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NORMAL VARIANCE-MEAN MIXTURES (I) AN INEQUALITY BETWEEN SKEWNESS AND KURTOSIS

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Abstract: It is shown that the ratio of squared skewness to kurtosis for the normal variance-mean mixture model is bounded above by the same ratio for the mixing distribution. Illustrations include the generalized hyperbolic distribution, the normal tempered stable distribution and a mixture model with log-normal mixing distribution.

Keywords: skewness and kurtosis inequality; generalized hyperbolic; normal tempered stable; normal-inverse Gaussian; hyperbolic; variance-gamma; hyperbolic skew t

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1. Introduction

The topic of moment inequalities is prominent in Probability and Statistics. For example, such inequalities determine the conditions under which random variables on a given range with given moments exist. For given moments up to the order four, the solution to this existence problem is found in Jansen et al. (1986), Hürlimann (2008), Theorem I.4.1. A general proof for the existence of random variables with known moments up to a given order is in De Vylder (1996), II.3.3. It is important to discuss the relationship between skewness and kurtosis in more specific but still general classes of distributions. From a practical point of view, the application of such inequalities to statistical inference is useful because it yields

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necessary conditions under which a given statistical model can be fitted to data. In the realm of Quantitative Finance, where skewness and kurtosis play a key role, one is interested in large classes of non-Gaussian distributions, which are able to supersede the ubiquitous Black-Scholes model. A first choice is the normal variance-mean (NVM) mixture model, which has even been proposed as theoretical foundation for a semi-parametric approach to financial modelling (e.g. Bingham and Kiesel (2001)). In particular, the NVM model includes two five parameter families of distributions, namely the generalized hyperbolic (GH) distribution and the normal tempered stable (NTS) distribution. Important members of the GH distribution are the normal-inverse Gaussian (NIG), the hyperbolic (HYP), the variance-gamma (VG) and the hyperbolic skew t (HST). The NTS family also includes the NIG distribution. As a main result, we show that the ratio of squared skewness to kurtosis for the NVM model is bounded above by the same ratio for the mixing distribution. A short account of the content follows.

Section 2 recalls the normal variance-mean (NVM) mixture model and briefly discusses the moment equations associated to it. Section 3 derives the general inequality between skewness and kurtosis for the NVM class. Section 4 is devoted to selected examples and a comparative study of such inequalities.

2. Moment equations for the normal variance-mean mixture model

The normal variance-mean (NVM) random variable is defined to be the mixture of a normal random variable of the type (e.g. Barndorff-Nielsen et al. (1982))

$$X = \upsilon + \beta \cdot W + \sqrt{W} \cdot Z, \quad \upsilon, \beta \in R, \tag{2.1}$$

where $Z \sim N(0,1)$ is a standard normal random variable, W is a non-negative mixing random variable with cumulant generating function (cgf) $C_W(t)$, and Z, W are independent. We assume that the first four cumulants of W exist and summarize them into a vector $\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$. A short hand notation for the random variable (2.1) is

$X \sim NVM(\upsilon, \beta, \kappa)$. One knows that the cgf of the NVM model is given by (e.g. Feller (1971), Section II.5)

$$C_X(t) = \upsilon \cdot t + C_W(\beta \cdot t + \tfrac{1}{2}t^2). \tag{2.2}$$

The mean, standard deviation, skewness and excess kurtosis of X are denoted throughout by μ, σ, S, K. Through differentiation of the cgf (2.2) one obtains the moment equations

$$\begin{aligned} &\mu = \upsilon + \kappa_1\beta, \quad \sigma^2 = \kappa_1 + \kappa_2\beta^2, \\ &S\sigma^3 = 3\kappa_2\beta + \kappa_3\beta^3, \quad K\sigma^4 = 3\kappa_2 + 6\kappa_3\beta^2 + \kappa_4\beta^4. \end{aligned} \tag{2.3}$$

A concrete specification of the cgf $C_W(t)$ and the vector $\kappa = \kappa(\theta)$ depends upon a multi-parameter vector $\theta = (\theta_1, \theta_2, ..., \theta_m),\ m \ge 2$, and equivalently to the above one writes $X \sim NVM(\upsilon, \beta, \theta)$. Since the degree of freedom of the system (2.3) is $m-2$, a solution to it will necessarily depend upon $m-2$ of the parameters θ_i, say θ_i $i = 1,...,m-2$. For short-hand notation we set $\alpha = (\theta_1, \theta_2, ..., \theta_{m-2})$. In applications, one parameter, say $\theta_m > 0$, will be a scale parameter such that the distribution and cgf of W/θ_m is independent of θ_m. We assume this and rename the remaining parameters as $\gamma = \theta_{m-1}, \delta^2 = \theta_m$. The moment problem for the NVM model consists to find necessary and sufficient conditions so that the system of equations (2.3), with known μ, σ, S, K and arbitrary but fixed α in the feasible parameter space, has a unique solution $(\upsilon, \beta, \gamma, \delta)$. In the following, a simple general strategy to solve (2.3) is presented.

First of all, taking into account the mean equation, it suffices to determine (β, γ, δ). Then, one has $\upsilon = \mu - \kappa_1\beta$. Let $\rho = (\kappa_2/\kappa_1) \cdot \beta^2 > 0$ be an auxiliary unknown parameter chosen such that the variance equation reads $(1+\rho)\cdot\kappa_1 = \sigma^2$. Inserting into the squared skewness equation, one finds the expression

$$S^2(1+\rho)^3=\frac{\kappa_2}{\kappa_1^2}\rho(3+\frac{\kappa_1\kappa_3}{\kappa_2^2}\rho)^2. \tag{2.4}$$

Similarly, the kurtosis equation can be rewritten as

$$K(1+\rho)^2=3\frac{\kappa_2}{\kappa_1^2}(1+2\frac{\kappa_1\kappa_3}{\kappa_2^2}\rho+\frac{\kappa_1^2\kappa_4}{3\kappa_2^3}\rho^2). \tag{2.5}$$

Now, consider the functions of the parameter vector $\theta=(\alpha,\gamma,\delta)$ defined by

$$Q_2=\frac{\kappa_2}{\kappa_1^2},\quad Q_3=\frac{\kappa_3}{\kappa_1\kappa_2},\quad Q_4=\frac{\kappa_4}{3\kappa_2^2}. \tag{2.6}$$

Since δ is a scale parameter, these functions depend only on (α,γ) and (2.4), (2.5) is equivalent to the following system of two equations in the unknowns (γ,ρ):

$$S^2Q_2(1+\rho)^3=\rho\cdot(3Q_2+Q_3\rho)^2,\quad K(1+\rho)^2=3\cdot(Q_2+2Q_3\rho+Q_4\rho^2). \tag{2.7}$$

In general, the second equation is quadratic in ρ and it has the unique positive solution

$$\rho=\frac{K-3Q_3+\sqrt{9(Q_3^2-Q_2Q_4)+3(Q_2+Q_4-2Q_3)K}}{3Q_4-K},\quad 3Q_2\le K\le 3Q_4. \tag{2.8}$$

Insert this into the first equation in (2.7) to get an implicit equation for γ (by fixed α). Therefore, the NVM moment problem reduces to find the necessary and sufficient conditions such that the latter non-linear equation has a unique solution. In case γ has been found numerically, the remaining parameters (β,δ) are obtained from the definition of the parameter ρ and the variance equation as

$$\delta = \frac{\sigma}{\sqrt{(1+\rho)\tilde{\kappa}_1}}, \quad \beta = \operatorname{sgn}(S)\cdot\sqrt{\frac{\kappa_1}{\kappa_2}\rho}\,. \tag{2.9}$$

This strategy yields even closed-form solutions for the variance-gamma (VG) distribution (reformulation of Theorem 3.1 in Hürlimann (2013)), as well as for the normal inverse Gaussian (NIG) distribution. A more complex example, for which this strategy is successful, is the generalized skew t (GST) distribution (see Proposition 2.5 in Ghysels and Wang (2011)). A further discussion of the general moment problem is postponed to later. The focus of the present note is solely on a simple general relationship between skewness and kurtosis in the NVM model.

3. An inequality between squared skewness and kurtosis

Under the technical conditions of Lemma 3.1 below, the ratio of squared skewness to kurtosis for the NVM model is bounded above by the same ratio for the mixing distribution. To derive this upper bound, let us divide the first and second equations in (2.7) to get the expression

$$\begin{gathered}\frac{S^2}{K} = \frac{1}{3Q_2} f(\rho), \quad f(\rho) = \frac{\rho}{1+\rho} g(\rho), \quad g(\rho) = \frac{p(\rho)^2}{q(\rho)} \\ p(\rho) = 3Q_2 + Q_3\rho, \quad q(\rho) = Q_2 + 2Q_3\rho + Q_4\rho^2.\end{gathered} \tag{3.1}$$

To derive the maximum of this ratio, an analysis of the function $f(\rho)$ is required.

Lemma 3.1. If the quantities $A = 3Q_3 - 2Q_2$ and $B = Q_3Q_4 - 3Q_2Q_4 + Q_3^2$ are non-negative, then the function $f(\rho)$ is strictly monotone increasing for all $\rho > 0$.

Proof. The derivative of the function $f(\rho)$ is of the form

$$f'(\rho)=\frac{1}{(1+\rho)^2}g(\rho)+\frac{\rho}{1+\rho}g'(\rho),\quad g'(\rho)=-\frac{2p(\rho)}{q(\rho)^2}\cdot\{2Q_2Q_3+(3Q_2Q_4-Q_3^2)\rho\}.$$

It follows that

$$f'(\rho)=\frac{2p(\rho)}{(1+\rho)^2q(\rho)^2}\cdot h(\rho),$$

$$h(\rho)=p(\rho)q(\rho)-\rho(1+\rho)\{2Q_2Q_3+(3Q_2Q_4-Q_3^2)\rho\}=3Q_2^2+5Q_2Q_3\rho+A\cdot Q_3\rho^2+B\cdot\rho^3,$$
$$A=3Q_3-2Q_2,\quad B=Q_3Q_4-3Q_2Q_4+Q_3^2.$$

Now, if $A,B\ge 0$ then $f'(\rho)>0$ for all $\rho>0$ because $Q_2,Q_3,Q_4>0$ (non-degenerate random variable $W\ge 0$). ◊

Theorem 3.1. (*NVM inequality between skewness and kurtosis*) Let $X\sim NVM(\upsilon,\beta,\kappa)$ be a normal variance-mean mixture with finite first four cumulants $\kappa=(\kappa_1,\kappa_2,\kappa_3,\kappa_4)$ of the non-negative mixing random variable W. Assume the conditions of Lemma 3.1 hold. Then, the skewness and kurtosis pair (S,K) of X satisfies the inequality

$$\frac{S^2}{K}\le\frac{S_W^2}{K_W},\tag{3.2}$$

where (S_W,K_W) is the skewness and kurtosis pair of the mixing distribution.

Proof. The equation (2.8) with its restriction on the kurtosis implies that $\rho\in(0,\infty)$. With (3.1), Lemma 3.1 and (2.6), the result follows easily as follows:

$$\max\left(\frac{S^2}{K}\right)\le\frac{1}{3Q_2}\cdot\lim_{\rho\to\infty}f(\rho)=\frac{Q_3^2}{3Q_2Q_4}=\frac{\kappa_3^2}{\kappa_2^3}\cdot\frac{\kappa_2^2}{\kappa_4}=\frac{S_W^2}{K_W}.\quad ◊$$

4. Selected examples and comparisons

The inequality between skewness and kurtosis is illustrated at three NVM mixture models, namely the NVM model with log-normal mixing distribution, the normal tempered stable

(NTS) distribution, and the generalized hyperbolic (GH) distribution. The GH has been widely discussed (e.g. Eberlein and Keller (1995), Prause (1999). Eberlein (2001), Eberlein and Prause (2002), Bibby and Sorensen (2003), Eberlein and Hammerstien (2004), McNeil et al. (2005), etc.). In particular, the variance-gamma (VG) subfamily of the GH is very popular in Finance. It has been introduced by Madan and Seneta (1990) (see also Madan and Milne (1991), Madan et al. (1998), Madan (2001), Carr et al. (2002), Geman (2002), Fu et al. (2006), etc.).

Example 4.1: Log-normal (lnN) mixing distribution

Let $W \sim \ln N(m,\tau^2)$ be a log-normal mixing random variable with skewness and excess kurtosis

$$S_W = (e^{\tau^2}+2)\cdot\sqrt{e^{\tau^2}-1}, \quad K_W = e^{4\tau^2}+2e^{3\tau^2}+3e^{2\tau^2}-6\,. \tag{4.1}$$

A calculation shows that

$$\frac{S_W^2}{K_W} = \frac{(e^{\tau^2}+2)^2}{e^{3\tau^2}+3e^{2\tau^2}+6e^{\tau^2}+6}\,. \tag{4.2}$$

The maximum of the ratio (4.2) is determined by the function

$$f(x) = \frac{(x+2)^2}{x^3+3x^2+6x+6}, \quad x = e^{\tau^2} > 1\,.$$

Since $f'(x) = \dfrac{-x\cdot(x+2)\cdot(x^2+6x+6)}{(x^3+3x^2+6x+6)^2} < 0$ for all $x > 1$, one obtains

$$\max_{\tau>0}\left\{\frac{S_W^2}{K_W}\right\} < f(1) = \frac{9}{16} = 0.5625\,. \tag{4.3}$$

On the other hand one has

$$Q_2 = x-1, \quad Q_3 = (x-1)(x+2), \quad Q_4 = \tfrac{1}{3}(x-1)(x^3+3x^2+6x+6).$$

It follows that $A = 3Q_3 - 2Q_2 = (x-1)(3x+4) > 0$, $x > 1$, and

$$B = Q_3Q_4 - 3Q_2Q_4 + Q_3^2 = \tfrac{1}{3}(x-1)^2(42+24x+6x^2-x^3-2x^4)$$

is non-negative over a limited range of volatilities $\tau \in (0, \tau_{\max}]$. In this situation, the inequality (3.2) will hold. A first application of this mixing distribution is Clark (1973). A recent application of the (generalized) log-normal mixing distribution within the context of the semi-parametric multivariate NVM mixture model is found in Cui (2012).

Example 4.2: Classical tempered stable (CTS) mixing distribution

The classical tempered stable (CTS) subordinator $W \sim CTS(\alpha,\delta,\gamma)$ is determined by the cgf

$$C_W(t) = \alpha^{-1} \cdot \{\gamma^{\alpha} - (\gamma^2 - 2\delta^2 t)^{\frac{\alpha}{2}}\}, \quad \alpha \in (0,2), \delta, \gamma > 0. \tag{4.4}$$

The corresponding NVM mixture is called normal tempered stable (NTS) model. A calculation shows that

$$C_W^{(k)}(t) = \delta^{2k} (\gamma^2 - 2\delta^2 t)^{\frac{\alpha}{2}-k} \cdot \prod_{j=1}^{k-1}(2j-\alpha), \quad k \ge 1,$$

where an empty product is one. It follows that $\kappa_k = \delta^{2k}\gamma^{\alpha-2k} \cdot \prod_{j=1}^{k-1}(2j-\alpha), k \ge 1,$ in particular

$$Q_2 = \frac{2-\alpha}{\gamma^{\alpha}}, \quad Q_3 = \frac{4-\alpha}{\gamma^{\alpha}}, \quad Q_4 = \frac{(4-\alpha)(6-\alpha)}{3(2-\alpha)\gamma^{\alpha}}.$$

The conditions of Lemma 3.1

$$A = 3Q_3 - 2Q_2 = (8-\alpha)\gamma^{-\alpha} > 0, \quad B = Q_3Q_4 - 3Q_2Q_4 + Q_3^2 = \frac{(4-\alpha)\{8+(2-\alpha)^2\}}{3(2-\alpha)}\gamma^{-2\alpha} > 0$$

are fulfilled. Therefore, the NTS distribution satisfies the inequality

$$\frac{S^2}{K} \le \frac{S_W^2}{K_W} = \frac{4-\alpha}{6-\alpha}. \tag{4.5}$$

The special case $\alpha = 1$ of the normal inverse Gaussian (NIG) is well-known from the literature (e.g. de Beus et al. (2003), Appendix 1, Ghysels and Wang (2011), Proposition 2.3, Hürlimann (2013), Appendix 3) (see also Example 4.3).

Example 4.3: Generalized inverse Gaussian (GIG) mixing distribution

An important class of NVM models is the generalized hyperbolic (GH) distribution. It belongs to the generalized inverse Gaussian (GIG) mixing random variable $W \sim GIG(\lambda, \delta, \gamma)$ with cgf

$$C_W(t) = \frac{1}{2}\lambda \cdot \ln\left\{ \frac{\gamma^2}{\gamma^2 - 2t} \right\} + \ln\left\{ \frac{K_\lambda(\delta\sqrt{\gamma^2 - 2t})}{K_\lambda(\delta\gamma)} \right\},$$

where $K_\lambda(x)$ is the modified Bessel function of the third kind. The domain of variation of the parameters depends upon three cases.

Case 1: generic GH distribution with $-\infty < \lambda < \infty, \quad \delta > 0, \quad \gamma > 0$

Case 2: variance-gamma (VG) distribution with $\lambda > 0, \quad \delta = 0, \quad \gamma > 0$

Case 3: skew hyperbolic t (SHT) distribution with $\lambda < 0, \quad \delta > 0, \quad \gamma = 0$

In the limiting Case 2 the mixing distribution reduces to a gamma distribution and in Case 3 one has an inverse gamma distribution. We discuss the three cases separately.

Case 1: generic case

It is convenient to re-parameterize the GIG by setting $\alpha = \delta\gamma > 0$. Then the cgf rewrites as

$$C_W(t) = \frac{1}{2}\lambda \cdot \ln\left\{ \frac{\alpha^2}{\alpha^2 - 2\delta^2 t} \right\} + \ln\left\{ \frac{K_\lambda(\sqrt{\alpha^2 - 2\delta^2 t})}{K_\lambda(\alpha)} \right\}. \quad (4.6)$$

The special case $\lambda = -\frac{1}{2}$ is the normal inverse Gaussian (NIG) and $\lambda = 1$ is the hyperbolic (HYP) distribution. Let us first discuss the simple NIG case.

Normal inverse Gaussian (NIG)

The cgf (4.6) simplifies to $C_W(t) = \alpha - \sqrt{\alpha^2 - 2\delta^2 t}$, from which one gets the first four cumulants

$$\kappa_1 = \frac{\delta^2}{\alpha}, \kappa_2 = \frac{\delta^4}{\alpha^3}, \kappa_3 = \frac{3\delta^6}{\alpha^5}, \kappa_4 = \frac{15\delta^8}{\alpha^7},$$

as well as the quantities

$$Q_2 = \alpha^{-1}, \quad Q_3 = 3\alpha^{-1}, \quad Q_4 = 5\alpha^{-1}.$$

The conditions of Lemma 3.1

$$A = 3Q_3 - 2Q_2 = 7\alpha^{-1} > 0, \quad B = Q_3Q_4 - 3Q_2Q_4 + Q_3^2 = 9\alpha^{-2} > 0,$$

are fulfilled. Therefore, the NIG distribution satisfies the inequality (known to be strict)

$$\frac{S^2}{K} < \frac{S_W^2}{K_W} = \frac{3}{5}.$$

General case

For fixed $-\infty < \lambda < \infty$, the moments of the GIG mixing distribution are (e.g. Paolella (2007))

$$E[W^k] = \left(\frac{\delta^2}{\alpha}\right)^k R_{\lambda,k}(\alpha), \quad R_{\lambda,k}(\alpha) = \frac{K_{\lambda+k}(\alpha)}{K_\lambda(\alpha)}. \tag{4.7}$$

Applying the relationships between moments and cumulants one obtains

$$\kappa_1 = \frac{\delta^2}{\alpha} R_{\lambda,1}(\alpha), \quad \kappa_2 = \frac{\delta^4}{\alpha^2}\{R_{\lambda,2}(\alpha) - R_{\lambda,1}^2(\alpha)\}, \quad \kappa_3 = \frac{\delta^6}{\alpha^3}\{R_{\lambda,3}(\alpha) - 3R_{\lambda,1}(\alpha)R_{\lambda,2}(\alpha) + 2R_{\lambda,1}^3(\alpha)\},$$
$$\kappa_4 = \frac{\delta^8}{\alpha^4}\{R_{\lambda,4}(\alpha) - 4R_{\lambda,1}(\alpha)R_{\lambda,3}(\alpha) - 3R_{\lambda,2}^2(\alpha) + 12R_{\lambda,1}^2(\alpha)R_{\lambda,2}(\alpha) - 6R_{\lambda,1}^4(\alpha)\}.$$

The associated quantities (2.6) are independent from the scale parameter δ and given by

$$Q_2(\alpha) = \frac{R_{\lambda,2}(\alpha) - R_{\lambda,1}^2(\alpha)}{R_{\lambda,1}^2(\alpha)}, \quad Q_3(\alpha) = \frac{R_{\lambda,3}(\alpha) - 3R_{\lambda,1}(\alpha)R_{\lambda,2}(\alpha) + 2R_{\lambda,1}^3(\alpha)}{R_{\lambda,1}(\alpha)\cdot\{R_{\lambda,2}(\alpha) - R_{\lambda,1}^2(\alpha)\}},$$
$$Q_4(\alpha) = \frac{1}{3}\frac{R_{\lambda,4}(\alpha) - 4R_{\lambda,1}(\alpha)R_{\lambda,3}(\alpha) - 3R_{\lambda,2}^2(\alpha) + 12R_{\lambda,1}^2(\alpha)R_{\lambda,2}(\alpha) - 6R_{\lambda,1}^4(\alpha)}{\{R_{\lambda,2}(\alpha) - R_{\lambda,1}^2(\alpha)\}^2}.$$

In general, not much is known about the analytical properties of the preceding quantities. The HYP special case might illustrate what can happen.

<u>Hyperbolic distribution (HYP)</u>

A numerical calculation with $\lambda = 1$ shows that the functions $A(\alpha) = 3Q_3(\alpha) - 2Q_2(\alpha)$ and $B(\alpha) = Q_3(\alpha)Q_4(\alpha) - 3Q_2(\alpha)Q_4(\alpha) + Q_3^2(\alpha)$ are non-negative, hence the conditions of Lemma 3.1 are fulfilled. An application of Theorem 3.1 and a further numerical evaluation shows the HYP inequality

$$\max\left(\frac{S^2}{K}\right) \le \max_{\alpha>0}\left(\frac{S_W^2(\alpha)}{K_W(\alpha)}\right) = \max_{\alpha>0}\left(\frac{Q_3^2(\alpha)}{3Q_2(\alpha)Q_4(\alpha)}\right) \le \frac{2}{3}. \qquad (4.8)$$

<u>Case 2</u>: variance-gamma (VG)

The cumulants of the gamma distributed mixing random variable $W \sim \Gamma(\alpha, \delta^{-2})$ are $\kappa_1 = \alpha\delta^2$, $\kappa_2 = \alpha\delta^4$, $\kappa_3 = 2\alpha\delta^6$, $\kappa_4 = 6\alpha\delta^8$, and one has

$$Q_2 = \alpha^{-1}, \quad Q_3 = 2\alpha^{-1}, \quad Q_4 = 2\alpha^{-1}.$$

The conditions $A = 3Q_3 - 2Q_2 = 4\alpha^{-1} > 0$, $B = Q_3 Q_4 - 3Q_2 Q_4 + Q_3^2 = 2\alpha^{-2} > 0$, are fulfilled. Therefore, the VG distribution satisfies the inequality

$$\frac{S^2}{K} \le \frac{S_W^2}{K_W} = \frac{2}{3}. \tag{4.9}$$

This inequality is sharp (e.g. Hürlimann (2013), Case 2 of Theorem 3.1). The present simple proof of this inequality is new. A different derivation, which is valid for the more general bilateral gamma (BG) convolution is Hürlimann (2013), Theorem A2.2.

Case 3: skew hyperbolic t (SHT)

The cumulants of the inverse gamma distributed mixing random variable $W \sim I\Gamma(\alpha, \delta^2)$ exist only for $\alpha > 4$ and are given by

$$\kappa_1 = \frac{\delta^2}{2(\alpha-1)}, \quad \kappa_2 = \frac{\delta^4}{4(\alpha-1)^2(\alpha-2)},$$
$$\kappa_3 = \frac{\delta^6}{2(\alpha-1)^3(\alpha-2)(\alpha-3)}, \quad \kappa_4 = \frac{3(5\alpha-11)\delta^8}{8(\alpha-1)^4(\alpha-2)^2(\alpha-3)(\alpha-4)}.$$

It follows that

$$Q_2 = \frac{1}{\alpha-2}, \quad Q_3 = \frac{4}{\alpha-3}, \quad Q_4 = \frac{2(5\alpha-11)}{(\alpha-3)(\alpha-4)}, \quad \text{as well as}$$

$$A = 3Q_3 - 2Q_2 = \frac{2(5\alpha-9)}{(\alpha-2)(\alpha-3)} > 0, \quad B = Q_4(Q_3 - 3Q_2) + Q_3^2 = Q_4 \frac{\alpha+1}{(\alpha-2)(\alpha-3)} + Q_3^2 > 0.$$

Since the conditions of Theorem 3.1 are fulfilled one obtains the inequalities

$$\max\left(\frac{S^2}{K}\right) \le \max_{\alpha>4}\left(\frac{Q_3^2(\alpha)}{3Q_2(\alpha)Q_4(\alpha)}\right) = \max_{\alpha>4}\left(\frac{8(\alpha-2)(\alpha-4)}{3(\alpha-3)(5\alpha-11)}\right) \le \frac{8}{15}. \tag{4.10}$$

Let us conclude this note with a brief comparison of some skewness and kurtosis inequalities.

Comparison with the domain of maximum size

Recall the general inequality between skewness and kurtosis for arbitrary distributions on

$(-\infty,\infty)$, namely

$$\frac{S^2}{K} \le 1 + \frac{2}{K}, \tag{4.11}$$

which is sharp and attained at a biatomic random variable with support $\{\omega, \overline{\omega} = -\omega^{-1}\}$, where $\omega = \frac{1}{2}(S - \sqrt{4+S^2})$ (Pearson (1916), Wilkins (1944), Guiard (1980), Hürlimann (2008), Theorem I.4.1). A family of distributions, which is able to model any admissible pair (S, K), is the Johnson system introduced in Johnson (1949) (see also Johnson et al. (1994), George (2007) among others). Note that for distributions with a finite range $[A, B]$, $-\infty < A < B < \infty$, the inequality (4.11) extends to a two-sided inequality (further information is found in Hürlimann (2008), Chap.I.4). Clearly, the domain of variation of skewness and kurtosis for the selected examples is more restricted than the domain of maximum size prescribed by the inequality (4.11). The selected examples have a maximum ratio of squared skewness to kurtosis equal to 2/3 for the gamma mixing distribution (VG special case of the GH). This ratio is also closely approximated by a CTS mixing distribution with $\alpha \to 0$. The maximum ratio of 2/3 coincides with the corresponding ratio for the five parameter bilateral gamma (BG) convolution in Hürlimann (2013), Theorem A2.2. In fact, the BG bound $S^2 \le \frac{2}{3}K$ is sharp and attained for limiting left- and right-tail gamma distributions (op.cit.).

Comparison of the VG, HYP, NIG and SHT boundaries

The domain of variation between skewness and kurtosis is larger for the VG/HYP than for the NIG. Indeed, the NIG domain $S^2 \le \frac{3}{5}K$ is contained in the VG/HYP domain $S^2 \le \frac{2}{3}K$. For the VG this result is also found in Ghysels and Wang (2011), p.8. These authors also show that the NIG domain contains the feasible domain of the skew hyperbolic t (SHT), called generalized skew t by them (applications of the SHT are found in Frecka and Hopwood (1983), Theodossiu (1998), Aas and Haff (2006), Hürlimann (2009), etc.).

Comparison of the VG with Hansen's generalized t

Hansen (1994) considers another generalization of the Student t distribution, simply called generalized t (GT) by Jondeau and Rockinger (2003). The boundaries of maximum skewness by given kurtosis for the VG are delimited by the two curves $S = \pm\sqrt{\frac{2}{3}K}$. In this situation, let $S_{VG}^2(K) = \frac{2}{3}K$ denote the maximum squared skewness as a function of K. The GT skewness and kurtosis boundary has been determined in Jondeau and Rockinger (2003), Section 2.2, Fig. 5 (note that the excess kurtosis is obtained by subtracting the constant 3 from the expression (3) in Section 2.1). Let $S_{GT}^2(K)$ denote the corresponding maximum squared skewness. The GT domain is contained in the VG domain for kurtosis higher than some relatively moderate value:

$$S_{VG}^2(K) \le S_{GT}^2(K), \quad \forall\, K \le K_0 = 2.774, \quad S_{VG}^2(K) > S_{GT}^2(K), \quad \forall\, K > K_0. \qquad (4.12)$$

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