Available online at http://scik.org

Adv. Inequal. Appl. 2014, 2014:9

ISSN: 2050-7461

GENERALISATIONS OF STEFFENSEN'S INEQUALITY VIA FINK IDENTITY AND RELATED RESULTS

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Abstract. Starting from the result given in [1] where the authors gave the extension of weighted Montgomery identity using Fink identity from [5], we give the generalizations of Steffensen's inequality. Also, we investigate the exponential convexity of differences of the left-hand and right-hand side of these inequalities. Using these differences, we produce new exponentially convex functions. They are used in studying Stolarsky type means.

Keywords: Steffensen inequality, Montgomery identity, Fink identity, *n*-convex functions, *n*-exponentially convexity, means.

2000 AMS Subject Classification: Primary 26D15, Secondary 26D10.

1. INTRODUCTION

Steffensen's inequality is one about which there are many related results and it is still the subject of the investigation by many mathematicians. This inequality was first given and proved by J.F. Steffensen in 1918 in paper [15]. The well-known Steffensen inequality reads:

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Received May 26, 2013

Theorem 1.1. *Suppose that f is decreasing and g is integrable on $[a,b]$ with $0 \le g \le 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int_{b-\lambda}^{b} f(t)dt \le \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt. \tag{1.1}$$

The inequalities are reversed for f increasing.

In [3] P. Cerone generalized Steffensen's inequality for general subintervals:

Theorem 1.2. *Let $f,g : [a,b] \to \mathbb{R}$ be integrable mappings on $[a,b]$ and let f be nonincreasing. Further, let $0 \le g(t) \le 1$ and $\lambda = \int_a^b g(t)dt = d_i - c_i$, where $[c_i,d_i] \subset [a,b]$ for $i = 1,2$ and $d_1 \le d_2$.*

Then the result

$$\int_{c_2}^{d_2} f(t)dt - r(c_2,d_2) \le \int_a^b f(t)g(t)dt \le \int_{c_1}^{d_1} f(t)dt + R(c_1,d_1), \tag{1.2}$$

holds where,

$$r(c_2,d_2) = \int_{d_2}^{b} (f(c_2) - f(t))g(t)dt \ge 0$$

and

$$R(c_1,d_1) = \int_a^{c_1} (f(t) - f(d_1))g(t)dt \ge 0.$$

Remark 1.1. *If we put $c_1 = a$, $d_2 = b$, $d_1 = a+\lambda$ and $c_2 = b-\lambda$ in (1.2), we get (1.1).*

A. M. Fink in [5] obtained the following identity:

$$\begin{aligned} \frac{1}{n}\left(f(x) + \sum_{k=1}^{n-1} F_k(x)\right) &- \frac{1}{b-a}\int_a^b f(t)dt \\ &= \frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)dt, \end{aligned} \tag{1.3}$$

where

$$F_k(x) = \frac{n-k}{k!}\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a},$$

$$k(t,x) = \begin{cases} t-a, & a \le t \le x \le b, \\ t-b, & a \le x < t \le b. \end{cases} \tag{1.4}$$

Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$, and $f':[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Then the Montgomery identity holds [10]:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\int_a^b P(x,t)\,f'(t)\,dt,$$

where $P(x,t)$ is the Peano kernel, defined by

$$P(x,t)=\begin{cases}\frac{t-a}{b-a}, & a\le t\le x,\\ \\ \frac{t-b}{b-a} & x<t\le b.\end{cases}$$

Now, let's suppose $w:[a,b]\to[0,\infty\rangle$ and $W(t)=\int_a^t w(x)\,dx$ for $t\in[a,b]$, $W(t)=0$ for $t<a$ and $W(t)=1$ for $t>b$. The following identity (given by Pečarić in [11]) is the weighted generalization of Montgomery identity:

$$f(x)=\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)\,f(t)\,dt+\int_a^b P_w(x,t)\,f'(t)\,dt, \tag{1.5}$$

where the weighted Peano kernel is

$$P_w(x,t)=\begin{cases}\frac{W(t)}{W(b)}, & a\le t\le x,\\ \\ \frac{W(t)}{W(b)}-1 & x<t\le b.\end{cases} \tag{1.6}$$

In [1], the authors gave the extension of weighted Montgomery identity (1.5) using identity (1.3), and further they obtained some new generalizations of the estimations of the difference of two weighted integral means. First is when $[c,d]\subseteq[a,b]$ and the second when $[a,b]\cap[c,d]=[c,b]$. Other two possible cases, when $[a,b]\cap[c,d]\neq\emptyset$ we simply get by change $a\leftrightarrow c$, $b\leftrightarrow d$.

Theorem 1.3. *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous function on $[a,b]$ for some $n>1$, and let $w:[a,b]\to[0,\infty\rangle$ and $u:[c,d]\to[0,\infty\rangle$. Then if $[a,b]\cap[c,d]\neq\emptyset$ and $x\in[a,b]\cap[c,d]$, we have*

$$\begin{aligned}\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)\,f(t)\,dt &- \frac{1}{\int_c^d u(t)dt}\int_c^d u(t)\,f(t)\,dt-T_{w,n}^{[a,b]}(x)+T_{u,n}^{[c,d]}(x)\\ &= \int_{\min\{a,c\}}^{\max\{b,d\}} K_n(x,y)\,f^{(n)}(y)\,dy,\end{aligned} \tag{1.7}$$

where

$$T_{w,n}^{[a,b]}(x)=\sum_{k=1}^{n-1}F_k^{[a,b]}(x)-\frac{1}{\int_a^b w(t)dt}\sum_{k=1}^{n-1}\int_a^b w(t)\,F_k^{[a,b]}(t)dt,$$

in case $[c,d]\subseteq[a,b]$,

$$K_n(x,y)=\begin{cases}\frac{-1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)\,(t-y)^{n-2}k^{[a,b]}(y,t)dt\right], & y\in[a,c],\\[2ex] \frac{-1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)\,(t-y)^{n-2}k^{[a,b]}(y,t)dt\right] \\ +\frac{1}{(n-2)!(d-c)}\left[\int_c^d P_u(x,t)\,(t-y)^{n-2}k^{[c,d]}(y,t)dt\right], & y\in\langle c,d],\\[2ex] \frac{-1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)\,(t-y)^{n-2}k^{[a,b]}(y,t)dt\right], & y\in\langle d,b],\end{cases}$$

and in case $[a,b]\cap[c,d]=[c,b]$,

$$K_n(x,y)=\begin{cases}\frac{-1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)\,(t-y)^{n-2}k^{[a,b]}(y,t)dt\right], & y\in[a,c],\\[2ex] \frac{-1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)\,(t-y)^{n-2}k^{[a,b]}(y,t)dt\right] \\ +\frac{1}{(n-2)!(d-c)}\left[\int_c^d P_u(x,t)\,(t-y)^{n-2}k^{[c,d]}(y,t)dt\right], & y\in\langle c,b],\\[2ex] \frac{1}{(n-2)!(d-c)}\left[\int_c^d P_u(x,t)\,(t-y)^{n-2}k^{[c,d]}(y,t)dt\right], & y\in\langle b,d].\end{cases}$$

Theorem 1.4. *Assume* (p,q) *is a pair of conjugate exponents, that is* $1\le p,q\le\infty$, $1/p+1/q=1$. *Let* $\left|f^{(n)}\right|^p:[a,b]\to\mathbb{R}$ *be an R-integrable function for some* $n>1$. *Then we have*

$$\left|\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)\,f(t)\,dt-\frac{1}{\int_c^d u(t)dt}\int_c^d u(t)\,f(t)\,dt-T_{w,n}^{[a,b]}(x)+T_{u,n}^{[c,d]}(x)\right|$$

$$(1.8)\qquad \le\left(\int_{\min\{a,c\}}^{\max\{b,d\}}\left|K_n(x,y)\right|^q dy\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p,$$

for every $x\in[a,b]\cap[c,d]$. *The constant* $\left(\int_{\min\{a,c\}}^{\max\{b,d\}}\left|K_n(x,y)\right|^q dy\right)^{1/q}$ *in the inequality (1.8) is sharp for* $1<p\le\infty$ *and the best possible for* $p=1$.

In this note we consider n-convex functions.

Definition 1.1. *Let f be a real-valued function defined on the segment $[a,b]$. The* **divided difference** *of order n of the function f at distinct points $x_0,...,x_n \in [a,b]$, is defined recursively (see* [2], [12]*) by*

$$f[x_i] = f(x_i),\ (i = 0,\dots,n)$$

and

$$f[x_0,\dots,x_n] = \frac{f[x_1,\dots,x_n] - f[x_0,\dots,x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0,\dots,x_n]$ is independent of the order of the points $x_0,\dots,x_n$.
The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x,\dots,x}_{j-times}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \tag{1.9}$$

The notion of ***n*-convexity** goes back to Popoviciu ([14]). We follow the definition given by Karlin ([7]):

Definition 1.2. *A function $f : [a,b] \to \mathbb{R}$ is said to be n-convex on $[a,b]$, $n \geq 0$, if for all choices of $(n+1)$ distinct points in $[a,b]$, $n-th$ order divided difference of f satisfies*

$$f[x_0,...,x_n] \geq 0.$$

In fact, Popoviciu proved that each continuous n-convex function on $[0,1]$ is the uniform limit of the sequence of corresponding Bernstein's polynomials (see for example [12, p. 293]). Also, Bernstein's polynomials of continuous n-convex function are also n-convex functions. Therefore, when stating our results for a continuous n-convex function f, without any loss in generality we assume that $f^{(n)}$ exists and is continuous.

Many related results, as well as some important inequalities due to Favard, Berwald and Steffensen can be found in [8].

The aim of this paper is to give the generalizations of Steffensen's inequality (1.1) using the Theorem 1.3. Also, we will give mean value theorems using that inequalities. We will introduce the notion of n-exponentially convex functions and deduce a method of producing n-exponentially convex functions use some known families of functions of the same type.

2. Generalization of Steffensen's inequality via Fink identity

Theorem 2.1. *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be n-convex on $[a,b]$ for some $n>1$ and let $w:[a,b]\to[0,\infty\rangle$ and $u:[c,d]\to[0,\infty\rangle$. Then if $[a,b]\cap[c,d]\neq\emptyset$, $x\in[a,b]\cap[c,d]$ and*

$$K_n(x,y)\geq 0, \tag{2.1}$$

we have

$$\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)\,dt - T_{w,n}^{[a,b]}(x) \geq \frac{1}{\int_c^d u(t)dt}\int_c^d u(t)f(t)\,dt - T_{u,n}^{[c,d]}(x). \tag{2.2}$$

If the reversed inequality in (2.1) is valid, then the reversed inequality in (2.2) is also valid.

Proof. Directly from Theorem 1.3. □

Remark 2.1. *For $u(t)=1$ and $\lambda=\int_a^b w(t)dt=d-c$ in inequality (2.2) we get the inequality related to left-hand side of inequality (1.2).*

For $a\leftrightarrow c,\ b\leftrightarrow d, w\leftrightarrow u$, $u(t)=1$ and $\lambda=\int_a^b w(t)dt=d-c$ in inequality (2.2) we get the inequality related to right-hand side of inequality (1.2).

Corollary 2.1. *Let $f:[a,b]\cup[a,a+\lambda]\to\mathbb{R}$ be n-convex on $[a,b]\cup[a,a+\lambda]$ for some $n>1$ and $w:[a,b]\to[0,\infty\rangle$. Then if $\lambda>0$, $x\in[a,b]\cap[a,a+\lambda]$ and*

$$K_n(x,y)\geq 0, \tag{2.3}$$

we have

$$\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)\,dt - T_{w,n}^{[a,b]}(x) \geq \frac{1}{\lambda}\int_a^{a+\lambda} f(t)\,dt - T_{1,n}^{[a,a+\lambda]}(x), \tag{2.4}$$

where, in case $(a+\lambda)\leq b$,

$$K_n(x,y)=\begin{cases} \frac{-1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)(t-y)^{n-2}k^{[a,b]}(y,t)dt\right] \\ +\frac{1}{\lambda(n-2)!}\left[\int_a^{a+\lambda} P_1(x,t)(t-y)^{n-2}k^{[a,a+\lambda]}(y,t)dt\right], & y\in[a,a+\lambda], \\ \\ \frac{-1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)(t-y)^{n-2}k^{[a,b]}(y,t)dt\right], & y\in\langle a+\lambda,b], \end{cases}$$

and in case $(a+\lambda) \geq b$,

$$K_n(x,y) = \begin{cases} \frac{-1}{(n-2)!(b-a)} \left[\int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t)dt \right] \\ + \frac{1}{\lambda(n-2)!} \left[\int_a^{a+\lambda} P_1(x,t)(t-y)^{n-2} k^{[a,a+\lambda]}(y,t)dt \right], \quad y \in [a,b], \\ \\ \frac{1}{\lambda(n-2)!} \left[\int_a^{a+\lambda} P_1(x,t)(t-y)^{n-2} k^{[a,a+\lambda]}(y,t)dt \right], \quad y \in \langle b, a+\lambda]. \end{cases}$$

If the reversed inequality in (2.3) is valid, then the reversed inequality in (2.4) is also valid.

Proof. We put $c = a,\ d = a+\lambda$ and $u(t) = 1$ in inequality (2.2) to get inequality (2.4). □

Corollary 2.2. *Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $1/p + 1/q = 1$. *Let* $\left| f^{(n)} \right|^p : [a,b] \to \mathbb{R}$ *be an R-integrable function for some* $n > 1$. *Then we have*

$$\left| \frac{1}{\int_a^b w(t)dt} \int_a^b w(t) f(t)\,dt - \frac{1}{\lambda} \int_a^{a+\lambda} f(t)\,dt - T_{w,n}^{[a,b]}(x) + T_{1,n}^{[a,a+\lambda]}(x) \right|$$

$$\leq \left(\int_a^{\max\{b,a+\lambda\}} |K_n(x,y)|^q\,dy \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p, \tag{2.5}$$

for every $x \in [a,b] \cap [a,a+\lambda]$. *The constant* $\left(\int_a^{\max\{b,a+\lambda\}} |K_n(x,y)|^q\,dy \right)^{1/q}$ *in the inequality (2.5) is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Proof. We put $c = a,\ d = a+\lambda$ and $u(t) = 1$ in inequality (1.8) to get inequality (2.5). □

Remark 2.2. *For* $n = 1$ *and* $\lambda \leq (b-a)$, $K_1(x,y)$ *becomes:*

$$K_1(x,y) = \begin{cases} \frac{y-a}{\lambda} - \frac{1}{\int_a^b w(t)dt} \int_a^y w(t)dt, & y \in [a,a+\lambda], \\ \frac{1}{\int_a^b w(t)dt} \int_y^b w(t)dt, & y \in \langle a+\lambda, b]. \end{cases}$$

So, if $\lambda \int_a^y w(t)dt \leq (y-a) \int_a^b w(t)dt$ *and* $f'(x) \geq 0$, *inequality (2.4) becomes*

$$\frac{1}{\int_a^b w(t)dt} \int_a^b w(t) f(t)\,dt \geq \frac{1}{\lambda} \int_a^{a+\lambda} f(t)\,dt,$$

which is the right-hand side of reversed generalized Steffensen inequality proved by Mitrinović and Pečarić in [9].

Remark 2.3. *For $n=1$ and $\int_a^b w(t)dt=\lambda$, $K_1(x,y)$ becomes:*

$$K_1(x,y)=\begin{cases} \frac{1}{\lambda}\int_a^y (1-w(t))dt, & y\in[a,a+\lambda],\\ \frac{1}{\lambda}\int_y^b w(t)dt, & y\in\langle a+\lambda,b].\end{cases}$$

So, if $w(t)\le 1$ and $f'(x)\ge 0$, inequality (2.4) becomes

$$\int_a^b w(t)f(t)\,dt\ge\int_a^{a+\lambda} f(t)\,dt,$$

which is the right-hand side of reversed Steffensen inequality (1.1).

Corollary 2.3. *Let $f:[a,b]\cup[b-\lambda,b]\to\mathbb{R}$ be n-convex on $[a,b]\cup[b-\lambda,b]$ for some $n>1$ and $w:[a,b]\to[0,\infty\rangle$. Then if $\lambda>0$, $x\in[a,b]\cap[b-\lambda,b]$ and*

$$K_n(x,y)\ge 0, \tag{2.6}$$

we have

$$\frac{1}{\lambda}\int_{b-\lambda}^b f(t)\,dt-T_{1,n}^{[b-\lambda,b]}(x)\ge\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)\,dt-T_{w,n}^{[a,b]}(x), \tag{2.7}$$

where, in case $(b-\lambda)\le a$,

$$K_n(x,y)=\begin{cases}\frac{-1}{\lambda(n-2)!}\left[\int_{b-\lambda}^b P_1(x,t)(t-y)^{n-2}k^{[b-\lambda,b]}(y,t)dt\right], & y\in[b-\lambda,a],\\ \frac{-1}{\lambda(n-2)!}\left[\int_{b-\lambda}^b P_1(x,t)(t-y)^{n-2}k^{[b-\lambda,b]}(y,t)dt\right] & \\ +\frac{1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)(t-y)^{n-2}k^{[a,b]}(y,t)dt\right], & y\in\langle a,b],\end{cases}$$

and in case $a\le(b-\lambda)$,

$$K_n(x,y)=\begin{cases}\frac{1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)(t-y)^{n-2}k^{[a,b]}(y,t)dt\right] & y\in[a,b-\lambda]\\ \frac{-1}{\lambda(n-2)!}\left[\int_{b-\lambda}^b P_1(x,t)(t-y)^{n-2}k^{[b-\lambda,b]}(y,t)dt\right] & \\ +\frac{1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)(t-y)^{n-2}k^{[a,b]}(y,t)dt\right], & y\in\langle b-\lambda,b].\end{cases}$$

If the reversed inequality in (2.6) is valid, then the reversed inequality in (2.7) is also valid.

Proof. We change $a\leftrightarrow c,\ b\leftrightarrow d, w\leftrightarrow u$, and put $c=b-\lambda,\ d=b,\ u(t)=1$ in inequality (2.2) to get inequality (2.7). $\square$

Corollary 2.4. *Assume (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $1/p+1/q=1$. Let $\left|f^{(n)}\right|^p : [a,b] \to \mathbb{R}$ be an R-integrable function for some $n>1$. Then we have*

$$\left|\frac{1}{\lambda}\int_{b-\lambda}^{b} f(t)\,dt - \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)\,dt - T_{1,n}^{[b-\lambda,b]}(x) + T_{w,n}^{[a,b]}(x)\right|$$

$$\le \left(\int_{\min\{a,b-\lambda\}}^{b} |K_n(x,y)|^q\,dy\right)^{\frac{1}{q}} \left\|f^{(n)}\right\|_p, \tag{2.8}$$

for every $x \in [a,b] \cap [b-\lambda,b]$. The constant $\left(\int_{\min\{a,b-\lambda\}}^{b} |K_n(x,y)|^q\,dy\right)^{1/q}$ in the inequality (2.8) is sharp for $1<p\le\infty$ and the best possible for $p=1$.

Proof. We change $a \leftrightarrow c,\ b \leftrightarrow d, w \leftrightarrow u$, and put $c=b-\lambda,\ d=b,\ u(t)=1$ in inequality (1.8) to get inequality (2.8). □

Remark 2.4. *For $n=1$ and $\lambda \le (b-a)$, $K_1(x,y)$ becomes:*

$$K_1(x,y) = \begin{cases} \frac{1}{\int_a^b w(t)dt}\int_a^y w(t)dt, & y \in [a,b-\lambda], \\ \frac{b-y}{\lambda} - \frac{1}{\int_a^b w(t)dt}\int_y^b w(t)dt, & y \in \langle b-\lambda,b]. \end{cases}$$

So, if $\lambda\int_y^b w(t)dt \le (b-y)\int_a^b w(t)dt$ and $f'(x)\ge 0$, inequality (2.7) becomes

$$\frac{1}{\lambda}\int_{b-\lambda}^{b} f(t)\,dt \ge \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)\,dt,$$

which is the left-hand side of reversed generalized Steffensen inequality proved by Mitrinović and Pečarić in [9].

Remark 2.5. *For $n=1$ and $\int_a^b w(t)dt=\lambda$, $K_1(x,y)$ becomes:*

$$K_1(x,y) = \begin{cases} \frac{1}{\lambda}\int_a^y w(t)dt, & y \in [a,b-\lambda], \\ \frac{1}{\lambda}\int_y^b (1-w(t))dt, & y \in \langle b-\lambda,b]. \end{cases}$$

So, if $w(t)\le 1$ and $f'(x)\ge 0$, inequality (2.7) becomes

$$\int_{b-\lambda}^{b} f(t)\,dt \ge \int_a^b w(t)f(t)\,dt,$$

which is the left-hand side of reversed Steffensen inequality (1.1).

3. n-EXPONENTIAL CONVEXITY OF STEFFENSEN'S INEQUALITY BY FINK IDENTITY

Motivated by the inequalities (2.2), (2.4) and (2.7) we define functionals $\Phi_1(f)$, $\Phi_2(f)$ and $\Phi_3(f)$ by

$$\Phi_1(f) = \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)\,dt - T_{w,n}^{[a,b]}(x) - \frac{1}{\int_c^d u(t)dt}\int_c^d u(t)f(t)\,dt + T_{u,n}^{[c,d]}(x), \tag{3.1}$$

$$\Phi_2(f) = \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)\,dt - T_{w,n}^{[a,b]}(x) - \frac{1}{\lambda}\int_a^{a+\lambda} f(t)\,dt + T_{1,n}^{[a,a+\lambda]}(x) \tag{3.2}$$

and

$$\Phi_3(f) = \frac{1}{\lambda}\int_{b-\lambda}^b f(t)\,dt - T_{1,n}^{[a,b]}(x) - \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)dt f(t)\,dt + T_{w,n}^{[a,b]}(x). \tag{3.3}$$

Also, we define $I_1 = [a,b]\cup[c,d]$, $I_2 = [a,b]\cup[a,a+\lambda]$, $I_3 = [a,b]\cup[b-\lambda,b]$, $\tilde{I}_1 = [a,b]\cap[c,d]$, $\tilde{I}_2 = [a,b]\cap[a,a+\lambda]$ and $\tilde{I}_3 = [a,b]\cap[b-\lambda,b]$.

Theorem 3.1. *Let $f : I_i \to \mathbb{R}$ ($i = 1,2,3$) be such that $f \in C^n(I_i)$, $n > 1$. If for $x \in \tilde{I}_i$ inequalities in (2.1) ($i = 1$), (2.3) ($i = 2$) and (2.6) ($i = 3$) hold, then there exists $\xi \in \tilde{I}_i$ such that*

$$\Phi_i(f) = f^{(n)}(\xi)\cdot\Phi_i(\varphi), \tag{3.4}$$

where $\varphi(x) = \frac{x^n}{n!}$.

Proof. Let us denote $m = \min f^{(n)}$ and $M = \max f^{(n)}$. We first consider the following function $\phi_1(x) = \frac{Mx^n}{n!} - f(x)$. Then $\phi_1^{(n)}(x) = M - f^{(n)}(x) \geq 0$, $x \in \tilde{I}_i$, $i = 1,2,3$, so ϕ_1 is a n-convex function. Similarly, a function $\phi_2(x) = f(x) - \frac{mx^n}{n!}$ is a n-convex function. Now, we use inequalities from Theorem 2.1, Corollary 2.1 and Corollary 2.3 for n-convex functions ϕ_1 and ϕ_2. So, we can conclude that there exists $\xi \in \tilde{I}_i$, $i = 1,2,3$, that we are looking for in (3.4). □

Corollary 3.1. *Let $f,h : I_i \to \mathbb{R}$, $i = 1,2,3$, such that $f,h \in C^n(I_i)$. If for $x \in \tilde{I}_i$ inequalities in (2.1), (2.3) and (2.6) hold, then there exists $\xi \in \tilde{I}_i$ such that*

$$\frac{\Phi_i(f)}{\Phi_i(h)} = \frac{f^{(n)}(\xi)}{h^{(n)}(\xi)},\ i = 1,2,3, \tag{3.5}$$

provided that the denominator of the left-hand side is non-zero.

Proof. We use the following standard technique: Let us define the linear functional $L(\chi) = \Phi_i(\chi),\ i = 1,2,3$. Next, we define $\chi(t) = f(t)L(h) - h(t)L(f)$. According to Theorem 3.1, applied on χ, there exists $\xi \in \tilde{I}_i$ so that

$$L(\chi) = \chi^{(n)}(\xi)\Phi_i(\varphi),\ \varphi(x) = \frac{x^n}{n!},\ i = 1,2,3.$$

From $L(\chi) = 0$, it follows $f^{(n)}(\xi)L(h) - h^{(n)}(\xi)L(f) = 0$ and (3.5) is proved. □

Now, let us recall some definitions and facts about exponentially convex functions (see [6]):

Definition 3.1. *A function $\psi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, \psi\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

hold for all choices $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and all choices $x_1, \ldots, x_n \in I$.

A function $\psi : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 3.1. *It is clear from the definition that* 1*-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex function in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N},\ k \leq n$.*

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition:

Proposition 3.1. *If ψ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is positive semi-definite matrix for all $k \in \mathbb{N},\ k \leq n$. Particularly,* $\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0$ *for all $k \in \mathbb{N},\ k \leq n$.*

Definition 3.2. *A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.*

A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 3.2. *It is known (and easy to show) that $\psi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if*

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geq 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

Proposition 3.2. *If f is a convex function on I and if $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, the inequality is reversed.

We use an idea from [6] to give an elegant method of producing an n-exponentially convex functions and exponentially convex functions applying the above functionals on a given family with the same property (see [13]):

Theorem 3.2. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $I_i,\ i = 1,2,3$, in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0, \ldots, z_l]$ is n-exponentially convex in the Jensen sense on J for every $(l+1)$ mutually different points $z_0, \ldots, z_l \in I_i,\ i = 1,2,3$. Let $\Phi_i(f),\ i = 1,2,3$, be linear functional defined as in (3.1)-(3.3). Then $s \mapsto \Phi_i(f_s)$ is an n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R},\ i = 1, \ldots, n$ and $s_i \in J,\ i = 1, \ldots, n$, we define the function

$$g(z) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i+s_j}{2}}(z).$$

Using the assumption that the function $s \mapsto f_s[z_0, \ldots, z_l]$ is n-exponentially convex in the Jensen sense, we have

$$g[z_0, \ldots, z_l] = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i+s_j}{2}}[z_0, \ldots, z_l] \geq 0,$$

which in turn implies that g is a l-convex function on J, so it is $\Phi_k(g) \geq 0,\ k = 1,2,3$, hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_k \left(f_{\frac{s_i+s_j}{2}} \right) \geq 0.$$

We conclude that the function $s \mapsto \Phi_k(f_s)$ is n-exponentially convex on J in the Jensen sense.

If the function $s \mapsto \Phi_k(f_s)$ is also continuous on J, then $s \mapsto \Phi_k(f_s)$ is n-exponentially convex by definition. □

The following corollaries are an immediate consequences of the above theorem:

Corollary 3.2. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0, \ldots, z_l]$ is exponentially convex in the Jensen sense on J for every $(l+1)$ mutually different points $z_0, \ldots, z_l \in I_i$. Let $\Phi_i(f),\ i = 1,2,3,$ be linear functional defined as in (3.1)-(3.3). Then $s \mapsto \Phi_i(f_s)$ is an exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is exponentially convex on J.*

Corollary 3.3. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0, \ldots, z_l]$ is 2-exponentially convex in the Jensen sense on J for every $(l+1)$ mutually different points $z_0, \ldots, z_l \in I_i$. Let $\Phi_i(f),\ i = 1,2,3,$ be linear functional defined as in (3.1)-(3.3). Then the following statements hold:*

(i) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is 2-exponentially convex function on J. If $s \mapsto \Phi_i(f_s)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:*

$$[\Phi_i(f_s)]^{t-r} \leq [\Phi_i(f_r)]^{t-s} [\Phi_i(f_t)]^{s-r} \tag{3.6}$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is strictly positive and differentiable on J, then for every $s, q, u, v \in J$, such that $s \leq u$ and $q \leq v$, we have*

$$\mu_{s,q}(\Phi_i, \Upsilon) \leq \mu_{u,v}(\Phi_i, \Upsilon), \tag{3.7}$$

where

$$\mu_{s,q}(\Phi_i,\Upsilon)=\begin{cases}\left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(\frac{\frac{d}{ds}\Phi_i(f_s)}{\Phi_i(f_q)}\right), & s=q,\end{cases} \tag{3.8}$$

for $f_s,f_q\in\Upsilon$.

Proof. (i) This is an immediate consequence of Theorem 3.2 and Remark 3.2.

(ii) Since by (i) the function $s\mapsto\Phi_i(f_s),\ i=1,2,3$, is log-convex on J, that is, the function $s\mapsto\log\Phi_i(f_s)$ is convex on J. So, we get

$$\frac{\log\Phi_i(f_s)-\log\Phi_i(f_q)}{s-q}\leq\frac{\log\Phi_i(f_u)-\log\Phi_i(f_v)}{u-v}, \tag{3.9}$$

for $s\leq u, q\leq v, s\neq q, u\neq v$, and there form conclude that

$$\mu_{s,q}(\Phi_i,\Upsilon)\leq\mu_{u,v}(\Phi_i,\Upsilon).$$

Cases $s=q$ and $u=v$ follows from (3.9) as limit cases.

□

Remark 3.3. *Note that the results from above theorem and corollaries still hold when two of the points $z_0,\ldots,z_l\in I_i$ coincide, say $z_1=z_0$, for a family of differentiable functions f_s such that the function $s\mapsto f_s[z_0,\ldots,z_l]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $(l+1)$ points coincide for a family of l differentiable functions with the same property. The proofs are obtained by (1.9) and suitable characterization of convexity.*

4. Applications to Stolarsky type means

In this section, we present several families of functions which fulfil the conditions of Theorem 3.2, Corollary 3.2, Corollary 3.3 and Remark 3.3. This enable us to construct a large families of functions which are exponentially convex. For a discussion related to this problem see [4].

Example 4.1. *Consider a family of functions*

$$\Omega_1=\{l_s:\mathbb{R}\to\mathbb{R}:s\in\mathbb{R}\}$$

defined by

$$l_s(x) = \begin{cases} \frac{e^{sx}}{s^n}, & s \neq 0, \\ \frac{x^n}{n!}, & s = 0. \end{cases}$$

We have $\frac{d^n l_s}{dx^n}(x) = e^{sx} > 0$ *which shows that* l_s *is n-convex on* $\mathbb{R}$ *for every* $s \in \mathbb{R}$ *and* $s \mapsto \frac{d^n l_s}{dx^n}(x)$ *is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 3.2 we also have that* $s \mapsto l_s[z_0,\dots,z_n]$ *is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 3.2 we conclude that* $s \mapsto \Phi_i(l_s), i = 1,2,3,$ *are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping* $s \mapsto l_s$ *is not continuous for* $s = 0$*), so it is exponentially convex.*

For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_1),\ i = 1,2,3,$ *from (3.8), becomes*

$$\mu_{s,q}(\Phi_i,\Omega_1) = \begin{cases} \left(\frac{\Phi_i(l_s)}{\Phi_i(l_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(id\cdot l_s)}{\Phi_i(l_s)} - \frac{n}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{1}{n+1}\frac{\Phi_i(id\cdot l_0)}{\Phi_i(l_0)}\right), & s = q = 0. \end{cases}$$

Now, using (3.7) it is monotonous function in parameters s and q.

We observe here that $\left(\frac{\frac{d^n l_s}{dx^n}}{\frac{d^n l_q}{dx^n}}\right)^{\frac{1}{s-q}}(\ln x) = x$ *so using Corollary 3.1 it follows that:*

$$M_{s,q}(\Phi_i,\Omega_1) = \ln \mu_{s,q}(\Phi_i,\Omega_1),\ i = 1,2,3$$

satisfy

$$\min\{a,c,b-\lambda\} \leq M_{s,q}(\Phi_i,\Omega_1) \leq \max\{b,d,a+\lambda\},\ i = 1,2,3.$$

So, $M_{s,q}(\Phi_i,\Omega_1)$ *is monotonic mean.*

Example 4.2. *Consider a family of functions*

$$\Omega_2 = \{f_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)\cdots(s-n+1)}, & s \notin \{0,1,\dots,n-1\}, \\ \frac{x^j \ln x}{(-1)^{n-1-j} j!(n-1-j)!}, & s = j \in \{0,1,\dots,n-1\}. \end{cases}$$

Here, $\frac{d^n f_s}{dx^n}(x) = x^{s-n} = e^{(s-n)\ln x} > 0$ *which shows that* f_s *is n-convex for* $x > 0$ *and* $s \mapsto \frac{d^n f_s}{dx^n}(x)$ *is exponentially convex by definition. Arguing as in Example 4.1 we get that the mappings*

$s \mapsto \Phi_i(f_s), i = 1,2,3$ are exponentially convex. In this case we assume that $I_i \in \mathbb{R}^+$. Function (3.8) is now equal to:

$$\mu_{s,q}(\Phi_i,\Omega_2) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\Phi_i(f_0 f_s)}{\Phi_i(f_s)} + \sum_{k=0}^{n-1}\frac{1}{k-s}\right), & s = q \notin \{0,1,\ldots,n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\Phi_i(f_0 f_s)}{2\Phi_i(f_s)} + \sum_{\substack{k=0 \\ k\neq s}}^{n-1}\frac{1}{k-s}\right), & s = q \in \{0,1,\ldots,n-1\}. \end{cases}$$

We observe that $\left(\frac{\frac{d^n f_s}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{s-q}}(x) = x$, so if Φ_i $(i = 1,2,3)$ are positive, then Corollary 3.1 yield that there exist some $\xi \in \tilde{I}_i, i = 1,2,3$ such that

$$\xi^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)},\ i = 1,2,3.$$

Since the function $\xi \mapsto \xi^{s-q}$ is invertible for $s \neq q$, we then have

$$\min\{a,c,b-\lambda\} \leq \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \leq \max\{b,d,a+\lambda\},\ i = 1,2,3, \tag{4.1}$$

which shows that $\mu_{s,q}(\Phi_i,\Omega_2), i = 1,2,3$, is mean.

Example 4.3. *Consider a family of functions*

$$\Omega_3 = \{h_s : (0,\infty) \to \mathbb{R} : s \in (0,\infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{(-\ln s)^n}, & s \neq 1 \\ \frac{x^n}{n!}, & s = 1. \end{cases}$$

Since $\frac{d^n h_s}{dx^n}(x) = s^{-x}$ is the Laplace transform of a non-negative function (see [16]) it is exponentially convex. Obviously h_s are n-convex functions for every $s > 0$. For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_3), i = 1,2,3$, in this case for $I_i \in \mathbb{R}^+$, from (3.8) becomes

$$\mu_{s,q}(\Phi_i,\Omega_3) = \begin{cases} \left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot h_s)}{s\Phi_i(h_s)} - \frac{n}{s\ln s}\right), & s = q \neq 1, \\ \exp\left(-\frac{1}{n+1}\frac{\Phi_i(id \cdot h_1)}{\Phi_i(h_1)}\right), & s = q = 1. \end{cases}$$

This is monotone function in parameters s and q by (3.7). Using Corollary 3.1 it follows that

$$M_{s,q}(\Phi_i,\Omega_3) = -L(s,q)\ln \mu_{s,q}(\Phi_i,\Omega_3),\ i = 1,2,3$$

satisfy

$$\min\{a,c,b-\lambda\} \leq M_{s,q}(\Phi_i,\Omega_3) \leq \max\{b,d,a+\lambda\},\ i=1,2,3.$$

So $M_{s,q}(\Phi_i,\Omega_3)$ *is monotonic mean.* $L(s,q)$ *is logarithmic mean defined by*

$$L(s,q) = \begin{cases} \frac{s-q}{\log s - \log q}, & s \neq q \\ s, & s = q. \end{cases}$$

Example 4.4. *Consider a family of functions*

$$\Omega_4 = \{k_s : (0,\infty) \to \mathbb{R} : s \in (0,\infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{(-\sqrt{s})^n}.$$

Since $\frac{d^n k_s}{dx^n}(x) = e^{-x\sqrt{s}}$ *is the Laplace transform of a non-negative function (see* [16]*) it is exponentially convex. Obviously* k_s *are* n*-convex functions for every* $s > 0$. *For this family of functions,* $\mu_{s,q}(\Phi_i,\Omega_4), i=1,2,3,$ *in this case for* $I_i \in \mathbb{R}^+$, *from (3.8) becomes*

$$\mu_{s,q}(\Phi_i,\Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id\cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{n}{2s}\right), & s = q. \end{cases}$$

This is monotone function in parameters s *and* q *by (3.7). Using Corollary 3.1 it follows that*

$$M_{s,q}(\Phi_i,\Omega_4) = -(\sqrt{s}+\sqrt{q}) \ln \mu_{s,q}(\Phi_i,\Omega_4),\ i=1,2,3$$

satisfy

$$\min\{a,c,b-\lambda\} \leq M_{s,q}(\Phi_i,\Omega_4) \leq \max\{b,d,a+\lambda\},\ i=1,2,3.$$

So $M_{s,q}(\Phi_i,\Omega_4)$ *is monotonic mean.*

Acknowledgements. The research of the authors was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grant $117-1170889-0888$ (first and third author) and $114-0000000-3145$ (second author).

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