (and continuous also for functionals  $\Lambda_k, k = 3, 4, 11, 12$ ).

$\mathcal{D}_3 = \{f_t : t \in J\}$  be a class of twice differentiable functions such that the function  $t \mapsto [z_0, z_0, z_0; f_t \circ \psi^{-1}]$  is  $n$ -exponentially convex in the Jensen sense on  $J$  for any point  $z_0 \in [0, \infty)$  where the function  $\psi$  is strictly increasing (and continuous also for functionals  $\Lambda_k, k = 3, 4, 11, 12$ ).

**Theorem 2.13.** *Let  $\Lambda_k$  be linear functionals for  $k = 3, 4, 7, 8, 11, 12, 15, 16$  as defined in (A3), (A4), (A7), (A8), (A11). Let  $J$  be an interval in  $\mathbb{R}$  and  $f_t \in \mathcal{D}_j; j = 1, 2, 3; t \in J$ . Then the following statements are valid for  $\Lambda_k; k = 3, 4, 7, 8, 11, 12, 15, 16$ :*

- (a) *The function  $t \mapsto \Lambda_k(f_t)$  is an  $n$ -exponentially convex function in the Jensen sense on  $J$ .*
- (b) *If the function  $t \mapsto \Lambda_k(f_t)$  is continuous on  $J$ , then the function  $t \mapsto \Lambda_k(f_t)$  is an  $n$ -exponentially convex function on  $J$ .*

**Proof.**

(a) Fix  $k = 3, 4, 7, 8, 11, 12, 15, 16$ .

Let us define the function

$$\Omega(z) = \sum_{l,m=1}^n b_l b_m f_{\frac{t_l+t_m}{2}}(z),$$

where  $\frac{t_l+t_m}{2} \in J$ ;  $b_l \in \mathbb{R}$ ;  $l = 1, 2, \dots, n$ .

which implies that

$$\Omega \circ \psi^{-1}(z) = \sum_{l,m=1}^n b_l b_m f_{\frac{t_l+t_m}{2}} \circ \psi^{-1}(z),$$

Since the function  $t \rightarrow [z_0, z_1, z_2; f_t \circ \psi^{-1}]$  is  $n$ -exponentially convex in the Jensen sense, we have

$$[z_0, z_1, z_2; \Omega \circ \psi^{-1}] = \sum_{l,m=1}^n b_l b_m [z_0, z_1, z_2; f_{\frac{t_l+t_m}{2}} \circ \psi^{-1}] \geq 0,$$

which implies that  $\Omega \circ \psi^{-1}$  is convex function on  $[0, \infty)$  and therefore we have

$$\Lambda_k(\Omega) \geq 0.$$

Hence

$$\sum_{l,m=1}^n b_k b_l \Lambda_k(f_{\frac{t_l+t_m}{2}}) \geq 0.$$

We conclude that the function  $t \rightarrow \Lambda_k(f_t)$  is an  $n$ -exponentially convex function on  $J$  in Jensen sense.

(b) This part is easily followed by definition of  $n$ -exponentially convex function.

As a consequence of the above theorem we can give the following corollaries for different classes of functions.

$\dot{D}_1 = \{f_t : t \in J\}$  be a class of functions such that the function

$t \mapsto [z_0, z_1, z_2; f_t \circ \psi^{-1}]$  is an exponentially convex in the Jensen sense on  $J$  for every three mutually distinct points  $z_0, z_1, z_2 \in [0, \infty)$  where the function  $\psi$  is strictly increasing (and continuous also for functionals  $\Lambda_k$ ,  $k = 3, 4, 11, 12$ ).

$\dot{D}_2 = \{f_t : t \in J\}$  be a class of differentiable functions such that the function

$t \mapsto [z_0, z_0, z_2; f_t \circ \psi^{-1}]$  is an exponentially convex in the Jensen sense on  $J$  for any two distinct points  $z_0, z_2 \in [0, \infty)$  where the function  $\psi$  is strictly increasing (and continuous also for functionals  $\Lambda_k$ ,  $k = 3, 4, 11, 12$ ).

$\tilde{D}_3 = \{f_t : t \in J\}$  be a class of twice differentiable functions such that the function  $t \mapsto [z_0, z_0, z_0; f_t \circ \psi^{-1}]$  is an exponentially convex in the Jensen sense on  $J$  for any point  $z_0 \in [0, \infty)$  where the function  $\psi$  is strictly increasing (and continuous also for functionals  $\Lambda_k$ ,  $k = 3, 4, 11, 12$ ).

**Corollary 2.14.** *Let  $\Lambda_k$  be linear functionals for  $k = 3, 4, 7, 8, 11, 12, 15, 16$  as defined in (A3), (A4), (A7), (A8), (A11). Let  $J$  be an interval in  $\mathbb{R}$  and  $f_t \in \tilde{D}_j$ ;  $j = 1, 2, 3$ ;  $t \in J$ . Then the following statements are valid for  $\Lambda_k$ ;  $k = 3, 4, 7, 8, 11, 12, 15, 16$ :*

- (a) *The function  $t \mapsto \Lambda_k(f_t)$  is an exponentially convex function in the Jensen sense on  $J$ .*
- (b) *If the function  $t \mapsto \Lambda_k(f_t)$  is continuous on  $J$ , then the function  $t \mapsto \Lambda_k(f_t)$  is an exponentially convex function on  $J$ .*

**Proof.** Proof follows directly from Theorem 2.13 and Proposition 2.3.

$\tilde{D}_1 = \{f_t : t \in J\}$  be a class of functions such that the function  $t \mapsto [z_0, z_1, z_2; f_t \circ \psi^{-1}]$  is 2-exponentially convex in the Jensen sense on  $J$  for every three mutually distinct points  $z_0, z_1, z_2 \in [0, \infty)$  where the function  $\psi$  is strictly increasing (and continuous also for functionals  $\Lambda_k$ ,  $k = 3, 4, 11, 12$ ).

$\tilde{D}_2 = \{f_t : t \in J\}$  be a class of differentiable functions such that the function  $t \mapsto [z_0, z_0, z_2; f_t \circ \psi^{-1}]$  is 2-exponentially convex in the Jensen sense on  $J$  for any two distinct points  $z_0, z_2 \in [0, \infty)$  where the function  $\psi$  is strictly increasing (and continuous also for functionals  $\Lambda_k$ ,  $k = 3, 4, 11, 12$ ).

$\tilde{D}_3 = \{f_t : t \in J\}$  be a class of twice differentiable functions such that the function  $t \mapsto [z_0, z_0, z_0; f_t \circ \psi^{-1}]$  is 2-exponentially convex in the Jensen sense on  $J$  for any point  $z_0 \in [0, \infty)$  where the function  $\psi$  is strictly increasing (and continuous also for functionals  $\Lambda_k$ ,  $k = 3, 4, 11, 12$ ).

**Corollary 2.15.** *Let  $\Lambda_k$  be linear functionals for  $k = 3, 4, 7, 8, 11, 12, 15, 16$  as defined in (A3), (A4), (A7), (A8), (A11). Let  $J$  be an interval in  $\mathbb{R}$  and  $f_t \in \tilde{D}_j$ ,  $j = 1, 2, 3$ ,  $t \in J$ . Then the following statements are valid for  $\Lambda_k$ ;  $k = 5, 6, 7, 8, 13, 14, 15, 16$ :*

- (a) *For  $t_1, \dots, t_m \in I$ , the matrix  $\left[ \Lambda_k \left( f_{\frac{t_i+t_j}{2}} \right) \right]_{i,j=1}^m$  is a positive semi-definite for all  $m \in \mathbb{N}$ ,  $m \leq n$ . Particularly,*

$$\det \left[ \Lambda_k \left( f_{\frac{t_i+t_j}{2}} \right) \right]_{i,j=1}^m \geq 0$$

for all  $m \in \mathbb{N}$ ,  $m \leq n$ .

(b) If the function  $t \mapsto \Lambda_k(f_t)$  is strictly positive continuous on  $J$ , then it is 2-exponentially convex on  $J$ , and thus log convex function.

(c) If the function  $t \mapsto \Lambda_k(f_t)$  is strictly positive and differentiable on  $J$ , then for every  $s, t, u, v \in J$ , such that  $s \leq u$  and  $t \leq v$ , we have

$$(5) \quad \mu_{s,t}(\Lambda_k, \tilde{D}_j) \leq \mu_{u,v}(\Lambda_k, \tilde{D}_j)$$

where

$$(6) \quad \mu_{s,t}(\Lambda_k, \tilde{D}_j) = \begin{cases} \left( \frac{\Lambda_k(f_s)}{\Lambda_k(f_t)} \right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp \left( \frac{\frac{d}{ds} \Lambda_k(f_s)}{\Lambda_k(f_s)} \right), & s = t \end{cases}$$

for  $f_s, f_t \in \tilde{D}_j$  for  $j = 1, 2, 3$ .

**Proof.**

(a) Direct consequence of Proposition 2.3.

(b) It follows directly form Theorem 2.13 and Remark 2.7.

(c) Similar to the proof of part-(c) of Corollary 2.12.

### 3. Cauchy Means

For the sake of completion we only state here two theorems which will be used in our examples. For the idea of the proof of the theorems see [11].

**Theorem 3.1.** Let  $\Lambda_k$  be linear functionals for  $k = 5, 6$  as defined in (A5), (A6) and  $\varphi \in C^2(I)$ , where  $I$  is a compact interval in  $\mathbb{R}$ . Then there exist  $\xi_k \in I$  such that

$$\Lambda_k(\varphi) = \frac{\varphi''(\xi_k)}{2} \Lambda_k(\varphi_0), \text{ where } \varphi_0(x) = x^2; \quad k = 5, 6.$$

**Theorem 3.2.** Let  $\Lambda_k$  be linear functionals for  $k = 5, 6$  as defined in (A5), (A6) and  $\varphi, \theta \in C^2(I)$ , where  $I$  is a compact interval in  $\mathbb{R}$ . Then there exist  $\xi_k \in I$  such that

$$\frac{\Lambda_k(\varphi)}{\Lambda_k(\theta)} = \frac{\varphi''(\xi_k)}{\theta''(\xi_k)}; \quad k = 5, 6.$$

provided that the denominator of the left-hand side is non-zero.

**Remark 3.3.** If the inverse of  $\frac{\varphi''}{\theta''}$  exists, then from the above mean value theorems we can give generalized means

$$(7) \quad \xi_k = \left( \frac{\varphi''}{\theta''} \right)^{-1} \left( \frac{\Lambda_k(\varphi)}{\Lambda_k(\theta)} \right); k = 5, 6.$$

**Remark 3.4.** For the functionals  $\Lambda_k; k = 1, 2, 3, 4, 7, 8, \dots, 16$  (as defined in (A1), (A2), (A3), (A4), (A7)(A8),  $\dots$ , (A16)) similar results as given in Theorems 3.1 and 3.2 can be find in [5] and [6]. In the similar way, we can use Remark 3.3 for these functionals as well.

## 4. Examples

In this section we will vary on choice of a family  $D = \{f_t : t \in J\}$  in order to construct different examples of exponentially convex functions and construct some means. Let us define  $M_s(\mathbf{x})$  for  $\mathbf{x} = (x_1, \dots, x_i, \dots, x_n)$  as follows:

$$M_s(\mathbf{x}) := \begin{cases} (\sum_{i=1}^n w_i x_i^s)^{\frac{1}{s}}, & s \neq 0; \\ \prod_{i=1}^n x_i^{w_i}, & s = 0. \end{cases}$$

This mean will be used in all examples.

**Example 4.1.** Let

$$\widetilde{D}_1 = \{\psi_t : \mathbb{R} \rightarrow [0, \infty) : t \in \mathbb{R}\}$$

be a family of functions defined by

$$\psi_t(x) = \begin{cases} \frac{1}{t^2} e^{tx}, & t \neq 0; \\ \frac{1}{2} x^2, & t = 0. \end{cases}$$

Here we observe that  $\psi_t$  is convex with respect to  $\psi(x) = x$  which is strictly increasing and continuous. Since,  $\psi_t(x)$  is a convex function on  $\mathbb{R}$  and  $t \rightarrow \frac{d^2}{dx^2} \psi_t(x)$  is exponentially convex function [3]. Using analogous arguing as in the proof of Theorems 2.10 and 2.13, we have that  $t \mapsto [y_0, y_1, y_2; \psi_t]$  is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 2.11 and 2.14 we conclude that  $t \mapsto \Lambda_k(\psi_t); k = 1, \dots, 16$  are exponentially convex in the Jensen sense. It is easy to see that these mappings are continuous, so they are



exponentially convex.

Assume that  $t \mapsto \Lambda_k(\psi_t) > 0$  for  $k = 1, 2, \dots, 16$ . By using convex functions  $\psi_t$  in (7) we obtain the following means:

For  $k = 1, 2, \dots, 16$

$$\mathfrak{M}_{s,t}(\Lambda_k, \widetilde{D}_1) = \begin{cases} \frac{1}{s-t} \ln \left( \frac{\Lambda_k(\psi_s)}{\Lambda_k(\psi_t)} \right), & s \neq t; \\ \frac{\Lambda_k(id, \psi_s)}{\Lambda_k(\psi_s)} - \frac{2}{s}, & s = t \neq 0; \\ \frac{\Lambda_k(id, \psi_0)}{3\Lambda_k(\psi_0)}, & s = t = 0. \end{cases}$$

In particular for  $k = 1$  we have

$$\begin{aligned} \mathfrak{M}_{s,t}(\Lambda_1, \widetilde{D}_1) &= \frac{1}{s-t} \ln \left( \frac{t^2 \cdot \frac{\sum_{i=1}^n w_i e^{s(\frac{b_i}{M_1(\mathbf{b})})} - \sum_{i=1}^n w_i e^{s(\frac{a_i}{M_1(\mathbf{a})})}}{s^2} - \frac{\sum_{i=1}^n w_i e^{t(\frac{b_i}{M_1(\mathbf{b})})} - \sum_{i=1}^n w_i e^{t(\frac{a_i}{M_1(\mathbf{a})})}}{s^2}}{\sum_{i=1}^n w_i e^{s(\frac{b_i}{M_1(\mathbf{b})})} - \sum_{i=1}^n w_i e^{s(\frac{a_i}{M_1(\mathbf{a})})}} \right); & s \neq t; s, t \neq 0; \\ \mathfrak{M}_{s,s}(\Lambda_1, \widetilde{D}_1) &= \frac{\sum_{i=1}^n w_i e^{s(\frac{b_i}{M_1(\mathbf{b})})} - \sum_{i=1}^n w_i e^{s(\frac{a_i}{M_1(\mathbf{a})})}}{\sum_{i=1}^n w_i e^{s(\frac{b_i}{M_1(\mathbf{b})})} - \sum_{i=1}^n w_i e^{s(\frac{a_i}{M_1(\mathbf{a})})}} - \frac{2}{s}; & s \neq 0; \\ \mathfrak{M}_{s,0}(\Lambda_1, \widetilde{D}_1) &= \frac{1}{s} \cdot \ln \left( \frac{\frac{2}{s^2} \cdot \frac{\sum_{i=1}^n w_i e^{s(\frac{b_i}{M_1(\mathbf{b})})} - \sum_{i=1}^n w_i e^{s(\frac{a_i}{M_1(\mathbf{a})})}}{\frac{M_2^2(\mathbf{b})}{M_1^2(\mathbf{b})} - \frac{M_2^2(\mathbf{a})}{M_1^2(\mathbf{a})}}}{\frac{M_2^2(\mathbf{b})}{M_1^2(\mathbf{b})} - \frac{M_2^2(\mathbf{a})}{M_1^2(\mathbf{a})}} \right); & s \neq 0, \\ \mathfrak{M}_{0,0}(\Lambda_1, \widetilde{D}_1) &= \frac{1}{3} \cdot \frac{\frac{M_3^2(\mathbf{b})}{M_1^2(\mathbf{b})} - \frac{M_3^2(\mathbf{a})}{M_1^2(\mathbf{a})}}{\frac{M_2^2(\mathbf{b})}{M_1^2(\mathbf{b})} - \frac{M_2^2(\mathbf{a})}{M_1^2(\mathbf{a})}}. \end{aligned}$$

Since  $\mathfrak{M}_{s,t}(\Lambda_k, \widetilde{D}_1) = \ln \mu_{s,t}(\Lambda_k, \widetilde{D}_1)$  ( $k = 1, 2, \dots, 16$ ), so by (1) these means are monotonic.

**Example 4.2.** Let

$$\widetilde{D}_2 = \{\varphi_t : (0, \infty) \rightarrow \mathbb{R} : t \in \mathbb{R}\}$$

be a family of functions defined by,

$$(8) \quad \varphi_t(x) = \begin{cases} \frac{x^t}{t(t-1)}; & t \neq 0, 1; \\ -\ln x; & t=0; \\ x \ln x, & t=1. \end{cases}$$

Since  $\varphi_t(x)$  is a convex function for  $x \in \mathbb{R}^+$  and  $t \rightarrow \frac{d^2}{dx^2} \varphi_t(x)$  is exponentially convex, so by the same arguments given in previous example we conclude that  $\Lambda_k(\varphi_t)$ ;  $k = 1, 2, \dots, 16$  are exponentially convex. We assume that  $\Lambda_k(\varphi_t) > 0$ ;  $k = 1, 2, \dots, 16$ .

For this family of convex functions we can give the following means:

For  $k = 1, 2, \dots, 16$

$$\mathfrak{M}_{s,t}(\Lambda_k, \widetilde{D}_2) = \begin{cases} \left( \frac{\Lambda_k(\varphi_s)}{\Lambda_k(\varphi_t)} \right)^{\frac{1}{s-t}}; & s \neq t; \\ \exp \left( \frac{1-2s}{s(s-1)} - \frac{\Lambda_k(\varphi_0 \varphi_s)}{\Lambda_k(\varphi_s)} \right); & s = t \neq 0, 1; \\ \exp \left( 1 - \frac{\Lambda_k(\varphi_0^2)}{2\Lambda_k(\varphi_0)} \right); & s = t = 0; \\ \exp \left( -1 - \frac{\Lambda_k(\varphi_0 \varphi_1)}{2\Lambda_k(\varphi_1)} \right); & s = t = 1. \end{cases}$$

In particular for  $k = 1$  we have

$$\begin{aligned} \mathfrak{M}_{s,t}(\Lambda_1, \widetilde{D}_2) &= \left( \frac{t(t-1)}{s(s-1)} \cdot \frac{\frac{M_s^s(\mathbf{b})}{M_1^s(\mathbf{b})} - \frac{M_s^s(\mathbf{a})}{M_1^s(\mathbf{a})}}{\frac{M_t^t(\mathbf{b})}{M_1^t(\mathbf{b})} - \frac{M_t^t(\mathbf{a})}{M_1^t(\mathbf{a})}} \right)^{\frac{1}{s-t}}; & s \neq t; s, t \neq 0; \\ \mathfrak{M}_{s,s}(\Lambda_1, \widetilde{D}_2) &= \exp \left( \frac{\sum_{i=1}^n w_i \left( \frac{b_i}{M_1(\mathbf{b})} \right)^s \ln \left( \frac{b_i}{M_1(\mathbf{b})} \right) - \sum_{i=1}^n w_i \left( \frac{a_i}{M_1(\mathbf{a})} \right)^s \ln \left( \frac{a_i}{M_1(\mathbf{a})} \right)}{\frac{M_s^s(\mathbf{b})}{M_1^s(\mathbf{b})} - \frac{M_s^s(\mathbf{a})}{M_1^s(\mathbf{a})}} - \frac{2s-1}{s(s-1)} \right); & s \neq 0, 1; \\ \mathfrak{M}_{s,0}(\Lambda_1, \widetilde{D}_2) &= \left( \frac{1}{s(1-s)} \cdot \frac{\frac{M_s^s(\mathbf{b})}{M_1^s(\mathbf{b})} - \frac{M_s^s(\mathbf{a})}{M_1^s(\mathbf{a})}}{\sum_{i=1}^n w_i \ln \left( \frac{b_i}{M_1(\mathbf{b})} \right) - \sum_{i=1}^n w_i \ln \left( \frac{a_i}{M_1(\mathbf{a})} \right)} \right)^{\frac{1}{s}}; & s \neq 0, 1; \\ \mathfrak{M}_{s,1}(\Lambda_1, \widetilde{D}_2) &= \left( \frac{1}{s(s-1)} \cdot \frac{\frac{M_s^s(\mathbf{b})}{M_1^s(\mathbf{b})} - \frac{M_s^s(\mathbf{a})}{M_1^s(\mathbf{a})}}{\sum_{i=1}^n w_i \left( \frac{b_i}{M_1(\mathbf{b})} \right) \ln \left( \frac{b_i}{M_1(\mathbf{b})} \right) - \sum_{i=1}^n w_i \left( \frac{a_i}{M_1(\mathbf{a})} \right) \ln \left( \frac{a_i}{M_1(\mathbf{a})} \right)} \right)^{\frac{1}{s-1}}; & s \neq 0, 1; \\ \mathfrak{M}_{0,0}(\Lambda_1, \widetilde{D}_2) &= \exp \left( \frac{1}{2} \cdot \frac{\sum_{i=1}^n w_i \ln^2 \left( \frac{b_i}{M_1(\mathbf{b})} \right) - \sum_{i=1}^n w_i \ln^2 \left( \frac{a_i}{M_1(\mathbf{a})} \right)}{\sum_{i=1}^n w_i \ln \left( \frac{b_i}{M_1(\mathbf{b})} \right) - \sum_{i=1}^n w_i \ln \left( \frac{a_i}{M_1(\mathbf{a})} \right)} + 1 \right); \\ \mathfrak{M}_{1,1}(\Lambda_1, \widetilde{D}_2) &= \exp \left( \frac{1}{2} \cdot \frac{\sum_{i=1}^n w_i \left( \frac{b_i}{M_1(\mathbf{b})} \right) \ln^2 \left( \frac{b_i}{M_1(\mathbf{b})} \right) - \sum_{i=1}^n w_i \left( \frac{a_i}{M_1(\mathbf{a})} \right) \ln^2 \left( \frac{a_i}{M_1(\mathbf{a})} \right)}{\sum_{i=1}^n w_i \left( \frac{b_i}{M_1(\mathbf{b})} \right) \ln \left( \frac{b_i}{M_1(\mathbf{b})} \right) - \sum_{i=1}^n w_i \left( \frac{a_i}{M_1(\mathbf{a})} \right) \ln \left( \frac{a_i}{M_1(\mathbf{a})} \right)} - 1 \right); \\ \mathfrak{M}_{1,0}(\Lambda_1, \widetilde{D}_2) &= \frac{\sum_{i=1}^n w_i \left( \frac{b_i}{M_1(\mathbf{b})} \right) \ln \left( \frac{b_i}{M_1(\mathbf{b})} \right) - \sum_{i=1}^n w_i \left( \frac{a_i}{M_1(\mathbf{a})} \right) \ln \left( \frac{a_i}{M_1(\mathbf{a})} \right)}{\sum_{i=1}^n w_i \ln \left( \frac{a_i}{M_1(\mathbf{a})} \right) - \sum_{i=1}^n w_i \ln \left( \frac{b_i}{M_1(\mathbf{b})} \right)}. \end{aligned}$$

Since  $\mathfrak{M}_{s,t}(\Lambda_k, \widetilde{D}_2) = \mu_{s,t}(\Lambda_k, \widetilde{D}_2)$  ( $k = 1, 2, \dots, 16$ ), so by (1) these means are monotonic.

**Example 4.3.** Let

$$\widetilde{D}_3 = \{\theta_t : (0, \infty) \rightarrow (0, \infty) : t \in (0, \infty)\}$$

be family of functions defined by

$$\theta_t(x) = \frac{e^{-x\sqrt{t}}}{t}.$$

Since  $t \rightarrow \frac{d^2}{dx^2} \theta_t(x) = e^{-x\sqrt{t}}$  is exponentially convex, being the Laplace transform of a non-negative function [3]. So by same argument given in Example 4.1 we conclude that  $\Lambda_k(\theta_t)$ ;  $k = 1, 2, \dots, 16$  are exponentially convex. We assume that  $\Lambda_k(\varphi_t) > 0$ ;  $k = 1, 2, \dots, 16$ .

For this family of functions we have the following possible cases of  $\mu_{s,t}(\Lambda_k, \widetilde{D}_3)$ :

For  $k = 1, 2, \dots, 16$

$$\mu_{s,t}(\Lambda_k, \widetilde{D}_3) = \begin{cases} \left( \frac{\Lambda_k(\theta_s)}{\Lambda_k(\theta_t)} \right)^{\frac{1}{s-t}}, & s \neq t; \\ \exp\left(-\frac{\Lambda_k(id.\theta_s)}{2\sqrt{s}\Lambda_k(\theta_s)} - \frac{1}{s}\right), & s = t; \end{cases}$$

In particular for  $k = 1$  we have

$$\begin{aligned} \mu_{s,t}(\Lambda_1, \widetilde{D}_3) &= \left( \frac{t \cdot \frac{\sum_{i=1}^n w_i e^{-(\frac{b_i}{M_1(\mathbf{b})})\sqrt{s}} - \sum_{i=1}^n w_i e^{-(\frac{a_i}{M_1(\mathbf{a})})\sqrt{s}}}{\sum_{i=1}^n w_i e^{-(\frac{b_i}{M_1(\mathbf{b})})\sqrt{t}} - \sum_{i=1}^n w_i e^{-(\frac{a_i}{M_1(\mathbf{a})})\sqrt{t}}}}{s} \right)^{\frac{1}{s-t}}, & s \neq t; \\ \mu_{s,s}(\Lambda_1, \widetilde{D}_3) &= \exp\left(-\frac{1}{2\sqrt{s}} \cdot \frac{\sum_{i=1}^n w_i (\frac{b_i}{M_1(\mathbf{b})}) e^{-\frac{b_i}{M_1(\mathbf{b})}\sqrt{s}} - \sum_{i=1}^n w_i (\frac{a_i}{M_1(\mathbf{a})}) e^{-\frac{a_i}{M_1(\mathbf{a})}\sqrt{s}}}{\sum_{i=1}^n w_i e^{-\frac{b_i}{M_1(\mathbf{b})}\sqrt{s}} - \sum_{i=1}^n w_i e^{-\frac{a_i}{M_1(\mathbf{a})}\sqrt{s}}} - \frac{1}{s}\right). \end{aligned}$$

Monotonicity of  $\mu_{s,t}((\Lambda_k, \widetilde{D}_3))$  is followed by (1). By (7)

$$\mathfrak{M}_{s,t}(\Lambda_k, \widetilde{D}_3) = -(\sqrt{s} + \sqrt{t}) \ln \mu_{s,t}(\Lambda_k, \widetilde{D}_3) \quad (k = 1, 2, \dots, 16)$$

defines a class of means.

**Example 4.4.** Let

$$\widetilde{D}_4 = \{\phi_t : (0, \infty) \rightarrow (0, \infty) : t \in (0, \infty)\}$$

be family of functions defined by

$$\phi_t(x) = \begin{cases} \frac{t^{-x}}{(\ln t)^2}, & t \neq 1; \\ \frac{x^2}{2}, & t = 1. \end{cases}$$

Since  $\frac{d^2}{dx^2} \phi_t(x) = t^{-x} = e^{-x \ln t} > 0$ , for  $x > 0$ , so by same argument given in Example 4.1 we conclude that  $t \rightarrow \Lambda_k(\phi_t)$ ;  $k = 1, 2, \dots, 16$  are exponentially convex. We assume that  $\Lambda_k(\phi_t) > 0$ ;  $k = 1, 2, \dots, 16$ .

For this family of functions we have the following possible cases of  $\mu_{s,t}(\Lambda_k, \widetilde{D}_4)$ :

For  $k = 1, 2, \dots, 16$

$$\mu_{s,t}(\Lambda_k, \widetilde{D}_4) = \begin{cases} \left( \frac{\Lambda_k(\phi_s)}{\Lambda_k(\phi_t)} \right)^{\frac{1}{s-t}}, & s \neq t; \\ \exp\left(-\frac{\Lambda_k(id.\phi_s)}{s\Lambda_k(\phi_s)} - \frac{2}{s \ln s}\right), & s = t \neq 1; \\ \exp\left(-\frac{1}{3} \frac{\Lambda_k(id.\phi_1)}{\Lambda_k(\phi_1)}\right), & s = t = 1; \end{cases}$$

In particular for  $k = 1$  we have

$$\begin{aligned} \mu_{s,t}(\Lambda_1, \widetilde{D}_4) &= \left( \frac{(\ln t)^2 \cdot \sum_{i=1}^n w_i s^{-\left(\frac{b_i}{M_1(\mathbf{b})}\right)} - \sum_{i=1}^n w_i s^{-\left(\frac{a_i}{M_1(\mathbf{a})}\right)}}{(\ln s)^2 \cdot \sum_{i=1}^n w_i t^{-\left(\frac{b_i}{M_1(\mathbf{b})}\right)} - \sum_{i=1}^n w_i t^{-\left(\frac{a_i}{M_1(\mathbf{a})}\right)}} \right)^{\frac{1}{s-t}}; & s \neq t; s, t \neq 1; \\ \mu_{s,s}(\Lambda_1, \widetilde{D}_4) &= \exp \left( -\frac{1}{s} \cdot \frac{\sum_{i=1}^n w_i \left(\frac{b_i}{M_1(\mathbf{b})}\right) s^{-\left(\frac{b_i}{M_1(\mathbf{b})}\right)} - \sum_{i=1}^n w_i \left(\frac{a_i}{M_1(\mathbf{a})}\right) s^{-\left(\frac{a_i}{M_1(\mathbf{a})}\right)}}{\sum_{i=1}^n w_i s^{-\left(\frac{b_i}{M_1(\mathbf{b})}\right)} - \sum_{i=1}^n w_i s^{-\left(\frac{a_i}{M_1(\mathbf{a})}\right)}} - \frac{2}{s \ln s} \right); & s \neq 1, \\ \mu_{s,1}(\Lambda_1, \widetilde{D}_4) &= \left( \frac{2}{(\ln s)^2} \cdot \frac{\left(\sum_{i=1}^n w_i s^{-\left(\frac{b_i}{M_1(\mathbf{b})}\right)} - \sum_{i=1}^n w_i s^{-\left(\frac{a_i}{M_1(\mathbf{a})}\right)}\right)^{\frac{1}{s-1}}}{\frac{M_2^2(\mathbf{b})}{M_1^2(\mathbf{b})} - \frac{M_2^2(\mathbf{a})}{M_1^2(\mathbf{a})}} \right)^{\frac{1}{s-1}}, \\ \mu_{1,1}(\Lambda_1, \widetilde{D}_4) &= -\frac{1}{3} \cdot \frac{\frac{M_3^3(\mathbf{b})}{M_1^3(\mathbf{b})} - \frac{M_3^3(\mathbf{a})}{M_1^3(\mathbf{a})}}{\frac{M_2^2(\mathbf{b})}{M_1^2(\mathbf{b})} - \frac{M_2^2(\mathbf{a})}{M_1^2(\mathbf{a})}}. \end{aligned}$$

Monotonicity of  $\mu_{s,t}(\Lambda_k, \widetilde{D}_4)$  is followed by (1). By (7)

$$\mathfrak{M}_{s,t}(\Lambda_k, \widetilde{D}_4) = -L(s, t) \ln \mu_{s,t}(\Lambda_k, \widetilde{D}_4) \quad (k = 1, 2, \dots, 16)$$

defines a class of means, where  $L(s, t)$  is Logarithmic mean defined as:

$$L(s, t) = \begin{cases} \frac{s-t}{\ln s - \ln t}, & s \neq t; \\ s, & s=t. \end{cases}$$

**Acknowledgement:** This research work is funded by Higher Education Commission Pakistan. The research of the third author was supported by the Croatian Ministry of Science, Education and Sports under the Research Grants 117-1170889-0888.

## REFERENCES

- [1] L. Berwald, Verallgemeinerung eines Mittelwertsatzes von J. Favard für positive konkave Funktionen, Acta Math 79(1947), 17-37.
- [2] J. Favard, Sur les valeurs moyennes, Bull. Sci. Math. 57(1933), 54-64.
- [3] J. Jakšetić and J. Pečarić, Exponential Convexity Method, J. Convex Anal., 20(2013), No. 1, 181-197.
- [4] S. Karlin, W. J. Studden and Tchebycheff Systems, With Applications in Analysis and Statistics, Interscience, New York, (1966).
- [5] N. Latif, J. Pečarić and I. Perić, On Discrete Favard and Berwald's Inequalities. Communication in Mathematical Analysis, 12, Number 2, pp. 34-57 (2012) ISSN 1938-9787.
- [6] N. Latif, J. Pečarić and I. Perić, On Majorization, Favard and Berwald's Inequalities. Annals of Functional Analysis, 2(2011), no. 1, 31-50, ISSN: 2008-8752.

- [7] L. Maligranda, J. Pečarić, and L. E. Persson, Weighted Favard's and Berwald's Inequalities, *J. Math. Anal. Appl.* 190 (1995), 248-262.
- [8] A. W. Marshall, I. Olkin and Barry C. Arnold, *Inequalities: Theory of Majorization and Its Applications* (Second Edition), Springer Series in Statistics, New York 2011.
- [9] C. P. Niculescu and L. E. Persson, *Convex functions and their applications. A contemporary Approach*, CMS Books in Mathematics, Vol. 23, Springer-Verlage, New York, 2006.
- [10] J. Pečarić and S. Abramovich, On New Majorization Theorems, *Rocky Mountains Journal of Mathematics*, 27, number 3, Summer 1997.
- [11] J. Pečarić and J. Perić, Improvements of the Giaccardi and the Petrović Inequality and Related Stolarsky type means, *Annals of the University of Craiova Mathematics and Computer Science Series*, Volume 39(1), 2012, pages 65-75.
- [12] J. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.