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ON THE q-VARIANT OF INTEGRAL BASKAKOV OPERATORS

ASHA RAM GAIROLA[1,*], GIRISH DOBHAL[1], KARUNESH KUMAR SINGH[2]

[1]Department of Mathematics, University of Petroleum & Energy Studies-Dehradun, India

[2]Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667 (Uttarakhand), India

Abstract. In this paper, we introduce a q-variant of integral Baskakov operator and study their approximation properties. We establish point wise and uniform convergence theorems in ordinary approximation. The rate of weighted approximation by means of Steklov functions in terms of a suitable modulus of smoothness is obtained.

Keywords: q-integrals; approximation by operators; rate of convergence.

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1. Introduction

A $q-$integral analogue of the $q-$Bernstein polynomials

$$B_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{n,k}(q;x),\ f \in C[0,1],$$

$p_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{r=0}^{n-k-1}(1-q^r x)$ defined by Phillips [9] was introduced by Derriennic [5] wherein she established some of their approximation properties. Motivated by the generalization in [5] we propose the operators $\mathscr{M}_{n,q}(f,x)$ as follows:

Let $C_B(R_0)$ be the class of bounded and continuous functions on $R_0 = [0,\infty)$. For $f \in C_B(R_0)$

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*Corresponding author

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we define

$$\mathscr{M}_{n,q}(f,x) = [n-1]_q \sum_{k=1}^{\infty} p_{n,k}(q;x) \int_0^{\infty/A} q^{k-1} p_{n,k-1}(q;u) f(u)\, d_q u + \frac{f(0)}{(1+x)_q^{n+k}},$$

$p_{n,k}(q;x) = \left[\begin{smallmatrix} n+k-1 \\ k \end{smallmatrix}\right]_q \frac{q^{k(k-1)/2} x^k}{(1+x)_q^{n+k}}$ whenever the integral exists in Jackson sense of improper integral [6]. The operators $\mathscr{M}_{n,q}$ are linear and positive.

In what follows, we shall use the notations $\varphi^2(x) = x(1+x)$, $N_0 = \mathbb{N} \cup \{0\}$ and the weights $w_0(x) = 1, w_m(x) = (1+x^m)^{-1}, m \in N_0$.

$B_m(R_0) := \{f : w_m(x)|f(x)| \leqslant M_1\}$,

$C_B(R_0) := \{f \in B_m(R_0) :\ \text{and}\, f\, \text{is}\, \text{continuous}\}$,

$C_m(R_0) := \left\{f \in C_m(R_0) :\ \text{and} \lim\limits_{x \to \infty} w_m(x) f(x) = M_2\right\}$,

where M_1 and M_2 depend on f only.

The $m-$th polynomial weighted spaces $C_m^*(R_0)$ are defined as follows:

$$C_m^*(R_0) = \{f : R_0 \to R |\, w_m f \,\text{is uniformly continuous and bounded on}\, R_0.\}$$

The space $C_m(R_0)$ is normed by $\|f\|_m := \sup_{x \in R_0} w_m(x)|f(x)|$. It is easy to see that $C_m^*(R_0) \subset C_m(R_0) \subset C_B(R_0) \subset B_m(R_0)$. The set $C_m(R_0)$ is a Banach space under the norm $\|.\|_m$.

We shall use the weighted modulus of continuity $\Omega(f,\delta)$. This modulus has the advantage over the usual modulus of continuity $\omega(f,\delta)$ that they tend to zero as $\delta \to 0$.

For $f \in C_m^*(R_0)$ the first and the second order weighted modulus of continuity are defined by

$$\Omega_m(f,\delta) = \sup_{|h| \leqslant \delta, x \in R_0} \frac{|\overrightarrow{\triangle}_h f(x)|}{(1+x^m)(1+h^m)} \,\text{and}\, \Omega_{m,2}(f,\delta) = \sup_{|h| \leqslant \delta, x \in R_0} \frac{|\overrightarrow{\triangle}_h^2 f(x)|}{(1+x^m)(1+h^m)},$$

respectively, where $\overrightarrow{\triangle}_h f(x)$ and $\overrightarrow{\triangle}_h^2 f(x)$ are the first and the second order forward differences for step size h.

We recall some definitions of $q-$calculus used in this paper which can be found in [7] and [10]. Let q be a real number satisfying $0 < q < 1$ and $\mathbb{N}$ the set of positive integers. For $n \in \mathbb{N}$, we define

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q} = 1 + q + \ldots + q^{n-1}, & q \neq 1 \\ n, & q = 1. \end{cases}$$

$$[n]_q! = \begin{cases} [n]_q[n-1]_q[n-2]_q.....[1]_q, & n = 1,2,...... \\ 1, & n = 0. \end{cases}$$

The $q-$ binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are given by the quotient $\frac{[n]_q!}{[k]_q![n-k]_q!}$, $0 \leqslant k \leqslant n$ and $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$. The $q-$rising product $(a+b)_q^n$ is defined by

$$(a+b)_q^n = \prod_{j=0}^{n-1}(a+q^j b).$$

The $q-$ Jackson integrals and $q-$ improper integrals are given by (see [6], [8])

$$\int_0^a f(x)\, d_q x = (1-q)\, a \sum_{n=0}^{\infty} f(aq^n)\, q^n$$

and

$$\int_0^{\infty/A} f(x)\, d_q x = (1-q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, A > 0$$

respectively. It is assumed that the sums converge absolutely. For any arbitrary real function $f : \mathbb{R} \to \mathbb{R}$ and $q \in (0,1)$ the $q-$ derivative $D_q f(t)$ is defined as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}; t \neq 0.$$

For $t = 0$ we take $D_q f(t) = \lim_{t\to 0} D_q f(t); t = 0$. The product formula for q-differentiation is given by

$$D_q(f(x)g(x)) = f(qx)D_q(g(x)) + g(x)D_q(f(x)).$$

Analogous to the classical gamma and beta functions the $q-$ gamma and $q-$ beta functions are introduced. The $q-$gamma function is given by the integral

$$\Gamma_q(t) = \int_0^{1/(1-q)} x^{t-1} E_q^{-qx}\, d_q x.$$

The $q-$beta function is given by

$$B_q(t,s) = K(A,t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}}\, d_q x, \tag{1.1}$$

where the function $K(x,t)$ is defined by

$$K(x,t)=\begin{cases} \frac{1}{1+x}x^t(1+\frac{1}{x})_q^t(1+x)_q^{1-t};\, t\in\mathbb{R} \\ 1;\, t=0 \\ q^{\frac{t(t-1)}{2}};\, t\in\mathbb{N}. \end{cases}$$

The functions $\Gamma_q(t)$ and $B_q(t,s)$ satisfy certain properties similar to those of $\Gamma(t)$ and $B(t,s)$ e.g. $B_q(t,s)=\frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}$ etc. and reduce to $\Gamma(t)$ and $B(t,s)$ respectively in the limit $q\to 1$. We discuss the convergence results with the help of Korovkin type approximation theorems. In the end section we obtain error estimates of weighted approximation in certain polynomial weighted space. Henceforth, we shall simply use $[n]$ in place of $[n]_q$ unless otherwise stated. Moreover, M will be a constant different at each occurrence and will be independent of n always, but may depend on q.

2. Preliminaries and Lemmas

Lemma 1. *Let us define* $\mu_{n,m}(x)=\mathscr{M}_{n,q}(t^m,x)$. *Then, we have*

$$\mu_{n,0}(x)=1, \mu_{n,1}(x)=\frac{[n]x}{q[n-2]} \text{ and } \mu_{n,2}(x)=\frac{[n][n+1]x^2+2q[n]x}{q^4[n-2][n-3]}.$$

Further, The following recurrence relation holds for $n>m+2$*:*

$$\mu_{n,m+1}(qx)=\frac{\Big([n]x+\big([m+1]q^{-1}-1\big)\Big)\mu_{n,m}(qx)+\varphi^2(x)D_q\mu_{n,m}(x)}{q^m[n-m-2]}. \tag{2.1}$$

Proof. Making use of the $q-$Taylor's formula $g(z)=\sum_{k=0}^{\infty}\frac{(z-x)_q^k}{[k]!}\left(D_q^k g(z)\right)_{z=x}$ for the function $g(z)=\frac{1}{(1+z)_q^{n+1}}$ at $z=0$ together the relation $(-x)_q^k=q^{k(k-1)/2}(-1)^k x^k$ we obtain

$$\begin{aligned} 1=g(0)= &\sum_{k=0}^{\infty}\frac{[n+k]...[n+1]}{[k]!}q^{k(k-1)/2}\frac{x^k}{(1+x)_q^{n+k+1}} \\ = &\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{k(k-1)/2}\frac{x^k}{(1+x)_q^{n+k+1}}. \end{aligned} \tag{2.2}$$

Therefore, we get from definitions of $\mu_{n,m}(x)$ and (1.1)

$$\begin{aligned}\mu_{n,0}(x) &= [n-1]\sum_{k=1}^{\infty} q^{k-1} p_{n,k}(q,x)\int_0^{\infty/A} p_{n,k-1}(q,t)\,d_q u+\frac{1}{(1+x)_q^{n+k}}\\ &= [n-1]\sum_{k=1}^{\infty}\begin{bmatrix} n+k-1\\ k\end{bmatrix}_q \frac{x^k}{(1+x)_q^{n+k}}\begin{bmatrix} n+k-2\\ k-1\end{bmatrix}_q \frac{B_q(k,n-1)}{K(A,k)} q^{k(k-1)}\\ &+ \frac{1}{(1+x)_q^{n+k}}\\ &= \sum_{k=0}^{\infty}\begin{bmatrix} n+k-1\\ k\end{bmatrix}_q q^{k(k-1)/2}\frac{x^k}{(1+x)_q^{n+k}}=\sum_{k=0}^{\infty} p_{n,k}(q,x)=1.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\mu_{n,1}(x) &= \sum_{k=0}^{\infty} q^{k-1}\begin{bmatrix} n+k-1\\ k\end{bmatrix}_q \frac{x^k}{(1+x)_q^{n+k}} q^{k(k-1)/2}\int_0^{\infty/A} p_{n,k-1}(q,t)t\,d_q t\\ &= \sum_{k=1}^{\infty} q^{k-1}\begin{bmatrix} n+k-1\\ k\end{bmatrix}_q \frac{x^k}{(1+x)_q^{n+k}} q^{k(k-1)/2}\frac{[k]q^{-2k+1}}{[n-2]}\\ &= \frac{x}{q[n-2]}\sum_{k=1}^{\infty} q^{k(k-1)/2}\frac{[n+k]!}{[k]![n-1]!}\frac{x^k}{(1+x)_q^{n+1+k}}\\ &= \frac{[n]x}{q[n-2]}.\end{aligned}$$

Now, using $q^k\varphi^2(x)D_q[p_{n,k}(q;x)]=\Big([k]-q^k[n]x\Big)p_{n,k}(q;qx)$, (see [4]) we obtain

$$\begin{aligned}&\varphi^2(x)D_q\mu_{n,m}(x)+[n]x\mu(qx)\\ &= [n-1]\sum_{k=0}^{\infty} q^{-k}[k]p_{n,k}(q;qx)\int_0^{\infty/A} q^{k-1}p_{n,k}(q;u)u^m\,d_q u\\ &= I_1+I_2+I_3 \text{ say,}\end{aligned}$$

where we have written the quantity $[k]$ as $q^k+\big([k-1]-q^{k-2}[n]t\big)+q^{k-2}[n]t$ and I_1, I_2 and I_3 correspond to these three quantities used in above integral. Clearly $I_1=\mu(qx)$, $I_3=[n]q^{-2}\mu_{m+1}(qx)$.

The transformation $t \to qu$ is valid in $q-$integration, therefore, we get

$$\begin{aligned}
I_2 &= q^m \sum_{k=1}^{\infty} p_{n,k}(qx) \int_0^{\infty/A} \Big([k-1] - q^{k-1}[n]u\Big)\, p_{n,k-1}(qu) u^m \, d_q u \\
&= q^m \sum_{k=1}^{\infty} p_{n,k}(qx) \int_0^{\infty/A} q^{k-1} \varphi^2(u) u^m \Big(D_q p_{n,k-1}(u)\Big)\, d_q u \\
&= q^m \sum_{k=1}^{\infty} p_{n,k}(qx) \int_0^{\infty/A} q^{k-1} \left(u^{m+1} + u^{m+2}\right) \Big(D_q p_{n,k-1}(u)\Big)\, d_q u.
\end{aligned}$$

Using $q-$integration by parts we get

$$\begin{aligned}
\int_0^{\infty/A} u^{m+1} \Big(D_q p_{n,k-1}(u)\Big)\, d_q u &= u^{m+1} p_{n,k-1}(u)\Big|_0^{\infty/A} - \int_0^{\infty/A} p_{n,k-1}(qu) \Big(D_q u^{m+1}\Big)\, d_q u \\
&= -[m+1] \int_0^{\infty/A} p_{n,k-1}(qu) u^m \, d_q u.
\end{aligned}$$

Hence,

$$I_2 = -\left([m+1] q^{-1} \mu_m(qx) + [m+2] q^{-2} \mu_{m+1}(qx)\right).$$

Combining these expressions we obtain (2.1). From this recurrence relation $\mu_{n,2}$ is easily obtained. □

Lemma 2. *The quantity*

$$A_n = 2q[n] - [n][n+1] - 2q^3[n][n-3] + q^4[n-2][n-3]$$

is negative for all q and $n \geqslant 4$.

Proof. We have

$$\begin{aligned}
A_4 &= 2q[4] - [4][5] - 2q^3[4] + q^4[2] \\
&= -1 - q^2 - 4q^3 - 3q^4 - 4q^5 - 4q^6 - q^7 < 0.
\end{aligned}$$

Suppose the lemma holds true for a certain n. We write A_n as follows

$$\begin{aligned} A_n &= 2q[n]-([3]+q^3[n-3])([3]+q^3[n-2])-2q^3([2]+q^2[n-2])[n-3] \\ &+ q^4[n-2][n-3] \\ &= 2q[n]-[3]^2-[3]q^3[n-3]-[3]q^3[n-2]-q^4(1-q)^2[n-2][n-3] \end{aligned}$$

so that

$$\begin{aligned} A_{n+1}-A_n &= 2q^{n+1}-[3]q^n-[3]q^{n+1}-q^4(1-q)^2[n-2]([n-1]-[n-3]) \\ &= -(1+2q^2+q^3)q^n-q^{n+1}(1-q)^2[n-2](1+q) \end{aligned}$$

which is negative. This completes the proof. □

Lemma 3. *For the functions $\psi_m(q,x)$ defined by $\psi_m(q,x)=\mathscr{M}_{n,q}((t-x)^m,x)$, we have*

$$\psi_0(q,x)=1,\ \psi_1(q,x)=\frac{(1+q^{n-1})x}{q[n-2]} \tag{2.3}$$

and there holds

$$\psi_2(q,x)\leqslant\frac{4}{q^4[n-2]}\delta_n^2(x),\,\forall x\in R_0,\,n>3,$$

where $\delta_n^2(x)=\left(\varphi^2(x)+\frac{3}{[n-3]}\right)$.

Proof. Since, $\mathscr{M}_{n,q}(f,x)$ are linear, (2.3) follows from (2.1) and direct calculations. Now, using the values of $\mu_{n,0}(x),\mu_{n,1}(x)$ and $\mu_{n,2}(x)$ we get

$$\psi_2(q,x) = \frac{c_0x^2+c_1x}{q^4[n-2][n-3]}=\frac{c_0\varphi^2(x)+(c_1-c_0)x}{q^4[n-2][n-3]},$$

where c_0 and c_1 are the coefficients in numerator given by $c_0=[n][n+1]-2q^3[n][n-3]+q^4[n-2][n-3]$ and $c_1=2q[n]$. From lemma 2 we have $c_1-c_0=A_n<0$. And we can write

$$c_0=a_0+a_1q+a_2q^2+...+a_{2n-1}q^{2n-1}.$$

It is observed that $a_j\leqslant 2: j=1,2....2n-1$. Hence,

$$c_0\leqslant 2(1+q+q^2......+q^{2n-1})=2[2n].$$

Now, $[2n]=(1+q^{n-3})[n-3]+q^{2n-6}[6]$. This gives

$$\begin{aligned}\psi_2(q,x) &\leqslant \frac{2}{q^4}\left(\frac{1+q^{n-3}}{[n-2]}+\frac{[6]q^{2n-6}}{[n-2][n-3]}\right)\varphi^2(x)\\ &\leqslant \frac{4}{q^4[n-2]}\delta_n^2(x).\end{aligned}$$

□

Lemma 4. *For $n>2m+1$, we have*

$$\mathscr{M}_{n,q}\left(\frac{(t-x)^2}{w_m(x)},x\right)\leqslant M\left(\psi_4(q,x)\right)^{1/2}\left(1+x^{2m}\right)^{1/2}.$$

Proof. Since $\mu_m(q,x)$ are polynomials of degree exactly m we can write $\mu_m(q,x)=\sum_{k=0}^m a_k^m x^k$. Using in the recurrence relation (2.1) we get

$$\begin{aligned}&q^m[n-m-2]\sum_{k=0}^{m+1}a_k^{m+1}(qx)^k\\ =\ &(x+x^2)\sum_{k=0}^{m}a_k^m[k](x)^{k-1}+\left([n]x+\left([m+1]q^{-1}-1\right)\right)\sum_{k=0}^{m}a_k^m(qx)^k\\ =\ &\sum_{k=0}^{m}a_k^m\left\{\left([k]+\left([m+1]q^{-1}-1\right)q^k\right)x^k+[n+k]x^{k+1}\right\}.\end{aligned}$$

Comparing coefficients on both sides, we get

$$a_m^m=\frac{1}{q^{m^2}}\prod_{j=0}^{m-1}\left(\frac{[n+j]}{[n-j-2]}\right)$$

which is the largest coefficient with respect to power of $[n]$. Similarly,

$$a_m^{m+1}=\frac{\left([2m]+q^{m-1}(1-q)\right)a_m^m+[n+m-1]a_{m-1}^m}{q^{2m}[n-m-2]}$$

etc. So that

$$a_{m-1}^m=\frac{[n][n+1]...[n+m-2]}{[n-2][n-3]...[n-m-1]}M_m(q).$$

Hence, we can write

$$\begin{aligned}\mu_m(q,x) &\leqslant M\left(x^m+\frac{[n][n+1]...[n+m-2]}{[n-2][n-3]...[n-m-1]}\sum_{j=0}^{m-1}b_jx^j\right)\\ &\leqslant M\left(x^m+\frac{1}{[n-m-1]}\sum_{j=0}^{m-1}b_jx^j\right)\end{aligned}$$

which implies

$$\mu_{2m}(x) \leqslant M\left(x^{2m} + O\left(\frac{1}{[n-2m-1]}\right)\right).$$

Therefore, using Hölders inequality we get

$$\begin{aligned} \mathscr{M}_{n,q}\left(\frac{(t-x)^2}{w_m(x)},x\right) &\leqslant \left(\mathscr{M}_{n,q}\left((t-x)^4,x\right)\right)^{1/2}\left(\mathscr{M}_{n,q}\left((1+x^m)^2,x\right)\right)^{1/2} \\ &\leqslant M\left(\psi_4(q,x)\right)^{1/2}\left(1+x^{2m}\right)^{1/2}. \end{aligned}$$

□

Remark 1. *For $m < m' \in \mathbb{N}$, we have*

$$\begin{aligned} \|\mathscr{M}_{n,q}(w_m^{-1},x)\|_{m'} &= \sup_{x\in R_0} \frac{\mathscr{M}_{n,q}(1+x^m,x)}{(1+x^{m'})} \\ &\leqslant \sup_{x\in R_0} \frac{\left(1+x^m+O\left(\frac{1}{[n+m-1]}\right)\right)}{(1+x^{m'})} \leqslant M'(m,q). \end{aligned}$$

Therefore, in view of the properties of positive linear operators (see [3]*)it follows that $\mathscr{M}_{n,q}$ maps $C_m(R_0)$ into $B_{m'}(R_0)$.*

In order to test the convergence of the operators $L_n : C_m \to B_m$ in weighted approximation we will use the following Korovkin type theorem.

Theorem 1. [1] *Let $L_n : C_m \to B_m$ be a sequence of positive linear operators such that* $\lim_{n\to\infty} \|L_n(e_r) - e_r\|_m = 0$ *for $e_r = t^r, r = 0,1,2$. Then,*

$$\lim_{n\to\infty} \|L_n(f) - f\|_m = 0$$

for $f \in C_m^*(R_0)$.

The following theorem due to Pop [11] will be used in our asymptotic results.

Theorem 2. *Let $I \subset R$ be an interval, $x \in I, r \in N$ and the function $f : I \to R$, f is r times derivable in x. According to Taylor's expansion theorem for the function f around x, we have*

$$f(t) = \sum_{k=0}^{r} \frac{(t-x)^k}{k!} f^{(k)}(x) + (t-x)^r \mu(t-x),$$

where μ is a bounded function and $lim_{t\to x}\mu(t-x) = 0$. If $f^{(r)}$ is a continuous function on I, then for any $\delta > 0$ $|\mu(t-x)| \leqslant \frac{1}{r!}\left(1+\delta^{-2}(t-x)^2\right)\omega\left(f^{(r)},\delta\right)$.

3. Convergence

It is obvious from Lemma 1 that the operators $\mathscr{M}_{n,q}$ do not satisfy the conditions of the Bohman-Korovkin theorem in case $0 < q < 1$. To make this theorem applicable we can choose a sequence (q_n) in place of the number q such that $\lim_{n\to\infty} q_n = 1$. With this modification we obtain following Korovkin type theorem.

Theorem 3. *Let (q_n), $0 < q_n < 1$ be a real sequence. Then, the sequence $\mathscr{M}_{n,q_n}(f,x)$ converge uniformly to f for any $f \in C_B(R_0)$ iff* $\lim_{n\to\infty} q_n = 1$.

Proof. Suppose there holds the limit $\lim_{n\to\infty} q_n = 1$. Then, from the definition of $q-$integers we get $\lim_{n\to\infty}[n]_{q_n} = \infty$. Therefore, $\lim_{n\to\infty} \psi_m(q,x) = 0, m = 0,1,2$. Thus, using Bohman-Korovkin theorem it follows that $\mathscr{M}_{n,q_n}(f,x) \rightrightarrows f(x)$. Let, if possible $\lim_{n\to\infty} q_n \neq 1$. Since (q_n) is monotonically increasing and bounded by 1, it has a subsequence (q_{n_k}) converges to some q_0 in $(0,1)$. Also we get $\lim_{n\to\infty}[n]_{q_n} = \frac{1}{1-q_0}$. Consequently, it follows that

$$\begin{aligned}\lim_{n\to\infty} \mu_2(q,x) &= \lim_{n\to\infty} \frac{[n]_{q_n}[n+1]_{q_n}x^2}{q_n^4[n-1]_{q_n}[n-2]_{q_n}} + \frac{2q_n[n]_{q_n}}{q_n^4[n-1]_{q_n}[n-2]_{q_n}}x \\ &= q_0^{-4}x^2 + \frac{2(1-q_0)}{q_0^3}x \neq x^2\end{aligned}$$

which is contradiction. This completes the proof. □

Theorem 4. *Let (q_n), $0 < q_n < 1$ be a real sequence such that* $\lim_{n\to\infty} q_n = 1$. *Then, the sequence $\mathscr{M}_{n,q_n}(f,x)$ converge uniformly to f for any $f \in C_B(R_0)$ in ρ_m norm.*

Proof. Clearly, we have $\|\mathscr{M}_{n,q_n}e_0 - e_0\|_m$=0 and we have $\mathscr{M}_{n,q_n}(e_1,x) - e_1 = \psi_1(q_n,x) = \frac{1}{q_n[n-1]_{q_n}} + \frac{([2]_{q_n}-q_n^n)x}{q_n^2[n-1]_{q_n}}$ so that

$$\begin{aligned}\|\mathscr{M}_{n,q_n}e_1 - e_1\|_m &= \sup_{x\in R_0} \frac{1}{q_n[n-1]_{q_n}}\left(\frac{1}{1+x^m} + \frac{([2]_{q_n}-q_n^n)x}{1+x^m}\right) \\ &\leqslant \frac{1}{q_n[n-1]_{q_n}}\left(1 + \frac{([2]_{q_n}-q_n^n)(m-1)^{1-1/m}}{m}\right), m \neq 1.\end{aligned}$$

Consequently,

$$\|\mathscr{M}_{n,q_n}e_1 - e_1\|_m \to 0.$$

Next, we have $\mathscr{M}_{n,q_n}(e_2,x)-e_2=\frac{[2]_{q_n}q_n^3}{q_n^4[n-1]_{q_n}[n-2]_{q_n}}+\frac{q_n[2]_{q_n}^2[n]_{q_n}x}{q_n^4[n-1]_{q_n}[n-2]_{q_n}}+\left(\frac{[n]_{q_n}[n+1]_{q_n}}{q_n^4[n-1]_{q_n}[n-2]_{q_n}}-1\right)x^2$. Therefore, we get

$$\begin{aligned}
&\|\mathscr{M}_{n,q_n}(e_2,x)-e_2\|_m\\
\leqslant\ & \frac{[2]_{q_n}}{q_n[n-1]_{q_n}[n-2]_{q_n}}+\frac{q_n[2]_{q_n}^2[n]_{q_n}(m-1)^{1-1/m}}{mq_n^4[n-1]_{q_n}[n-2]_{q_n}}\\
+\ & \left|\frac{[n]_{q_n}[n+1]_{q_n}}{q_n^4[n-1]_{q_n}[n-2]_{q_n}}-1\right|\frac{2^{2/m}(m-2)^{1-2/m}}{m}\\
\leqslant\ & \frac{[2]_{q_n}}{q_n[n-1]_{q_n}[n-2]_{q_n}}+\frac{q_n[2]_{q_n}^2[n]_{q_n}(m-1)^{1-1/m}}{mq_n^4[n-1]_{q_n}[n-2]_{q_n}}\\
+\ & \left|\frac{[2]_{q_n}^2}{q_n^4[n-1]_{q_n}[n-2]_{q_n}}+\frac{[2]_{q_n}}{q_n^2[n-1]_{q_n}}+\frac{[2]_{q_n}}{q_n^2[n-2]_{q_n}}\right|\frac{2^{2/m}(m-2)^{1-2/m}}{m}
\end{aligned}$$

Hence,

$$\|\mathscr{M}_{n,q_n}(e_2,x)-e_2\|_m\to 0.$$

Therefore, the proof follows from these limits and Theorem 1. □

Theorem 5. (Voronovskaya-type) *If* $f,f',f''\in C_B(R_0)$, *and* q_n *be a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty}q_n=1$, *then we have*

$$\lim_{n\to\infty}\Big(\mathscr{M}_{n,q_n}(f,x)-f(x)\Big)=xf'(x)+x^2f''(x).$$

Proof. The proof follows from $\lim_{n\to\infty}\mathscr{M}_{n,q_n}((t-x)^j,x)$, $j=1,2$. Using Theorem 2 and the limit $\lim_{q_n\to 1}[n]_{q_n}=n$, we get

$$\begin{aligned}
&\lim_{n\to\infty}\left(\mathscr{M}_{n,q_n}(f,x)-f(x)-\psi_1(q_n,x)f'(x)-\frac{1}{2}\psi_2(q_n,x)f''(x)\right)\\
=\ &\lim_{n\to\infty}\mathscr{M}_{n,q_n}((t-x)^2\mu(t-x),x).
\end{aligned}$$

Now, we have

$$\begin{aligned}
&\left|\mathscr{M}_{n,q_n}((t-x)^2\mu(t-x),x)\right|\\
\leqslant\ & M_1\psi_4^{1/2}(q,x)\left(\mathscr{M}_{n,q_n}(\mu^2(t-x),x)\right)^{1/2}.
\end{aligned}$$

Since $\mu(t-x) \in C_B(R_0)$ it follows that $\lim_{n\to\infty} \mathscr{M}_{n,q_n}\left((\mu^2(t-x),x)\right) = \mu^2(x-x) = 0$. Therefore, $\lim_{n\to\infty} \mathscr{M}_{n,q_n}((t-x)^2\mu(t-x),x) = 0$ implies

$$\lim_{n\to\infty}\left(\mathscr{M}_{n,q_n}(f,x) - f(x)\right) = \lim_{n\to\infty}\left(\psi_1(q_n,x)f'(x) + \frac{1}{2}\psi_2(q_n,x)f''(x)\right). \tag{3.1}$$

We obtain

$$\begin{aligned}\lim_{n\to\infty}\psi_1(q_n,x) &= \lim_{n\to\infty}\frac{[n]_{q_n}x}{q_n[n-2]_{q_n}}\\ &= \lim_{n\to\infty}\frac{[2]_{q_n}}{q_n[n-2]_{q_n}} + q_n x = x\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\psi_2(q_n,x)\\ &\quad == \lim_{n\to\infty}\left(\mu_2(x) - 2x\mu_1(x) + x^2\mu_0(x)\right)\\ &\quad = \lim_{n\to\infty}\left[\frac{[n]_{q_n}x\left([n+1]_{q_n}x + 2q_n\right)}{q_n^4[n-2]_{q_n}[n-3]_{q_n}} - 2x\frac{[n]_{q_n}x}{q_n[n-2]_{q_n}} + x^2\right]\\ &\quad = \lim_{n\to\infty}\left[\left\{1 + [3]_{q_n}q_n^{n-3}\left(\frac{1}{q_n^3[n-2]_{q_n}} + \frac{1}{q_n^4[n-2]_{q_n}}\right)\right.\right.\\ &\qquad \left. + \frac{q_n^{2n-5}[3]_{q_n}^2}{q_n^4[n-2]_{q_n}[n-3]_{q_n}}\right\}x^2\\ &\qquad + \frac{2\left([3]_{q_n}q_n^{n-3} + [n-3]_{q_n}\right)}{q_n^3[n-2]_{q_n}[n-3]_{q_n}}x - \frac{2\left([3]_{q_n}q_n^{n-3} + [n-3]_{q_n}\right)}{q_n^4[n-2]_{q_n}[n-3]_{q_n}}x^2 + x^2 = 2x^2.\end{aligned} \tag{3.3}$$

Therefore, the proof follows from (3.1) to (3.3). □

4. Local Approximation

The error estimates similar to those in [4], can be obtained by methods used therein. In order to make the paper complete, we mention two of them without proof.

Theorem 6. *Let $f \in C_B(R_0)$, $q \in (0,1)$ and $n \geqslant 3$. We have*

$$|\mathscr{M}_{n,q}(f,x) - f(x)| \leqslant M\omega_2\left(f, \frac{\delta_n}{\sqrt{q^4[n-2]}}\right) + \omega\left(f, \frac{\left(1+q^{n-1}\right)x}{q[n-2]}\right).$$

for every $x \in [0,\infty)$ and $f \in C_B(R_0)$.

Theorem 7. *Let $f \in Lip_M \alpha$, $\alpha \in (0,1]$ for $x \in [0,A]$ $A > 0$. Then,*

$$|\mathscr{M}_{n,q}(f) - f| \leqslant M\left(\frac{2\delta_n(x)}{\sqrt{q^4[n-2]}}\right)^{\alpha/2}.$$

5. Weighted Approximation

Theorem 8. *For $f \in C_m^*(R_0), m \in N_0, n \geqslant 3$ there holds*

$$w_m(x)\left|\mathscr{M}_{n,q}(f,x) - f(x)\right| \leqslant 8\left(1 + \left(\frac{2^{m-1}\delta_n(x)}{\sqrt{q^4[n-2]}}\right)^m\right)\Omega_m\left(f, \frac{2^{m-1}\delta_n(x)}{\sqrt{q^4[n-2]}}\right).$$

Proof. We use Steklov functions $f_h = \frac{2}{h}\int\limits_0^{h/2} f(x+u)\,du$ as members of interpolation space between $C_m^*(R_0)$ and $C_m(R_0)$.

$$\begin{aligned}
&w_m(x)\left(\mathscr{M}_{n,q}(f,x) - f(x)\right)\\
&= w_m(x)\Big[\left(\mathscr{M}_{n,q}(f - f_h, x)\right) + \left(\mathscr{M}_{n,q}(f_h,x) - f_h(x)\right) + (f_h(x) - f(x))\Big]\\
&= E_1 + E_2 + E_3, \text{ say.}
\end{aligned}$$

It is sufficient to compute E_2 because $w_m(x)|E_1| \leqslant w_m(x)|E_3| \leqslant \|f_h - f\|_m$. Using the smoothness of f_h by writing $f_h(t) = f_h(x) + (t-x)f_h'(\theta)$ where θ lies between t and x we get

$$\begin{aligned}
w_m(x)|E_2| &= w_m(x)\left|\mathscr{M}_{n,q}\left((t-x)f_h'(\theta), x\right)\right|\\
&\leqslant \|f_h'\|_m \mathscr{M}_{n,q}(|t-x|, x) \leqslant \sqrt{\frac{4}{q^4[n-2]}}\,\delta_n(x)\|f_h'\|_m.
\end{aligned}$$

By direct calculations we get $\|f_h - f\|_m \leqslant 2(1+h^m)\Omega_m(f,h)$ and $\|f_h'\|_m \leqslant \frac{2^m}{h}(1+h^m)\Omega_m(f,h)$. Therefore, we obtain

$$\begin{aligned}
w_m(x)\left|\mathscr{M}_{n,q}(f,x) - f(x)\right| &\leqslant \left(4(1+h^m) + \frac{2^m}{h}(1+h^m)\delta_n(x)\sqrt{\frac{4}{q^4[n-2]]}}\right)\Omega_m(f,h)\\
&\leqslant 4(1+h^m)\left(1 + \frac{2^{m-1}}{h}\delta_n(x)\sqrt{\frac{1}{q^4[n-2]}}\right)\Omega_m(f,h).
\end{aligned}$$

The proof follows by choosing $h = \frac{2^{m-1}\delta_n(x)}{\sqrt{q^4[n-2]}}$. $\square$

Theorem 9. *If $f \in C_m^*(R_0), m \in N_0$ and $n \geqslant 3$ then, there holds*

$$\begin{aligned} & w_m(x)\left|\mathscr{M}_{n,q}(f,x)-f(x)\right| \\ \leqslant \;& M\left(1+\|\psi_1^m\|\right)\left(3\Omega_m(f,\|\psi_1\|)+\frac{9}{h^2}\left(\psi_2(q,x)+\frac{\psi_1^2(q,x)}{2}+\sqrt{\psi_4(q,x)}\right)\Omega_{m,2}(f,\|\psi_1\|)\right), \end{aligned}$$

where $M = M(q,m)$.

Proof. Let $f_{2,h}$ be the Steklov function of order two corresponding to f given by

$$f_{2,h}(x)=\frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\left(2f(x+s+t)-f(x+2s+2t)\right)ds\,dt.$$

It is known that ([2])

$$\|f''_{2,h}\|_m \leqslant \frac{9}{h^2}(1+h^m)\Omega_{m,2}(f,h)$$

and

$$\|f-f_{2,h}\|_m \leqslant 2(1+h^m)\Omega_m(f,h).$$

We define the operator $\overline{\mathscr{M}}_{q,n}(f,x)=\mathscr{M}_{q,n}(f,x)-f(x+\psi_1(q,x))+f(x)$ so that we can write

$$\begin{aligned} \mathscr{M}_{q,n}(f,x)-f(x) &= [\overline{\mathscr{M}}_{q,n}(f-f_{2,h},x)]+[\overline{\mathscr{M}}_{q,n}(f_{2,h},x)-f_{2,h}] \\ &+ [f(x+\psi_1(q,x))-f(x)], \end{aligned}$$

Using the smoothness of $f_{2,h}$ we write

$$f_{2,h}(t)=f_{2,h}(x)+(t-x)f'_{2,h}+R_2(f_{2,h},t,x),$$

$$R_2(f_{2,h},t,x)=\int_x^t (t-u)f''_{2,h}(u)\,du.$$

It follows that

$$\overline{\mathscr{M}}_{q,n}(f_{2,h},x)-f_{2,h}(x)=\mathscr{M}_{q,n}\left(R_2(f_{2,h},t,x),x\right)-\int_x^{x+\psi_1(q,x)}(x+\psi_1(q,x)-u)f''_{2,h}(u)\,du.$$

Therefore,

$$\begin{aligned} w_m(x)\left|\overline{\mathscr{M}}_{q,n}(f_{2,h},x)-f_{2,h}(x)\right| &\leqslant w_m(x)\left|\mathscr{M}_{q,n}\left(R_2(f_{2,h},t,x),x\right)\right| \\ &+ w_m(x)\left|\int\limits_x^{x+\psi_1(q,x)}(x+\psi_1(q,x)-u)f''_{2,h}(u)\,du\right| \\ &= F_1+F_2, \text{ say.} \end{aligned}$$

Now, using Lemma 3 and Lemma 4, we get

$$\begin{aligned} F_1 &= w_m(x)\left|\mathscr{M}_{q,n}\left(R_2(f_{2,h},t,x),x\right)\right| \\ &\leqslant w_m(x)\mathscr{M}_{q,n}\left(\left|\int\limits_x^t(t-u)f''_{2,h}(u)\,du\right|,x\right) \\ &\leqslant \|f''_{2,h}\|_m\mathscr{M}_{q,n}\left(\left(1+\frac{w_m(x)}{w_m(t)}\right)(t-x)^2,x\right) \\ &\leqslant \|f''_{2,h}\|_m\left(\psi_2(q,x)+w_m(x)\mathscr{M}_{q,n}\left(\frac{(t-x)^2}{w_m(t)},x\right)\right) \\ &\leqslant M_1\|f''_{2,h}\|_m\left(\psi_2(q,x)+w_m(x)\left((\psi_4(q,x))\left(1+x^{2m}\right)\right)^{1/2.}\right) \\ &\leqslant M_2\|f''_{2,h}\|_m\left(\psi_2(q,x)+\sqrt{\psi_4(q,x)}\right). \end{aligned}$$

And

$$F_2\leqslant\frac{1}{2}\|f''_{2,h}\|_m\left(\psi_1(q,x)\right)^2.$$

Next

$$\begin{aligned} w_m(x)|f(x+\psi_1(q,x))-f(x)| &\leqslant w_m(x)\sup_{\psi_1(q,x)}\frac{\left|\overrightarrow{\Delta}_{\psi_1(q,x)}f(x)\right|}{1+\psi_1^m(q,x)}(1+\psi_1^m(q,x)) \\ &\leqslant (1+\psi_1^m(q,x))\Omega_m(f,\psi_1(q,x)). \end{aligned}$$

Combining these estimates we obtain

$$\begin{aligned} & w_m(x)\left|\mathscr{M}_{n,q}(f,x)-f(x)\right| \\ \leqslant\ & M_3\|f-f_{2,h}\|_m\Omega_m(f,h)+\frac{9}{h^2}(1+h^m)\times \\ \times\ & \Omega_{m,2}(f,h)\left(\psi_2(q,x)+\frac{\psi_1^2(q,x)}{2}+\sqrt{\psi_4(q,x)}\right)+(1+\|\psi_1^m\|)\,\Omega_m(f,\|\psi_1\|). \end{aligned}$$

Choosing $h = \|\psi_1(q,x)\|$, gives

$$\begin{aligned} & w_m(x)\left|\mathscr{M}_{n,q}(f,x)-f(x)\right| \\ \leqslant\ & M_4\left(1+\|\psi_1^m\|\right)\Big(3\Omega_m(f,\|\psi_1\|)+\frac{9}{h^2}\times \\ \times\ & \left(\psi_2(q,x)+\frac{\psi_1^2(q,x)}{2}+\sqrt{\psi_4(q,x)}\right)\Omega_{m,2}(f,\|\psi_1\|)\Big). \end{aligned}$$

This completes the proof. □

Conflict of Interests

The author declares that there is no conflict of interests.

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