Available online at http://scik.org
Adv. Inequal. Appl. 2014, 2014:12
ISSN: 2050-7461

SOME NEW INTEGRAL INEQUALITIES USING HADAMARD FRACTIONAL INTEGRAL OPERATOR

VAIJANATH L. CHINCHANE[1,*] AND DEEPAK B. PACHPATTE[2]

[1]Department of Mathematics, Deogiri Institute of Engineering and Management Studies
Aurangabad-431005, INDIA
[2]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University
Aurangabad-431 004, INDIA

Abstract. The main objective of this paper is to develop certain type of fractional integral inequalities based on Hadamard fractional integral operator.

Keywords: fractional integral inequality; Hadamard fractional integral; integral inequality.

2000 AMS Subject Classification: 26D10, 26A33.

1. Introduction

In last few decades, considerable interests has been shown in developing various aspect of the fractional differential and integral equations both for their own sake and for their application in science and technology; for detail, see [2, 13, 14, 15] and the references therein. In recent years, many authors have worked on fractional differential and integral inequalities using

*Corresponding author
E-mail addresses: chinchane85@gmail.com (V.L. Chinchane), pachpatte@gmail.com (D.B. Pachpatte)
Received August 6, 2013

Riemann-Liouville and Caputo fractional integrals; see [3, 6, 7, 8, 9, 12, 13] and the references therein. Ahmed Anber *et al.* established fractional integral inequality using Riemann-Liouville fractional integration; for more details, see [1]. In the literature, few results were obtained on some fractional integral inequalities using Hadamard fractional integral; see [4, 5]. Motivated by the results presented in [1], we prove some new results using Hadamard fractional integral operator. The paper is organized as follows. In Section 2, basic definitions and propositions related to Hadamard fractional derivatives and integrals are given. In Section 3, the results on fractional integral inequality using fractional Hadamard integral are presented.

2. Preliminaries

The necessary details of fractional Hadamard calculus are given in Kilbas [14] and Samko *et al.* [15]. Here we present some definitions of Hadamard derivative and integral as given in [2].

Definition 2.1. The Hadamard fractional integral of order $\alpha \in R^{+}$ of function $f(x)$, for all $x > 1$ is defined as,

$$ {}_{H}D_{1,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{1}^{x} \ln(\frac{x}{t})^{\alpha-1} f(t) \frac{dt}{t}, \tag{2.1} $$

where $\Gamma(\alpha) = \int_{0}^{\infty} e^{-u} u^{\alpha-1} du.$

Definition 2.2. The Hadmard fractional derivative of order $\alpha \in [n-1, n)$, $n \in Z^{+}$, of function $f(x)$ is given as follows

$$ {}_{H}D_{1,x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} (x\frac{d}{dx})^{n} \int_{1}^{x} \ln(\frac{x}{t})^{n-\alpha-1} f(t) \frac{dt}{t}. \tag{2.2} $$

From the above definitions, we can see obviously the difference between Hadamard fractional and Riemann-Liouville fractional derivative and integrals, which include two aspects. The kernel in the Hadamard integral has the form of $\ln(\frac{x}{t})$ instead of the form of $(x-t)$, which is involves both in the Riemann-Liouville and Caputo integral. The Hadamard derivative has the operator$(x\frac{d}{dx})^{n}$, whose construction is well suited to the case of the half-axis and is invariant relation to dilation [15, p.330], while the Riemann-Liouville derivative has the operator $(\frac{d}{dx})^{n}$.

We give some image formulas under the operator (2.1) and (2.2), which would be used in the derivation of our main result.

Proposition 2.1. [2] If $0<\alpha<1$, the following equalities hold:

$$_HD_{1,x}^{-\alpha}(\ln x)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln x)^{\beta+\alpha-1}, \tag{2.3}$$

and

$$_HD_{1,x}^{\alpha}(\ln x)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln x)^{\beta-\alpha-1}, \tag{2.4}$$

respectively.

For the convenience of establishing the result, we give the semigroup property,

$$(_HD_{1,x}^{-\alpha})(_HD_{1,x}^{-\beta})f(x)=_H D_{1,x}^{-(\alpha+\beta)}f(x). \tag{2.5}$$

3. Main results

Now, we are in a position to give the main result.

Theorem 3.1. *Let $\alpha>0$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and let f, g be two positive functions on $[0,\infty[$, such that for all $t>0$, $_HD_{1,t}^{-\alpha}f(t)<\infty$, $_HD_{1,t}^{-\alpha}g(t)<\infty$. If $0<m\leq\frac{f(\tau)}{g(\tau)}\leq M<\infty$, $\tau\in[0,t]$, then we have the following*

$$\left[_HD_{1,t}^{-\alpha}f(t)\right]^{\frac{1}{p}}\left[_HD_{1,t}^{-\alpha}g(t)\right]^{\frac{1}{q}}\leq(\frac{M}{m})^{\frac{1}{pq}}\left[_HD_{1,t}^{-\alpha}(f(t))^{\frac{1}{p}}(g(t))^{\frac{1}{q}}\right]. \tag{3.1}$$

Proof. Since $\frac{f(\tau)}{g(\tau)}\leq M$, $\tau\in[0,t[$, $t>0$, we find that

$$[g(\tau)]^{\frac{1}{p}}\geq M^{\frac{-1}{q}}[f(\tau)]^{\frac{1}{q}} \tag{3.2}$$

and

$$\begin{aligned}[f(\tau)]^{\frac{1}{p}}[g(\tau)]^{\frac{1}{q}}&\geq M^{\frac{-1}{q}}[f(\tau)]^{\frac{1}{q}}[f(\tau)]^{\frac{1}{p}}\\ &\geq M^{\frac{-1}{q}}[f(\tau)]^{\frac{1}{q}+\frac{1}{q}}\\ &\geq M^{\frac{-1}{q}}[f(\tau)].\end{aligned} \tag{3.3}$$

Multiplying both sides of (3.3) by $\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}}{\tau\Gamma(\alpha)}$, which is positive because $\tau\in(0,t)$, $t>0$, we integrate resulting identity with respect to τ from 1 to t ge that

$$\frac{1}{\Gamma(\alpha)}\int_1^t(\ln(\frac{t}{\tau}))^{\alpha-1}[f(\tau)]^{\frac{1}{p}}[g(\tau)]^{\frac{1}{q}}\frac{d\tau}{\tau}\geq M^{\frac{-1}{q}}\frac{1}{\Gamma(\alpha)}\int_1^t(\ln(\frac{t}{\tau}))^{\alpha-1}f(\tau)\frac{d\tau}{\tau}, \tag{3.4}$$

which implies that

$$ {}_H D_{1,t}^{-\alpha}\left[[f(t)]^{\frac{1}{p}}[g(t)]^{\frac{1}{q}}\right] \le M^{\frac{-1}{q}}\left[{}_H D_{1,t}^{-\alpha} f(t)\right]. \tag{3.5} $$

It follows that

$$ \left({}_H D_{1,t}^{-\alpha}\left[[f(t)]^{\frac{1}{p}}[g(t)]^{\frac{1}{q}}\right]\right)^{\frac{1}{p}} \le M^{\frac{-1}{pq}}\left[{}_H D_{1,t}^{-\alpha} f(t)\right]^{\frac{1}{p}}. \tag{3.6} $$

Notice that $mg(\tau) \le f(\tau)$, $\tau \in [0,t]$, $t > 0$. It follows that

$$ [f(\tau)]^{\frac{1}{p}} \ge m^{\frac{1}{p}}[g(\tau)]^{\frac{1}{p}}. \tag{3.7} $$

Multiplying the equation (3.7) by $[g(\tau)]^{\frac{1}{q}}$, we arrive at

$$ [f(\tau)]^{\frac{1}{p}}[g(\tau)]^{\frac{1}{q}} \ge m^{\frac{1}{p}}[g(\tau)]^{\frac{1}{q}}[g(\tau)]^{\frac{1}{p}} = m^{\frac{1}{p}}[g(\tau)] \tag{3.8} $$

Multiplying both sides of (3.8) by $\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}}{\tau\Gamma(\alpha)}$, which is positive because $\tau \in (0,t)$, $t > 0$, we integrate resulting identity with respect to τ from 1 to t obtaining that

$$ \frac{1}{\Gamma(\alpha)}\int_1^t (\ln(\frac{t}{\tau}))^{\alpha-1}[g(\tau)]^{\frac{1}{q}}[f(\tau)]^{\frac{1}{p}}\frac{d\tau}{\tau} \ge m^{\frac{1}{p}}\frac{1}{\Gamma(\alpha)}\int_1^t (\ln(\frac{t}{\tau}))^{\alpha-1}g(\tau)\frac{d\tau}{\tau}, \tag{3.9} $$

that is,

$$ {}_H D_{1,t}^{-\alpha}\left[[f(t)]^{\frac{1}{p}}[g(t)]^{\frac{1}{q}}\right] \le m^{\frac{1}{p}}\left[{}_H D_{1,t}^{-\alpha} g(t)\right]. \tag{3.10} $$

Hence we have

$$ \left({}_H D_{1,t}^{-\alpha}\left[[f(t)]^{\frac{1}{p}}[g(t)]^{\frac{1}{q}}\right]\right)^{\frac{1}{q}} \le m^{\frac{1}{pq}}\left[{}_H D_{1,t}^{-\alpha} g(t)\right]^{\frac{1}{q}}. \tag{3.11} $$

Multiplying the equation (3.6) and (3.11), we can draw the desired conclusion easily.

Lemma 3.2. *Let $\alpha > 0$, f and g be two positive functions on $[0,\infty[$, such that for all ${}_H D_{1,t}^{-\alpha} f^p(t) < \infty$, ${}_H D_{1,t}^{-\alpha} g^q(t) < \infty$, $t > 0$. If $0 < m \le \frac{f(\tau)^p}{g(\tau)^q} \le M < \infty$, $\tau \in [0,t]$, then we have*

$$ \left[{}_H D_{1,t}^{-\alpha} f^p(t)\right]^{\frac{1}{p}}\left[{}_H D_{1,t}^{-\alpha} g^q(t)\right]^{\frac{1}{q}} \le (\frac{M}{m})^{\frac{1}{pq}}\left[{}_H D_{1,t}^{-\alpha}(f(t)(g(t)\right], \tag{3.12} $$

where $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. Replacing $f(\tau)$ and $g(\tau)$ by $f(\tau)^p$ and $g(\tau)^q$, $\tau \in [0,t]$, $t > 0$ in theorem 3.1, we find the desired result. This completes the proof.

Lemma 3.3. *Let $\alpha > 0$ and let f, g be two positive functions on $[0,\infty[$, such that f is nondecreasing and g is non-increasing. Then*

$$
{}_H D_{1,t}^{-\alpha} f^{\gamma}(t) g^{\delta}(t) \leq \frac{\Gamma(\alpha+1)}{(\ln t)^{\alpha}})^{\alpha} {}_H D_{1,t}^{-\alpha} f^{\gamma}(t) {}_H D_{1,t}^{-\alpha} g^{\delta}(t). \tag{3.13}
$$

for any $t > 0$ $\gamma > 0$ $\delta > 0$.

Proof. Let $\tau, \rho \in [0,t]$, $t > 0$, for any $\delta > 0$, $\gamma > 0$. Then we have

$$
\left(f^{\gamma}(\tau) - f^{\gamma}(\rho)\right)\left(g^{\delta}(\rho) - g^{\delta}(\tau)\right) \geq 0 \tag{3.14}
$$

and

$$
f^{\gamma}(\tau) g^{\delta}(\rho) - f^{\gamma}(\tau) g^{\delta}(\tau) - f^{\gamma}(\rho)(g^{\delta}(\rho) + f^{\gamma}(\rho) g^{\delta}(\tau) \geq 0. \tag{3.15}
$$

It follows that

$$
f^{\gamma}(\tau) g^{\delta}(\tau) + f^{\gamma}(\rho)(g^{\delta}(\rho) \leq f^{\gamma}(\tau) g^{\delta}(\rho) + f^{\gamma}(\rho) g^{\delta}(\tau), \tag{3.16}
$$

Now, multiplying both sides of (3.16) by $\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}}{\tau\Gamma(\alpha)}$, which is positive because $\tau \in (0,t)$, $t > 0$, we integrate resulting identity with respect to τ from 1 to t find that

$$
{}_H D_{1,t}^{-\alpha} f^{\gamma}(t) g^{\delta}(t) + f^{\gamma}(\rho)(g^{\delta}(\rho) {}_H D_{1,t}^{-\alpha}(1) \leq g^{\delta}(\rho) {}_H D_{1,t}^{-\alpha} f^{\gamma}(t) + f^{\gamma}(\rho) {}_H D_{1,t}^{-\alpha} g^{\delta}(t), \tag{3.17}
$$

Again, multiplying both sides of (3.17) by $\frac{(\ln(\frac{t}{\rho}))^{\alpha-1}}{\rho\Gamma(\alpha)}$, which is positive because $\rho \in (0,t)$, $t > 0$, we integrate resulting identity with respect to ρ from 1 to t obtaining that

$$
\begin{aligned}
&{}_H D_{1,t}^{-\alpha} f^{\gamma}(t) g^{\delta}(t) {}_H D_{1,t}^{-\alpha}(1) + {}_H D_{1,t}^{-\alpha} f^{\gamma}(t)(g^{\delta}(t) {}_H D_{1,t}^{-\alpha}(1) \\
&\leq {}_H D_{1,t}^{-\alpha} g^{\delta}(t) {}_H D_{1,t}^{-\alpha} f^{\gamma}(t) + f^{\gamma}(t) {}_H D_{1,t}^{-\alpha} g^{\delta}(t).
\end{aligned}
$$

It follows that

$$
2 {}_H D_{1,t}^{-\alpha} f^{\gamma}(t) g^{\delta}(t) \leq \frac{1}{[{}_H D_{1,t}^{-\alpha}(1)]^{-1}} 2 {}_H D_{1,t}^{-\alpha} f^{\gamma}(t) {}_H D_{1,t}^{-\alpha} g^{\delta}(t)
$$

This completes the proof.

Theorem 3.4. *Let f, g be two positive functions on $[0,\infty[$, such that f is nondecreasing and g is non-increasing. Then for all $t > 0$ $\gamma > 0$ $\delta > 0$, we have*

$$
\begin{aligned}
&\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)}{}_H D_{1,t}^{-\alpha} f^{\gamma}(t) g^{\delta}(t) + \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_H D_{1,t}^{-\alpha} f^{\gamma}(t) g^{\delta}(t) \\
&\leq \left({}_H D_{1,t}^{-\alpha} f^{\gamma}(t)\right)\left({}_H D_{1,t}^{-\beta} g^{\delta}(t)\right) + \left({}_H D_{1,t}^{-\alpha} g^{\delta}(t)\right)\left({}_H D_{1,t}^{-\beta} f^{\gamma}(t)\right).
\end{aligned} \tag{3.18}
$$

Proof. Multiplying both sides of the equation (3.17) $\frac{(\ln(\frac{t}{\rho}))^{\beta-1}}{\rho\Gamma(\beta)}$, $\rho\in(0,t)$, $t>o$ which is positive, we integrate resulting identity with respective to ρ from 1 to t finding that

$$\begin{aligned}&{}_HD_{1,t}^{-\alpha}f^{\gamma}(t)g^{\delta}(t)\frac{1}{\Gamma(\beta)}\int_1^t(\ln(\frac{t}{\rho}))^{\beta-1}\frac{d\rho}{\rho}+\frac{1}{\Gamma(\beta)}\int_1^t(\ln(\frac{t}{\rho}))^{\beta-1}f^{\gamma}(\rho)g^{\delta}(\rho)\frac{d\rho}{\rho}\,{}_HD_{1,t}^{-\alpha}(1)\\&\leq\frac{1}{\Gamma(\beta)}\int_1^t(\ln(\frac{t}{\rho}))^{\beta-1}g^{\delta}(\rho)\frac{d\rho}{\rho}\,{}_HD_{1,t}^{-\alpha}f^{\gamma}(t)+\frac{1}{\Gamma(\beta)}\int_1^t(\ln(\frac{t}{\rho}))^{\beta-1}f^{\gamma}(\rho)\frac{d\rho}{\rho}\,{}_HD_{1,t}^{-\alpha}g^{\delta}(t).\end{aligned}\tag{3.19}$$

which implies (3.18). This completes proof.

Remark 3.1. Applying theorem 3.4 for $\alpha=\beta$, we obtain theorem 3.3 immediately.

Theorem 3.5. *Let $f\geq 0$, $g\geq 0$ be two functions defined on $[0,\infty[$, such that g is non-decreasing. If*

$$ {}_HD_{1,t}^{-\alpha}f(t)\geq {}_HD_{1,t}^{-\alpha}g(t), t>o. \tag{3.20}$$

then, for all $\alpha>0$, $\gamma>0$, $\delta>0$ and $\gamma-\delta>0$,

$$ {}_HD_{1,t}^{-\alpha}f^{\gamma-\delta}(t)\leq {}_HD_{1,t}^{-\alpha}f^{\gamma}(t)g^{-\delta}(t). \tag{3.21}$$

Proof. In view of the arithmetic-geometric inequality, for $\gamma>0$, $\delta>0$, we have

$$\frac{\gamma}{\gamma-\delta}f^{\gamma-\delta}(\tau)-\frac{\delta}{\gamma-\delta}g^{\gamma-\delta}(\tau)\leq f^{\gamma}(\tau)g^{-\delta}(\tau),\ \tau\in(0,t),t>0. \tag{3.22}$$

Now, multiplying both sides of (3.22) by $\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}}{\tau\Gamma(\alpha)}$, which is positive, we have

$$\frac{\gamma}{\gamma-\delta}\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}}{\tau\Gamma(\alpha)}f^{\gamma-\delta}(\tau)-\frac{\delta}{\gamma-\delta}\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}}{\tau\Gamma(\alpha)}g^{\gamma-\delta}(\tau)\leq\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}}{\tau\Gamma(\alpha)}f^{\gamma}(\tau)g^{-\delta}(\tau). \tag{3.23}$$

Integrating (3.23) with respective to τ from 1 to t, we obtain

$$\begin{aligned}&\frac{\gamma}{\gamma-\delta}\frac{1}{\Gamma(\alpha)}\int_1^t\ln(\frac{t}{\tau})^{\alpha-1}f^{\gamma-\delta}(\tau)\frac{d\tau}{\tau}-\frac{\delta}{\gamma-\delta}\frac{1}{\Gamma(\alpha)}\int_1^t\ln(\frac{t}{\tau})^{\alpha-1}g^{\gamma-\delta}(\tau)\frac{d\tau}{\tau}\\&\leq\frac{1}{\Gamma(\alpha)}\int_1^t\ln(\frac{t}{\tau})^{\alpha-1}f^{\gamma}(\tau)g^{-\delta}(\tau)\frac{d\tau}{\tau}.\end{aligned}\tag{3.24}$$

It follows that

$$\frac{\gamma}{\gamma-\delta}\,{}_HD_{1,t}^{-\alpha}f^{\gamma-\delta}(t)-\frac{\delta}{\gamma-\delta}\,{}_HD_{1,t}^{-\alpha}g^{\gamma-\delta}(t)\leq {}_HD_{1,t}^{-\alpha}f^{\gamma}(t)g^{-\delta}(t), \tag{3.25}$$

which implies that

$$\frac{\gamma}{\gamma-\delta}\,{}_HD_{1,t}^{-\alpha}f^{\gamma-\delta}(t) \leq {}_HD_{1,t}^{-\alpha}f^{\gamma}(t)g^{-\delta}(t)+\frac{\delta}{\gamma-\delta}\,{}_HD_{1,t}^{-\alpha}g^{\gamma-\delta}(t) \tag{3.26}$$

and

$${}_HD_{1,t}^{-\alpha}f^{\gamma-\delta}(t) \leq \frac{\gamma-\delta}{\gamma}\,{}_HD_{1,t}^{-\alpha}f^{\gamma}(t)g^{-\delta}(t)+\frac{\delta}{\gamma}\,{}_HD_{1,t}^{-\alpha}f^{\gamma-\delta}(t).$$

This completes the proof.

References

[1] A. Anber, Z. Dahmani, B. Bendoukha, New integral inequalities of Feng Qi type via Riemann-Liouville fractional integration, Facta Universitatis (NIS) Ser. Math. Imform. 27 (2012) 13-22.

[2] D. Baleanu, J. A. T.Machado, C. J. Luo, Fractional Dynamic and Control, Springer, (2012) 159-171.

[3] S. Belarbi, Z. Dahmani, On some new fractional integral inequality, J. Inequal. Pure Appl. Math. 10 (2009) 86.

[4] V. L. Chinchane, D. B. Pachpatte, A note on some integral inequalities via Hadamard integral, J. Fractional Calculus Appl. 4 (2013) 1-5.

[5] V. L. Chinchane, D. B. Pachpatte, On some integral inequalities using Hadamard fractional integral, Malaya J. Math. 1 (2012) 62-66.

[6] Z. Dahmani, On Minkowski and Hermit-Hadamard integral inequalities via fractional integration, Ann. Funct. Anal. 1 (2010) 51-58.

[7] Z. Dahmani, New inequalities in fractional integrals, Int. J. Nonlinear Sci. 9 (2010) 493-497.

[8] Z. Dahmani, The Riemann-Liouville operator to genarate some new inequalities, Int. J. Nonlinear Sci. 12 (2011) 452-455.

[9] Z. Dahmani, Some results associate with fractional integrals involving the extended Chebyshev, Acta Univ. Apulensis Math. Inform. 27 (2011) 217-224.

[10] Z. Dahmani, A. Benzidane, New inequalities using Q-fractional theory, Bull. Math. Anal. Appl. 4 (2012) 190-196.

[11] Z. Dahmani, L. Tabharit, S. Taf, New generalisations of Gruss inequality using Riemann-Liouville fractional integrals, Bull. Math. Anal. Appl. 2 (2010) 93-99.

[12] Z. Denton, A.S. Vatsala, Fractional integral inequalities and application, Comput. Math. Appl. 59 (2010) 1087-1094.

[13] G. A. Anastassiou, Fractional Differentiation Inequalities, Springer Publishing Company, New York, 2009.

[14] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Application of Fractional Differential Equations, Elsevier, Amersterdam, 2006.

[15] S. G. Somko, A. A. Kilbas, O. I. Marichev, Fractional Integral and Derivative Theory and Application, Gordon and Breach, Yverdon, Switzerland, 1993.

[16] W. Yang, Some new fractional quantum integral inequalities, Appl. Math. Lett. 25 (2012) 963-969.