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EXTENSION OF STOLARSKY MEANS BY EULER-RADAU EXPANSIONS

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Abstract. We present construction of exponentially convex functions via functionals that follow from some inequalities for convex functions. These inequalities are derived from expansions of Euler and Radau. Using fruitful properties of exponential convexity we construct various means that have nice monotone properties over defining parameters. We further show how known results about Cauchy means can be treated in a succinct way.

Keywords: General two-point formulae; Stolarsky means; exponential convexity; Cauchy means.

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1. Introduction

The well known Stolarsky means are defined in [12] as follows :

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$$E_{r,s}(a,b) = \begin{cases} \left(\frac{s(b^r-a^r)}{r(b^s-a^s)}\right)^{1/r-s}, & r \neq s, r \neq 0, \\ \left(\frac{b^r-a^r}{r(\ln b - \ln a)}\right)^{1/r}, & s = 0, r \neq 0, \\ e^{-\frac{1}{r}}\left(\frac{a^{a^r}}{b^{b^r}}\right)^{1/a^r-b^r}, & s = r \neq 0, \\ \sqrt{ab}, & r = s = 0, \end{cases} \tag{0.1}$$

where a and b are positive real numbers $a \neq b$, r and s are real numbers.
Stolarsky proved that the function $E_{r,s}(a,b)$ is increasing in both parameters r and s, that is for $r \geq p$ and $s \geq q$:

$$E_{r,s}(a,b) \geq E_{p,q}(a,b). \tag{0.2}$$

These means, since their invention, are generalized in various directions. However, in [2] Stolarsky means are recognized as application of the linear functional

$$f \mapsto \frac{f(x)-f(y)}{x-y},\ x \neq y \tag{0.3}$$

on the family of functions $\{\varphi_r : t \in \mathbb{R}\}$ (defined on $(0,\infty)$)

$$\varphi_r(x) = \begin{cases} x^r/r, & r \neq 0; \\ \log x, & r = 0. \end{cases} \tag{0.4}$$

Since functional defined above is nonnegative on monotonically increasing functions, and $\frac{d\varphi_r}{dx}(x) = x^{r-1} \geq 0,\ r \in \mathbb{R}$, then using Cauchy mean-value theorem and log-convexity we get construction and monotonicity property of Stolarsky means, as is showed in [2]. In that paper, this idea is further extended via application of Hermite-Hadamard functionals

$$f \mapsto \frac{1}{y-x}\int_x^y f(u)du - f\left(\frac{x+y}{2}\right) \tag{0.5}$$

and

$$f \mapsto \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(u)du \tag{0.6}$$

on a family of convex functions (see **Example** 3.), another two means of Stolarsky type are constructed and monotonicity property is proved again using log-convexity.

In this paper we generalize means from [2] in several directions. First, Hermite-Hadamard functionals are generalized using Euler and Radau expansions. Second, these functionals are applied on new families (aside of (0.4)) of convex functions which give us quite different means. Third, it is showed that log-convexity can be shifted on finer classes such as $n-$exponentially convex and exponentially convex functions. Also, our approach give us non-trivial examples of exponentially convex functions.

1. THEORY OVERVIEW AND AUXILIARY RESULTS

Exponentially convex functions are invented by Bernstein in [3] as a subclass of convex functions on a given open interval. These functions have many nice properties, for example, they are analytical on their domain. Although we will need only few of these properties we point here that very good reference on general results about exponential convexity is [1] and [7].

If not specified, in the sequel, I stands for an open interval in $\mathbb{R}$.

Definition 1.1. *For fixed $n \in \mathbb{N}$, a function $f : I \to \mathbb{R}$ is $n-$exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{s_i+s_j}{2}\right) \geq 0$$

for all choices of $\xi_i \in \mathbb{R},\ s_i \in I,\ i = 1, \ldots, n.$

A function $f : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

The notion of $n-$exponential convexity is introduced in [8].

Remark 1.2. *From Definition* 1.1 *it follows that* 1*-exponentially convex functions in the Jensen sense are in fact nonnegative functions. It is further obvious that n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $n \geq k$.*

By well known Sylvester criteria, we have following proposition.

Proposition 1.3. *If f is n-exponentially convex in the Jensen sense on I then the matrix*

$$\left[f\left(\frac{s_i+s_j}{2}\right)\right]_{i,j=1}^{n}$$

is positive semi-definite. Particularly

$$\det\left[f\left(\frac{s_i+s_j}{2}\right)\right]_{i,j=1}^{n} \geq 0,$$

where $s_i \in I$, $i=1,\ldots,n$.

Corollary 1.4.

(i) *If $f : I \to (0,\infty)$ is 2-exponentially convex in the Jensen sense then f is a log-convex function in the Jensen sense on I.*

(ii) *If $f : I \to (0,\infty)$ is 2-exponentially convex then f is a log-convex function on I.*

Proof. (i) From

$$\xi_1^2 f(x) + 2\xi_1\xi_2 f\left(\frac{x+y}{2}\right) + \xi_2^2 f(y) \geq 0,$$

for any $\xi_1, \xi_2 \in \mathbb{R}$ and all $x, y \in I$, we conclude

$$f^2\left(\frac{x+y}{2}\right) \leq f(x)f(y), \tag{1.1}$$

for all $x, y \in I$.

(ii) Since f is continuous function we have

$$f(\lambda x + (1-\lambda)y) \leq f(x)^{\lambda} f(y)^{1-\lambda}, \tag{1.2}$$

for all $x, y \in I$ and any $\lambda \in [0,1]$. □

Definition 1.5. *A function $f : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense on I for each $n \in \mathbb{N}$.*

A function $f : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Proposition 1.6. *Let $\mathscr{E}_I$ denote a set of all exponentially convex functions on open interval I.*

(i) *$\mathscr{E}_I$ is a convex cone i.e. if $f, g \in \mathscr{E}_I$ and $\alpha,\ \beta \geq 0$ then $\alpha f + \beta g \in \mathscr{E}_I$.*

(ii) *$\mathscr{E}_I$ is closed under multiplication i.e. if $f, g \in \mathscr{E}_I$ then $fg \in \mathscr{E}_I$.*

Proof. (i)-part follows from definition. (ii)-part follows from the next theorem (see [7]). □

One of main features of exponentially convex functions is its integral representation.

Theorem 1.7. *The function $f : I \to \mathbb{R}$ exponentially convex on I if and only if*

$$f(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(t), \;\; x \in I \tag{1.3}$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

Proof. See [1], p. 211. □

Corollary 1.8. *Assume that $f : I \to \mathbb{R}$ is an exponentially convex function on I. Then*

(i) *for any $k \in \mathbb{N}$ we have*

$$f^{(k)}(x) = \int_{-\infty}^{\infty} t^k e^{tx} d\sigma(t),$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is some non-decreasing function;

(ii) *for any $k \in \mathbb{N}$ the function $x \mapsto f^{(2k)}(x)$ is exponentially convex function on I.*

Proof. (i) This follows straightforward.

(ii) Using integral representation from (i)-part, for any choice $\xi_i \in \mathbb{R},\ s_i \in I,\ i = 1, \ldots, n$ we have

$$\sum_{i,j=1}^{n} \xi_i \xi_j f^{(2k)}\left(\tfrac{s_i+s_j}{2}\right) = \int_{-\infty}^{\infty} t^{2k} \left(\sum_{i=1}^{n} \xi_i e^{ts_i/2}\right)^2 \sigma(dt) \geq 0.$$

□

Let us point here two basic examples of exponentially convex functions.

Example 1. The function $x \mapsto e^{\gamma x}$ is exponentially convex on $\mathbb{R}$ for any $\gamma \in \mathbb{R}$. This can be checked directly:

$$\sum_{i,j=1}^{n} \xi_i \xi_j e^{\gamma \frac{s_i+s_j}{2}} = \left(\sum_{i=1}^{n} \xi_i e^{\gamma \frac{s_i}{2}}\right)^2 \geq 0,$$

or we can use integral representation (1.3)

$$e^{\gamma x} = \int_{-\infty}^{\infty} e^{tx} d\sigma(t)$$

with $\sigma(t) = 1_{[\gamma,\infty)}(t)$ as choice of non-decreasing function.

The next example is less trivial, and integral representation (1.3) is particularly useful here.

Example 2. For every $\alpha > 0$ the function

$$x \mapsto x^{-\alpha}$$

is exponentially convex on $(0,\infty)$, since

$$x^{-\alpha} = \int_0^\infty e^{-xt} \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt$$

(see [7] and [11] p. 210).

We observe here that the previous integral can be rearranged as $\int_{-\infty}^{\infty} e^{tx} d\sigma(t)$ where $\sigma : \mathbb{R} \to \mathbb{R}$ is non-decreasing function defined with $\sigma(t) = \frac{-(-t)^\alpha}{\alpha\Gamma(\alpha)} 1_{(-\infty,0)}(t)$.

Remark 1.9. *It is obvious that every exponentially convex function is n-exponentially convex. Converse is not, in general, true since for example $f(x) = e^{x^3-x}$ is 2-exponentially convex on $(0,1)$ and not exponentially convex function on $(0,1)$ (see [7] for details).*

The next theorem will play important role in our applications.

Theorem 1.10. *Let $f : I \to (0,\infty)$ be log-convex, derivable function.*
Let $M : I \times I \to (0,\infty)$ be defined with

$$M(x,y) = \begin{cases} \left(\frac{f(x)}{f(y)}\right)^{\frac{1}{x-y}}, & x \neq y; \\ \exp\left(\frac{f'(x)}{f(x)}\right), & x = y. \end{cases} \tag{1.4}$$

If $x_1 \leq x_2$, $y_1 \leq y_2$ then

$$M(x_1,y_1) \leq M(x_2,y_2). \tag{1.5}$$

Proof. See [7]. ☐

We recall the definition of divided difference, for more on this subject see [10].

Definition 1.11. *The second order divided difference of $f : [a,b] \to \mathbb{R}$, at mutually different knots $y_0, y_1, y_2 \in [a,b]$ is defined recursively by*

$$[y_i; f] = f(y_i), \ i = 0,1,2$$

$$[y_i, y_{i+1}; f] = \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i},\ i = 0, 1$$

$$[y_0, y_1, y_2; f] = \frac{[y_1, y_2; f] - [y_0, y_1; f]}{y_2 - y_0}.$$

Remark 1.12. *The value* $[y_0, y_1, y_2; f]$ *is independent of the order among knots* y_0, y_1, y_2. *This definition may be extended to include the case in which some or all knots coincide. Namely,*

$$[y_0, y_0, y_2; f] = \lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2},\ y_2 \neq y_0$$

provided f' *exists, and furthermore,*

$$[y_0, y_0, y_0; f] = \lim_{y_2 \to y_0} \lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = \frac{f''(y_0)}{2}$$

provided f'' *exists.*

In the sequel we will study linear functionals $A : C[a,b] \to \mathbb{R}$ that have property

$$f \in C[a,b] \text{ is convex } \Rightarrow Af \geq 0. \tag{1.6}$$

Hermite-Hadamard functionals (0.5) and (0.6) from introduction are examples of functionals A with property (1.6).

Theorem 1.13. *Let* $A : C[a,b] \to \mathbb{R}$ *be linear functional that satisfies* (1.6)*, let* I *be any open interval in* $\mathbb{R}$ *and let* n *be any positive integer. Assume that* $\mathbf{F} = \{f_t : t \in I\}$ *is the family of functions from* $C[a,b]$ *such that* $t \mapsto [y_0, y_1, y_2; f_t]$ *is* n*-exponentially convex in the Jensen sense on* I *for every choice of three distinct knots* $y_0, y_1, y_2 \in [a,b]$. *Then*

(i) *the function* $t \mapsto A(f_t)$ *is* n*-exponentially convex in the Jensen sense on* I;

(ii) *if* $t \mapsto A(f_t)$ *is continuous on* I*, then it is* n*-exponentially convex on* I.

Proof. (i) For any $\xi_j \in \mathbb{R}$, $s_j \in I$ where $j = 1, \ldots, n$ we define

$$h(x) = \sum_{j,k=1}^{n} \xi_j \xi_k f_{\frac{s_j + s_k}{2}}(x).$$

Since $[y_0,y_1,y_2;f_{\frac{s_j+s_k}{2}}]$ is n-exponentially convex in the Jensen sense on $[\alpha,\beta]$, we conclude

$$[y_0,y_1,y_2;h]=\sum_{j,k=1}^{n}\xi_j\xi_k\left[y_0,y_1,y_2;f_{\frac{s_j+s_k}{2}}\right]\geq 0.$$

This means that h is a convex function from $C[a,b]$ and (1.6) implies

$$\sum_{j,k=1}^{n}\xi_j\xi_k A\left(f_{\frac{s_j+s_k}{2}}\right)=A(h)\geq 0,$$

hence $t\mapsto A(f_t)$ is n-exponentially convex in the Jensen sense on $[a,b]$.

(ii) Follows from (i)-part and Definition 1.1.

□

Corollary 1.14. *Let $A:C[a,b]\to\mathbb{R}$ be linear functional that satisfies* (1.6) *and let I be any open interval. Assume that $\mathbf{F}=\{f_t : t\in I\}$ is the family of functions from $C[a,b]$ such that $t\mapsto[y_0,y_1,y_2;f_t]$ is exponentially convex in the Jensen sense on I for every choice of three distinct knots $y_0,y_1,y_2\in[a,b]$. Then*

(i) *the function $t\mapsto A(f_t)$ is also exponentially convex in the Jensen sense on I;*

(ii) *if $t\mapsto A(f_t)$ is continuous function on I, then the function $t\mapsto A(f_t)$ is exponentially convex on the I.*

Corollary 1.15. *Let $A:C[a,b]\to\mathbb{R}$ be linear functional that satisfies* (1.6) *and let I be any open interval. Assume that $\mathbf{F}=\{f_t : t\in I\}$ is the family of functions from $C[a,b]$ such that $t\mapsto[y_0,y_1,y_2;f_t]$ is log-convex in the Jensen sense on I for every choice of three distinct knots $y_0,y_1,y_2\in[a,b]$. Then*

(i) *the function $t\mapsto A(f_t)$ is also log-convex in the Jensen sense on I;*

(ii) *if $t\mapsto A(f_t)$ is continuous positive function on I, then the function $t\mapsto A(f_t)$ is log-convex on the I;*

(iii) *if $t\mapsto A(f_t)$ is positive, derivable function on I, then for any $p\leq u,\ q\leq v;\ p,q,u,v\in I$, we have*

$$M_{p,q}(A,\mathbf{F})\leq M_{u,v}(A,\mathbf{F}) \tag{1.7}$$

where

$$M_{p,q}(A,\mathbf{F}) = \begin{cases} \left(\frac{A(f_p)}{A(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\left(A(f_p)\right)}{A(f_p)}\right), & p = q. \end{cases} \tag{1.8}$$

Proof. (i) and (ii) parts follow from Theorem 1.13, (iii) follows from Theorem 1.10. □

Remark 1.16. *Note that the results from Theorem* 1.13 *and Corollary* 1.14 *still hold when two of the knots y_0, y_1, y_2 are coincide, say y_0, y_1, for family of differentiable functions such that the function $t \mapsto [y_0, y_1, y_2; f_t]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense,* log*-convex in the Jensen sense), and furthermore, they still hold when all three knots are coincide for family of twice differentiable functions with the same property.*

Definition 1.17. *For $M_{p,q}(A,\mathbf{F})$ defined with* (1.8) *we will refer as mean if*

$$a \leq M_{p,q}(A,\mathbf{F}) \leq b,$$

for $p,q \in I$. Otherwise we will refer it as quasi-mean with monotonicity property (1.7).

1.1. Further examples, generating families.

Here we list some families of functions $\mathbf{F} = \{f_t : t \in I\}$ from [7] for which we will use Corollaries 1.14 and 1.15 in order to construct exponentially convex functions and then means. Of course, we also need linear functional with property (1.6), and these functionals we construct in the next section via application of Euler-Radau expansions.

Example 3. Let a,b be positive real numbers, $I = \mathbb{R}$ and family $\mathbf{F}_1 = \{f_t : t \in I_1\}$ of functions defined with

$$f_t(x) = \begin{cases} \frac{x^t}{t(t-1)}, & t \neq 0,1, \\ -\log x, & t = 0, \\ x\log x, & t = 1. \end{cases} \tag{1.9}$$

Since $\frac{d^2}{dx^2}f_t(x) = x^{t-2} = e^{(t-2)\ln x}$; $t \mapsto \frac{d^2}{dx^2}f_t(x)$ is exponentially convex function by Example 1. Using Remark 1.16 we then conclude that we can apply conclusions of Corollaries 1.14 and 1.15.

Example 4. Let a,b real numbers, $I=\mathbb{R}$ and family $\mathbf{F}_2 = \{f_t : t \in I_2\}$ of functions defined with

$$f_t(x) = \begin{cases} \frac{e^{tx}}{t^2}, & t \neq 0, \\ \frac{x^2}{2}, & t = 0 \end{cases} \tag{1.10}$$

Since $\frac{d^2}{dx^2}f_t(x) = e^{tx}$, $t \mapsto \frac{d^2}{dx^2}f_t(x)$ is exponentially convex function by Example 1.

Example 5. Let a,b positive real numbers, $I=(0,\infty)$ and family $\mathbf{F}_3 = \{f_t : t \in I_3\}$ of functions defined on $C[a,b]$ with

$$f_t(x) = \begin{cases} \frac{t^{-x}}{\log^2 t}, & t \neq 1, \\ \frac{x^2}{2}, & t = 1. \end{cases} \tag{1.11}$$

Since $t \mapsto \frac{d^2}{dx^2}f_t(x) = t^{-x}$, by Example 2. we know that $t \mapsto \frac{d^n}{dx^n}f_t(x)$ is exponentially convex function on $I=(0,\infty)$.

Example 6. Let a,b be positive real numbers, $I=(0,\infty)$ and family $\mathbf{F}_4 = \{f_t : t \in I\}$ of functions defined on $C[a,b]$ with

$$f_t(x) = \frac{e^{-x\sqrt{t}}}{t}. \tag{1.12}$$

In [7] it is showed that $t \mapsto \frac{d^2}{dx^2}f_t(x) = e^{-x\sqrt{t}}$ is exponentially convex function on $I=(0,\infty)$.

Remark 1.18. *Families* $\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_3,\mathbf{F}_4$ *are not independent: if we substitute* $t \to -\log t$, *for* $t>0$, *in* (1.10) *we get family* $\mathbf{F}_3$ *and if we substitute* $t \to -\sqrt{t}$, *for* $t>0$, *in* (1.10) *we get family* $\mathbf{F}_4$. *Observe that* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *are constructed as antiderivatives of basic examples of exponentially convex functions:* $t \mapsto x^{t-2} = e^{(t-2)\log x}$ *and* $t \mapsto e^{tx}$ *(see Example 1.).*

1.2. **Three mean value results.**

In this subsection we give a few results headed to means that we will need in later part of paper.

Theorem 1.19. *Let $f \in C^2[a,b]$ and let $A : C[a,b] \to \mathbb{R}$ be a linear functional that have property* (1.6). *Then there exists $\xi \in [a,b]$ such that*

$$A(f) = f''(\xi)A(g_0), \tag{1.13}$$

where $g_0(x) = x^2/2$.

Proof. Let $m = \min_{x\in[a,b]} f''(x)$, $M = \max_{x\in[a,b]} f''(x)$. Let us observe that function $\varphi(x) = M\frac{x^2}{2} - f(x) = Mg_0(x) - f(x)$ is convex function since $\varphi''(x) = M - f''(x) \geq 0$. Hence, $A(\varphi) \geq 0$ and we conclude

$$A(f) \leq MA(g_0).$$

Similarly,

$$mA(g_0) \leq A(f) \leq MA(g_0).$$

Now we have (1.13). □

Remark 1.20. *If we denote powers with $e_i(x) = x^i$, $i = 0,1,2,\ldots$, from* (1.13) *it follows that* $A(e_0) = A(e_1) = 0$.

Corollary 1.21. *Let $f,g \in C^2[a,b]$, let $A : C[a,b] \to \mathbb{R}$ be a linear and functional which satisfies* (1.6). *Then there exists $\xi \in [a,b]$ such that*

$$\frac{f''(\xi)}{g''(\xi)} = \frac{A(f)}{A(g)}, \tag{1.14}$$

assuming both denominators not equal zero.

Proof. This is standard proof as in Cauchy mean value theorem. □

Remark 1.22. *If* $\frac{f''}{g''}$ *is an invertible function then unique number* ξ,

$$\xi = \left(\frac{f''}{g''}\right)^{-1}\left(\frac{A(f)}{A(g)}\right), \tag{1.15}$$

represents well-known Cauchy mean. It is obvious $a \le \xi \le b$.

Corollary 1.23. *Let* I *be an open interval in* $\mathbb{R}$, $a,b \in \mathbb{R}$ *and* $A : C[a,b] \to \mathbb{R}$ *linear functional which satisfies* (1.6). *Let* $\mathbf{F} = \{f_t : t \in I\}$ *be a family of functions in* $C^2[a,b]$ *such that* $t \mapsto \frac{d^2 f_t}{dx^2}$ *is a log-convex function on* I. *If*

$$a \le \left(\frac{\frac{d^2 f_p}{dx^2}}{\frac{d^2 f_q}{dx^2}}\right)^{\frac{1}{p-q}}(x) \le b, \tag{1.16}$$

for $x \in [a,b]$, $p,q \in I$, *then* $M_{p,q}(A,\mathbf{F})$ *is a mean.*

Remark 1.24. *We observe that family* $\mathbf{F}_1$ *do satisfy condition* (1.16) *and families* $\mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_3$ *don't satisfy it.*

2. Euler-Radau means and exponential convexity

2.1. **Euler two-point formulae.** We start with Euler two-point formula (see [9] p. 558.)

Theorem 2.1. *Let* $f : [0,1] \to \mathbf{R}$ *be such that* f' *is a continuous function of bounded variation on* $[0,1]$. *Then for each* $s \in [0,1/2]$

$$\int_0^1 f(t)\mathrm{d}t = \frac{1}{2}\left[f(s) + f(1-s)\right] + \frac{1}{4}\int_0^1 F_2^s(t)\,\mathrm{d}f'(t), \tag{2.1}$$

where

$$F_2^s(t) = \begin{cases} 2t^2, & 0 \le t \le s, \\ 2t^2 - 2t + 2s, & s < t \le 1-s, \\ 2t^2 - 4t + 2, & 1-s < t \le 1. \end{cases} \tag{2.2}$$

In the [6] the previous theorem is used to prove the following result.

Theorem 2.2. *Let $f \in C^2[a,b]$. Then for each $t \in \{a\} \cup [\frac{3a+b}{4}, \frac{a+b}{2}]$ there exist some $\xi \in [a,b]$, such that*

$$\frac{f(t)+f(a+b-t)}{2} - \frac{1}{b-a}\int_a^b f(u)du = f''(\xi)R(a,b;t) \tag{2.3}$$

where

$$R(a,b;t) = \frac{6t^2 - 6t(a+b) + a^2 + b^2 + 4ab}{12}. \tag{2.4}$$

It is then observed that if $a < b$ then $R(a,b;t) > 0$ for $t = a$ and $R(a,b;t) < 0$ for $t \in [\frac{a+b}{2}, \frac{a+3b}{4}]$. In the end the following corollary is stated.

Corollary 2.3. *Let $f \in C^2[a,b]$ be a convex function. Then*

$$\frac{1}{b-a}\int_a^b f(u)du \geq \frac{f(t)+f(a+b-t)}{2} \tag{2.5}$$

for each $t \in [\frac{3a+b}{4}, \frac{a+b}{2}]$. For $t = a$ the above inequality is reversed.

We now define linear functional $A_1 : C[a,b] \to \mathbb{R}$ with

$$A_1(f) = \frac{1}{a-b}\int_a^b f(u)du - \frac{f(t)+f(a+b-t)}{2}, \tag{2.6}$$

$t \in \{a\} \cup [\frac{3a+b}{4}, \frac{a+b}{2}]$. According Corollary 2.7 linear functional A_1 satisfies (1.6) property for $t \in [\frac{3a+b}{4}, \frac{a+b}{2}]$.

2.2. **Radau-type quadratures.** We proceed with similar idea of constructing linear functional that will have property (1.6), this time using Radau-type quadratures given in [4, 5].

Theorem 2.4. *Let $f : [-1,1] \to \mathbb{R}$ be such that f'' is continuous on $[-1,1]$ and let $s \in (-1,0] \cup \{1\}$. Then there exists $\xi \in [-1,1]$ such that*

$$\int_{-1}^1 f(t)dt - \frac{2s}{1+s}f(-1) - \frac{2}{1+s}f(s) = \frac{1}{3}(1-s)f''(\xi) \tag{2.7}$$

and

$$\int_{-1}^{1} f(t)dt - \frac{2}{1+s}f(-s) - \frac{2s}{1+s}f(1) = \frac{1}{3}(1-s)f''(-\xi) \tag{2.8}$$

In [6] the previous theorem is used to prove the following result.

Theorem 2.5. *Let $\phi \in C^2[a,b]$. Then for each $t \in (a, \frac{a+b}{2}] \cup \{b\}$ there exist some $\xi \in [a,b]$, such that*

$$\tfrac{2}{b-a}\int_a^b \phi(u)du - \tfrac{2t-a-b}{t-a}\phi(a) - \tfrac{b-a}{t-a}\phi(t) = \phi''(\xi)R_1(a,b;t) \tag{2.9}$$

and for each $v \in [\frac{a+b}{2}, b) \cup \{a\}$ there exist some $\eta \in [a,b]$, such that

$$\tfrac{2}{b-a}\int_a^b \phi(t)dt - \tfrac{b-a}{b-v}\phi(v) - \tfrac{b+a-2v}{b-v}\phi(y) = \phi''(\eta)R_2(a,b;v) \tag{2.10}$$

where

$$R_1(a,b;t) = \frac{(4b+2a-6t)(b-a)}{12} \text{ and } R_2(a,b;v) = \frac{(6v-4a-2b)(b-a)}{12}.$$

Remark 2.6. *Assume $a < b$. Observe then $R_1(a,b;t) > 0$ for $t \in (a, \frac{a+b}{2}]$; and $R_1(a,b;t) < 0$ for $t = b$. Also $R_2(a,b;v) > 0$ for $v \in [\frac{a+b}{2}, b)$; and $R_2(a,b;v) < 0$ for $v = a$.*

Corollary 2.7. *Let $f \in C^2[a,b]$ be a convex function.*

(i) *For every $t \in (a, \frac{a+b}{2}]$*

$$\tfrac{2}{b-a}\int_a^b f(u)du \geq \tfrac{2t-a-b}{t-a}f(a) + \tfrac{b-a}{t-a}f(t).$$

For $t = b$ the above inequality is reversed.

(ii) *For every $v \in [\frac{a+b}{2}, b)$*

$$\tfrac{2}{b-a}\int_a^b f(t)dt \geq \tfrac{b-a}{b-v}f(v) + \tfrac{b+a-2v}{b-v}f(b).$$

For $v = a$ the above inequality is reversed.

Similar to linear functional A_1 defined with (2.6) in Euler case, we define two linear functionals A_2 and A_3 acting on $C[a,b]$ using Corollary 2.7:

$$A_2(f) = \tfrac{2}{b-a}\int_a^b f(u)du - \tfrac{2t-a-b}{t-a}f(a) - \tfrac{b-a}{t-a}f(t), \tag{2.11}$$

where $t \in \{b\} \cup (a, \frac{a+b}{2}]$;

$$A_3(f) = \frac{2}{b-a}\int_a^b f(t)dt - \frac{b-a}{b-v}f(v) - \frac{b+a-2v}{b-v}f(b), \tag{2.12}$$

where $v \in \{a\} \cup [\frac{a+b}{2}, b)$.

Corollary 2.7 confirms that linear functionals A_2 and A_3 do satisfy property (1.6) for $t \in (a, \frac{a+b}{2}]$ and $v \in [\frac{a+b}{2}, b)$ respectively.

2.3. **Euler-Radau means and quasi-means.** For each family $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4$, defined in Examples 3, 4, 5, 6 using Corollaries 1.14 and 1.14 we will construct first exponentially convex functions.

All constructed functions have domain $I = \mathbb{R}$ or $I = (0, \infty)$ depending on family $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ or $\mathbf{F}_4$. Define functions $\psi_{i,j} : I \to \mathbb{R}$ with

$$\psi_{i,j}(u) = A_i(f_u), \ \text{ for } f_u \in \mathbf{F}_j, \tag{2.13}$$

$i = 1,2,3;\ j = 1,2,3,4$. As above defined, linear functionals A_1, A_2, A_3 are considered for $t \in [\frac{3a+b}{4}, \frac{a+b}{2}]$, $t \in (a, \frac{a+b}{2}]$, $v \in [\frac{a+b}{2}, b)$ respectively.

Theorem 2.8. *Let $\psi_{i,j}$, $i = 1,2,3;\ j = 1,2,3,4$ be functions on I defined with* (2.13).

(i) *Functions $\psi_{i,j}$ are exponentially convex on I.*

(ii) *For all $t_m \in I$, $m = 1,2,\ldots,n$, matrix $\left[\psi_{i,j}\left(\frac{t_k+t_l}{2}\right)\right]_{k,l=1}^n$ is positive semi-definite matrix. Particularly*

$$\det\left[\psi_{i,j}\left(\frac{t_k+t_l}{2}\right)\right]_{k,l=1}^n \geq 0. \tag{2.14}$$

Proof. (i) According (i)-part of Corollary 1.14 we first conclude that $\psi_{i,1}, i = 1,2,3$ are exponentially convex on I in Jensen sense. Direct calculation shows that $\psi_{i,1}, i = 1,2,3$ are continuous on I, concluding its exponential convexity according (i)-part of Corollary 1.14.

(ii) This part follows from Proposition 1.3. □

Let us now define

$$M_{p,q}(A_i, \mathbf{F}_j) = \left(\frac{\psi_{i,j}(p)}{\psi_{i,j}(q)}\right)^{\frac{1}{p-q}}, \tag{2.15}$$

for $p,q\in I;\ i=1,2,3;\ j=1,2,3,4.$

Now we consider limit cases for each family separately, since there are some difference between expressions.

FAMILY $\mathbf{F}_1$:

$$M_{p,q}(A_i,\mathbf{F}_1)=\begin{cases}\left(\frac{A_i(f_p)}{A_i(f_q)}\right)^{1/(p-q)}, & p\neq q,\\ \exp\left(\frac{1-2q}{q(q-1)}-\frac{A_i(f_0f_q)}{A_i(f_q)}\right), & p=q\neq 0,1,\\ \exp\left(1-\frac{A_i(f_0^2)}{2A_i(f_0)}\right), & p=q=0,\\ \exp\left(-1-\frac{A_i(f_0f_1)}{2A_i(f_1)}\right), & p=q=1,\end{cases} \tag{2.16}$$

$f_p\in\mathbf{F}_1,\ i=1,2,3.$

FAMILY $\mathbf{F}_2$:

$$M_{p,q}(A_i,\mathbf{F}_2)=\begin{cases}\left(\frac{A_i(f_p)}{A_i(f_q)}\right)^{1/(p-q)}, & p\neq q,\\ \exp\left(\frac{A_i(e_1f_q)}{A_i(f_q)}-\frac{2}{q}\right), & p=q\neq 0,\\ \exp\left(\frac{A_i(e_1f_0)}{3A_i(f_0)}\right), & p=q=0,\end{cases} \tag{2.17}$$

$e_1(x)=x, f_p\in\mathbf{F}_2,\ i=1,2,3.$

FAMILY $\mathbf{F}_3$:

$$M_{p,q}(A_i,\mathbf{F}_3)=\begin{cases}\left(\frac{A_i(f_p)}{A_i(f_q)}\right)^{1/(p-q)}, & p\neq q,\\ \exp\left(\frac{-A_i(e_1f_1)}{A_i(f_1)}-\frac{2}{q\ln q}\right), & p=q\neq 1,\\ \exp\left(-\frac{A_i(e_1f_1)}{3A_i(f_1)}\right), & p=q=1,\end{cases} \tag{2.18}$$

$f_p\in\mathbf{F}_3,\ i=1,2,3.$

FAMILY $\mathbf{F}_4$:

$$M_{p,q}(A_i,\mathbf{F}_4)=\begin{cases}\left(\frac{A_i(f_p)}{A_i(f_q)}\right)^{1/(p-q)}, & p\neq q,\\ \exp\left(\frac{-A_i(e_1 f_q)}{2\sqrt{q}A_i(f_q)}-\frac{1}{q}\right), & p=q,\end{cases} \tag{2.19}$$

$f_p\in\mathbf{F}_4,\ i=1,2,3.$

Theorem 2.9. *Let* $p\le u,\ q\le v;\ p,q,u,v\in I$. *Then*

$$M_{p,q}(A_i,\mathbf{F}_j)\le M_{u,v}(A_i,\mathbf{F}_j), \tag{2.20}$$

for all $f_p\in\mathbf{F}_j,\ i=1,2,3;\ j=1,2,3,4.$

Proof. Follows from (iii)-part of Corollary 1.15. □

According Remark 1.24, with $M_{p,q}(A_i,\mathbf{F}_1)$ we defined means, $i=1,2,3$. $M_{p,q}(A_i,\mathbf{F}_j)$ are quasi-means for $i=1,2,3; j=2,3,4$ that can be easily converted to means using Remark 1.22:

$$\overline{M}_{p,q}(A_i,\mathbf{F}_2):=\log M_{p,q}(A_i,\mathbf{F}_2),\ \ i=1,2,3;$$

$$\overline{M}_{p,q}(A_i,\mathbf{F}_3):=-L(p,q)\log M_{p,q}(A_i,\mathbf{F}_3),\ \ i=1,2,3,$$

where $L(p,q)=\begin{cases}\frac{p-q}{\log p-\log q}, & p\neq q,\\ q, & p=q;\end{cases}$

$$\overline{M}_{p,q}(A_i,\mathbf{F}_4):=-(\sqrt{p}+\sqrt{q})\log M_{p,q}(A_i,\mathbf{F}_4),\ \ i=1,2,3,$$

are all means.

2.4. **Generalized Euler-Radau means and quasi-means.** As we have already seen exponentially convex functions are very base of our means and quasi-means. Now we generalize this construction adding one more parameter in our (quasi-)means and newly constructed (quasi-)means will retain monotonicity property. For that purpose we need one additional, simple, property of exponentially convex functions given in the next proposition.

Proposition 2.10. *Let $I=(0,\infty)$ or $I=\mathbb{R}$. If $\psi: I\to\mathbb{R}$ is exponentially convex function on I then for any $c\in I$ the function $x\mapsto \psi(cx)$ is also exponentially convex.*

Let us now make substitutions $p\to\frac{p}{s},\ q\to\frac{q}{s},\ a\to a^s,\ b\to b^s$ for $s>0$ and for the case $s<0$ $\ a\to b^s,\ b\to a^s$ in means $M_{p,q}(A_i,\mathbf{F}_j),\ i=1,2,3; j=1,2,3,4$. The parameter s that we introduce here will be in corresponding I.

For $spq(p-q)\neq 0$ we define new means with

$$M^t_{p,q;s}(A_i,\mathbf{F}_j)=\left(\frac{\psi_{i,j}(\frac{p}{s})}{\psi_{i,j}(\frac{q}{s})}\right)^{\frac{1}{p-q}}, \tag{2.21}$$

for $i=1,2,3; j=1,2,3,4$.

We extend (2.21) with limit cases for every of families $\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_3,\mathbf{F}_4$.

FAMILY $\mathbf{F}_1$:

$$M^t_{p,q;s}(A_i,\mathbf{F}_1)=\begin{cases}\left(\dfrac{\psi_{i,1}\left(\frac{p}{s}\right)}{\psi_{i,1}\left(\frac{q}{s}\right)}\right)^{\dfrac{1}{p-q}}, & p\neq q, s\neq 0,\ s\in I;\\ \exp\left(\dfrac{s-2ps}{p(p-s)}-\dfrac{A_i(f_0f_{\frac{p}{s}})}{sA_i(f_{\frac{p}{s}})}\right), & p=q,\ sp(p-s)\neq 0;\\ \exp\left(\frac{1}{s}-\dfrac{A_i(f_0^2)}{2sA_i(f_0)}\right), & p=q=0,\ s\neq 0;\\ \exp\left(-\frac{1}{s}-\dfrac{A_i(f_0f_1)}{2sA_i(f_1)}\right), & p=q=s,\ s\neq 0;\\ \left(\dfrac{\overline{A}_i(g_p)}{\overline{A}_i(g_q)}\right)^{\dfrac{1}{p-q}}, & p\neq q,\ s=0;\\ \exp\left(-\frac{2}{p}-\dfrac{\overline{A}_i(e_1g_p)}{\overline{A}_i(g_p)}\right), & p=q\neq 0,\ s=0;\\ \sqrt{xy}, & p=q=s=0,\end{cases}\tag{2.22}$$

where $i=1,2,3$;

$$g_p(x)=\begin{cases}\frac{e^{px}}{p^2}, & p\neq 0;\\ \frac{x^2}{2}, & p=0\end{cases}$$

and $\overline{A}_i,\ i=1,2,3$ stands for linear functional now acting on $C[\ln a,\ln b]$.

FAMILY $\mathbf{F}_2$:

$$M^t_{p,q;s}(A_i,\mathbf{F}_2)=\begin{cases}\left(\dfrac{\psi_{i,2}\left(\frac{p}{s}\right)}{\psi_{i,2}\left(\frac{q}{s}\right)}\right)^{\dfrac{1}{p-q}}, & p\neq q, s\neq 0,\ s\in I;\\ \exp\left(-\frac{2}{p}-\dfrac{A_i(e_sf_{\frac{p}{s}})}{sA_i(f_{\frac{p}{s}})}\right), & p=q,\ sp\neq 0;\\ \exp\left(\frac{1}{3s}\dfrac{A_i(e_sf_0)}{A_i(f_0)}\right), & p=q=0\ s\neq 0;\end{cases}\tag{2.23}$$

where $i=1,2,3$ and $e_s(x)=x^s$.

FAMILY $\mathbf{F}_3$:

$$M^t_{p,q;s}(A_i,\mathbf{F}_3)=\begin{cases}\left(\dfrac{\psi_{i,3}\left(\frac{p}{s}\right)}{\psi_{i,3}\left(\frac{q}{s}\right)}\right)^{\frac{1}{p-q}}, & p\neq q\neq s,\ s\in I;\\ \exp\left(-\frac{1}{3}\frac{A_i(e_s f_0)}{A_i(f_0)}\right), & p=q=s;\end{cases}\tag{2.24}$$

where $i=1,2,3$.

FAMILY $\mathbf{F}_4$:

$$M^t_{p,q;s}(A_i,\mathbf{F}_4)=\begin{cases}\left(\dfrac{\psi_{i,4}\left(\frac{p}{s}\right)}{\psi_{i,4}\left(\frac{q}{s}\right)}\right)^{\frac{1}{p-q}}, & p\neq q\neq s,\ p,q,s\in I;\\ \exp\left(\frac{-1}{p}-\frac{1}{2\sqrt{ps}}\frac{A_i(e_s f_{\frac{p}{s}})}{A_i(f_{\frac{p}{s}})}\right), & p=q,\ s\in I;\end{cases}\tag{2.25}$$

where $i=1,2,3$.

Remark 2.11. *Let us note that we can give the explicit version for means* (2.19) *are obtained in* [6].

Theorem 2.12. *Let $p\leq u,\ q\leq v;\ p,q,u,v,s\in I$. Then*

$$M^t_{p,q;s}(A_i,\mathbf{F}_j)\leq M^t_{u,v;s}(A_i,\mathbf{F}_j),\tag{2.26}$$

for $i=1,2,3$; $j=1,2,3,4$.

Proof. Follows from Proposition 2.10 and Theorem 2.9. □

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