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A NEW KIND OF HERMITE INTERPOLATION

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Abstract. In this paper, we study the convergence of Hermite interpolation polynomials on the nodes obtained by projecting vertically the zeros of $\left(1-x^2\right)P_n^{(\alpha,\beta)}(x)$, where $P_n^{(\alpha,\beta)}(x)$ stands for the Jacobi polynomial.

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1. Introduction

In a paper, Goodman and Sharma [3] considered convergence and divergence behaviour of Hermite interpolation in the circle of radius $\rho^{\frac{3}{2}}$. In [4], Goodman, Ivanov and Sharma considered the behaviour of the Hermite interpolation in the roots of unity. In [1], Bahadur and Mathur proved the convergence of quasi-Hermite interpolation on the nodes obtained by projecting vertically the zeros of $\left(1-x^2\right)P_n(x)$ on the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial. Later on convergence of Hermite interpolation was considered in [8] on the same set of nodes. Recently, Berriochoaa, Cachafeiros and Breyb [2] studied the convergence of the

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Hermite -Fejér and the Hermite interpolation polynomials, which are constructed by taking e-qually spaced nodes on the unit circle. As a consequence, they achieved some improvements on Hermite interpolation problems on the real line.

In [7], Mathur and Saxena Investigated the convergence of Quasi-Hermite -Fejér interpolation. In [11], Xie considered the regularity of $(0,1,2,.....,r-2,r)^{*}$-interpolation on the nodes obtained by projecting vertically the zeros of $\left(1-x^2\right)P_n^{(\alpha,\beta)}(x)$ onto the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for the Jacobi Polynomial. In [9], Szabo gave a generalization of Pál-type interpolation for the zeros of Jacobi Polynomial. In [6], Lenard studied the convergence of the modified $(0,2)$- interpolation procedure if the inner nodal points are the roots of the Ultraspherical polynomials with odd integer parameter.

In this paper, we consider the zeros of $\left(1-x^2\right)P_n^{(\alpha,\beta)}(x)$, which are projected vertically onto the unit circle. In Section 2, we give some preliminaries. In Section 3, we describe the problem and obtain the existence of interpolatory polynomials. In Section 4, explicit formulae of interpolatory polynomials are given. In Section 5 and Section 6, the estimation and convergence of interpolatory polynomials are considered, respectively.

2. Preliminaries

In this section, we give some well known results.

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is

$$(2.1)\quad (1-x^2)P_n^{(\alpha,\beta)''}(x)+\left(\beta-\alpha-(\alpha+\beta+2)x\right)P_n^{(\alpha,\beta)'}(x)+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0,$$

$$(2.2)\qquad W(z)=\prod_{k=1}^{2n}(z-z_k)\quad =K_nP_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n,$$

$$(2.3)\qquad R(z)=\left(z^2-1\right)W(z).$$

We shall require the fundamental polynomials of Lagrange interpolation based on the nodes as zeros of $R(z)$ is given by

$$(2.4)\qquad L_k\ (z)\quad=\quad\frac{R(z)}{R'(z_k)(z-z_k)},\qquad\qquad k\ =0(1)2n+1$$

We will also use the following results

$$(2.5)\qquad (-1)^nW'(z_{n+k})\ =W'(z_k)$$
$$=-\tfrac{1}{2}K_nP_n^{(\alpha,\beta)\prime}(x_k)\left(1-z_k^2\right)z_k^{n-2}\quad,\ k=1\,(1)\,n$$

$$(2.6)\qquad \begin{aligned}(-1)^{n-1} W''(z_{n+k}) &= W''(z_k)\\ = -\tfrac{1}{2} K_n P_n^{(\alpha,\beta)\prime}(x_k) \Big[-\big[2(\beta-\alpha) z_k - (\alpha+\beta+2)\left(1+z_k^2\right)\big] &+ 2n\left(1-z_k^2\right) - 2 \Big]\, z_k^{n-3}, \\ k &= 1(1)n\end{aligned}$$

$$(2.7)\qquad R'(z_k) = \left(z_k^2-1\right) W'(z_k)$$

$$(2.8)\qquad R''(z_k) = W'(z_k)\left[4z_k^2 + 2(\beta-\alpha) z_k - (\alpha+\beta+2)\left(1+z_k^2\right) - 2n\left(1-z_k^2\right) + 2\right] z_k^{-1}.$$

We will also use the following well known inequalities (see [6], [10])

$$(2.9)\qquad \left(1-x^2\right)^{\frac{1}{2}} P_n^{(\alpha,\beta)}(x) = o\left(n^{\alpha-1}\right), \quad \text{for} \quad \alpha > 0, \; x \in [-1,1]$$

$$(2.10)\qquad \left(1-x_k^2\right)^{-1} \sim \left(\tfrac{k}{n}\right)^{-2},$$

$$(2.11)\qquad \left| P_n^{(\alpha,\beta)\prime}(x_k)\right| \sim k^{-\alpha-\frac{3}{2}}\, n^{\alpha+2}.$$

3. THE PROBLEM AND REGULARITY

Let

$$(3.1)\qquad Z_n = \begin{cases} z_0 = 1, z_{2n+1} = -1 \\ z_k = \cos\theta_k + i\sin\theta_k,\, z_{n+k} = -z_k, k = 1\,(1)\,n \end{cases}$$

be the vertical projections on the unit circle of the zeros of $\left(1-x^2\right) P_n^{(\alpha,\beta)}(x)$, we determine the interpolatory polynomials $R_n(z)$ of degree $\le 4n+3$ satisfying the conditions:

$$(3.2)\qquad \begin{cases} R_n(z_k) = \alpha_k; & k = 0(1)2n+1 \\ R_n'(z_k) = \beta_k; & k = 0(1)2n+1 \end{cases}$$

where α_k and β_k are arbitrary complex numbers and establish the convergence theorem of $R_n(z)$.

Theorem 1. *Hermite interpolation is regular on Z_n.*

Proof. It is sufficient if we show the unique solution of (3.2) is $R_n(z) \equiv 0$, when all data $\alpha_k = \beta_k = 0$.Clearly in this case we have $R_n(z) = R(z)\, q(z)$, where $q(z)$ is a polynomial of degree $\le 2n+1$.

As $R_n'(z_k) = 0 \qquad k = 0(1)2n+1$

$R'(z_k)\, q(z_k) = 0.$

We have $q(z_k) = 0$. It follows that

$$q(z) = (az+b)W(z)$$

As $q(\pm 1) = 0$, we get $a = b = 0$, which gives that $q(z) \equiv 0$ leading to $R_n(z) \equiv 0$. This completes the proof.

4. EXPLICIT REPRESENTATION OF INTERPOLATORY POLYNOMIALS

We shall write $R_n(z)$ satisfying (3.2) as

(4.1) $$R_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) + \sum_{k=0}^{2n+1} \beta_k B_k(z),$$ where $A_k(z)$ and $B_k(z)$ are fundamental polynomials of the first and second type respectively , each of degree atmost $4n+3$ satisfying the conditions:

For $j,\ k = 0(1)2n+1$,

(4.2) $$\begin{cases} A_k(z_j) = \delta_{jk}, \\ A'_k(z_j) = 0 \end{cases}$$

(4.3) $$\begin{cases} B_k(z_j) = 0, \\ B'_k(z_j) = \delta_{jk.} \end{cases}$$

Theorem 2. *For $k = 0(1)2n+1$, we have*

(4.4) $$B_k(z) = \frac{R(z) L_k(z)}{R'(z_k)}.$$

Theorem 3. *For $k = 0(1)2n+1$, we have*

(4.5) $$A_k(z) = L_k^2(z) - 2L'_k(z_k) B_k(z).$$

One can prove theorems 2 and 3 owing to (4.3) *and* (4.2) *respectively.*

5. ESTIMATION OF FUNDAMENTAL POLYNOMIALS

Lemma 1. *Let $L_k(z)$ be given by* (2.4). *Then*

(5.1) $$\max_{|z|=1} \sum_{k=0}^{2n+1} | L_k(z) | \le c \log n$$

where c is a constant independent of n and z .

Proof. From maximal principal, we know

$$\lambda_n = \max_{|z|=1} \lambda_n(z)$$
$$\lambda_n = \sum_{k=0}^{2n+1} |L_k(z)|.$$

Let $z = x+iy$ and $|z| = 1$. Then we see, for $0 \le \arg z < \pi$ and $k = 1,2,......n$.

$$|L_k(z)| = \left| \frac{(z^2-1)W(z)}{(z_k^2-1)W\prime(z_k)(z-z_k)} \right|.$$

Using (2.2) and (2.5), we get that

$$|L_k(z)| = \left| \frac{(1-x^2)^{\frac{1}{2}} P_n^{(\alpha,\beta)}(x) \left[(1-xx_k) + (1-x^2)^{\frac{1}{2}} \ (1-x_k^2) \ ^{\frac{1}{2}}\right]^{\frac{1}{2}}}{\sqrt{2}(1-x_k^2) P_n^{(\alpha,\beta)\prime}(x_k)(x-x_k)} \right|$$

$$\le \frac{(1-x^2)^{\frac{1}{2}} P_n^{(\alpha,\beta)}(x)(1-xx_k)^{\frac{1}{2}}}{(1-x_k^2) P_n^{(\alpha,\beta)\prime}(x_k)(x-x_k)} = G_k(x).$$

Also, we have $|L_{n+k}(z)| \le G_k(x)$.

Similarly for $\pi \le \arg z < 2\pi$ and $k = 1,2,.........n$, we have

$$|L_k(z)| \le G_k(x), \qquad |L_{n+k}(z)| \le G_k(x).$$

For a fixed $z = x+iy$, $|z| = 1$ and $-1 < x < 1$, we see that

$$\lambda_n(z) \le 2\sum_{k=1}^{n} G_k(x) + |L_0(z)| + |L_{2n+1}(z)|$$
$$= 2 \sum_{|x_k-x| \ge \frac{1}{2}(1-x_k^2)} G_k(x) + 2 \sum_{|x_k-x| < \frac{1}{2}(1-x_k^2)} G_k(x) + 2$$

Using (2.9), (2.10) and (2.11), we get the desired result.

Lemma 2. *Let* $B_k(z)$ *be given by* (4.4). *Then*

(5.2) $$\sum_{k=0}^{2n+1} |B_k(z)| \le c \frac{\log n}{n}$$

where c is a constant independent of n and z .

Proof. In view of Lemma 1 and using (2.3), (2.7), (2.9), (2.10) and (2.11), we get the required result.

Lemma 3. *Let* $A_k(z)$ *be given by* (4.5). *Then*

(5.3) $$\sum_{k=0}^{2n+1} |A_k(z)| \le c \log n,$$

where c is a constant independent of n and z .

Proof. In view of Lemma 1 and 2, we find the desired result.

6.CONVERGENCE:

Let $f(z)$ be analytic for $|z| < 1$ and continuous for $|z| \leq 1$ and $\omega(f,\delta)$ be the modulus of continuity of $f\left(e^{ix}\right)$

THEOREM 4 : *Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Let the arbitrary numbers β_k's be such that*

$$(6.1) \qquad |\beta_k| = o\left(n\omega\left(f,n^{-1}\right)\right) \quad , k = 1(1)2n$$

Then R_n be defined by

$$(6.2) \qquad R_n(z) = \sum_{k=0}^{2n+1} f(z_k) A_k(z) + \sum_{k=0}^{2n+1} \beta_k B_k(z)$$

satisfies the relation

$$(6.3) \qquad |R_n(z) - f(z)| = o\left(\omega(f,n^{-1})\log n\right),$$

where $\omega\left(f,n^{-1}\right)$ is the modulus of continuity of $f(z)$.

To prove theorem 4, we shall need the following:

Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$.Then there exists a polynomial of degree $2n-2$ satisfying Jackson's inequality

$$(6.4) \qquad |f(z) - F_n(z)| \leq c\omega\left(f,n^{-1}\right) \quad , \qquad z = e^{i\theta}\,(0 < \theta \leq 2\pi)$$

and also an inequality due to O.Kiš [5]

$$(6.5) \qquad \left|F_n^{(m)}(z)\right| \leq cn^m \omega\left(f,n^{-1}\right) \quad \text{for } m = 1.$$

PROOF: Since $R_n(z)$ be given by (6.2) is a uniquely determined polynomial of degree $\leq 4n+3$, the polynomial $F_n(z)$ satisfying (6.4) and (6.5) can be expressed as

$$F_n(z) = \sum_{k=0}^{2n+1} F(z_k) A_k(z) + \sum_{k=0}^{2n+1} F_n'(z_k)\, B_k(z)$$

Then ,
$$\begin{aligned} |R_n(z) - f(z)| &\leq |R_n(z) - F_n(z)| + |F_n(z) - f(z)| \\ &\leq \sum_{k=0}^{2n+1} |f(z_k) - F_n(z_k)|\,|A_k(z)| \\ &+ \sum_{k=0}^{2n+1} \{|\beta_k| + F_n'(z_k)\}\,|B_k(z)| + |F_n(z) - f(z)| \end{aligned}$$

Using $z = e^{i\theta}$ $(0 < \theta \leq 2\pi)$, (6.1), (6.4), (6.5) and Lemma 2 and 3 , we get (6.3)

Conflict of Interests

The authors declare that there is no conflict of interests.

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