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AN EXTENSION TO PRICE'S INEQUALITY

SAMIH A. OBAID

Department of Mathematics and Statistics, San Jose State University, San Jose, California 95192-0103, USA

Abstract. We introduce a proof of an inequality motivated by Price's inequality. The new inequality involves hyperbolic functions instead of trigonometric functions.

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1. Introduction

In 2002 Price [1] derived the following inequality: Let $a \neq b \geq 0, \theta$ be real numbers and $n \geq 1$ an integer. Then

$$\frac{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}{a^2 + b^2 - 2ab\cos(\theta)} \leq \left(\frac{a^n - b^n}{a - b}\right)^2, \tag{1.1}$$

where equality holds when θ is zero. This inequality resulted from studying certain products of chords contained in an ellipse. Katsuura and Obaid [2] introduced three simpler proofs of the inequality. The inequality also led to other simple inequalities involving elementary functions of complex variables in [2].

E-mail address: samih.obaid@sjsu.edu

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By continuing this line of thought, one wonders whether the inequality (1) remains valid if we change the cosine function to a hyperbolic cosine function. It turns out this is not true However, we will prove the new inequality given as follows:

Let $a \neq b \geq 0$, and θ a real number such that $(b/a) \neq e^{\theta}$ or $e^{-\theta}$. Then

$$\Big(\frac{a^n - b^n}{a-b}\Big)^2 \leq \frac{a^{2n} + b^{2n} - 2a^n b^n \cosh(n\theta)}{a^2 + b^2 - 2ab\cosh(\theta)}, \tag{1.2}$$

where equality holds when θ is zero. The condition on θ is to avoid vanishing of the denominator on the right hand side.

2. Main results

We now prove inequality (1.2). The proof is by mathematical induction. It is sufficient instead to prove the following inequality for any $r \neq 1, r > 0$ and $r \neq e^{\theta}$ or $e^{-\theta}$:

$$\Big(\frac{r^n - 1}{r-1}\Big)^2 \leq \frac{r^{2n} + 1 - 2r^n \cosh(n\theta)}{r^2 + 1 - 2r\cosh(\theta)}. \tag{2.1}$$

The right side of the inequality (2.1) can be factored and thus inequality (2.1) becomes

$$\Big(\frac{r^n - 1}{r-1}\Big)^2 \leq \frac{(r^n - e^{n\theta})(r^n - e^{-n\theta})}{(r - e^{\theta})(r - e^{-\theta})}. \tag{2.2}$$

Let the right side of inequality (2.2) be R and let $\alpha = e^{\theta}$. Then

$$R = \Big[\frac{\big(\frac{r}{\alpha}\big)^n - 1}{\frac{r}{\alpha} - 1}\Big]\Big[\frac{\big(\alpha r\big)^n - 1}{\alpha r - 1}\Big] = \Big[\sum_{k=0}^{n-1}\big(\frac{r}{\alpha}\big)^k\Big]\Big[\sum_{k=0}^{n-1}\big(\alpha r\big)^k\Big], r \neq \alpha, r \neq 1/\alpha. \tag{2.3}$$

Then we need to show that

$$R \geq \Big(\frac{r^n - 1}{r-1}\Big)^2, r \neq 1, \alpha, \frac{1}{\alpha}. \tag{2.4}$$

We now prove (2.4) by mathematical induction. The case $n = 1$ is trivial. Suppose n is an integer such that the inequality (2.4) is valid. Thus we must show that (2.4) is valid when n is replaced by $n+1$. Let

$$S = \Big[\sum_{k=0}^{n}\big(\alpha r\big)^k\Big]\Big[\sum_{k=0}^{n}\big(\frac{r}{\alpha}\big)^k\Big] = \Big[\big(\alpha r\big)^n + \sum_{k=0}^{n-1}\big(\alpha r\big)^k\Big]\Big[\big(\frac{r}{\alpha}\big)^n + \sum_{k=0}^{n-1}\big(\frac{r}{\alpha}\big)^k\Big]. \tag{2.5}$$

Multiplying the above square brackets and using the induction hypothesis (2.4) yields

$$S \geq r^{2n} + \Big(\frac{r^n - 1}{r-1}\Big)^2 + r^n\Big[\frac{1}{\alpha^n}\sum_{k=0}^{n-1}\big(\alpha r\big)^k + \alpha^n\sum_{k=0}^{n-1}\big(\frac{r}{\alpha}\big)^k\Big]. \tag{2.6}$$

We now estimate the quantity in the square bracket of the last term in inequality (2.6). Then S may be rewritten in the form:

$$S \geq r^{2n} + \Big(\frac{r^n-1}{r-1}\Big)^2 + r^n \sum_{k=0}^{n-1} r^k \big(\alpha^{n-k} + \alpha^{-(n-k)}\big). \tag{2.7}$$

Since $\alpha = e^{\theta}$ and $\cosh[(n-k)\theta] \geq 1$, thus we have

$$S \geq r^{2n} + 2r^n \frac{r^n-1}{r-1} + (\frac{r^n-1}{r-1})^2 = \Big[r^n + \frac{r^n-1}{r-1}\Big]^2 = \Big[\frac{r^{n+1}-1}{r-1}\Big]^2, r \neq 1, \alpha, \frac{1}{\alpha}. \tag{2.8}$$

This completes the proof of the inequality (1.2).

Combining inequalities (1.1) and (1.2), we have for $r \neq 1, r > 0$ and $r = e^{\theta}$ or $e^{-\theta}$

$$\frac{a^{2n}+b^{2n}-2a^nb^n\cos(n\theta)}{a^2+b^2-2ab\cos(\theta)} \leq \frac{a^n-b^n}{a-b} \leq \frac{a^{2n}+b^{2n}-2a^nb^n\cosh(n\theta)}{a^2+b^2-2ab\cosh(\theta)}. \tag{2.9}$$

We conclude by indicating an alternate proof of the inequality (1.2). Expanding S in (2.5) by simple multiplication, we make an estimate for each term in the sum, it is Interesting that the coefficients are symmetric for the case n an even integer as seen below

$$S \geq 1 + 2r + 3r^2 + \cdots + nr^{n-1} + (n+1)r^n + \cdots + 2r^{2n-1} + r^{2n}.$$

Then we use the following identity which is valid for any $r > 0$ and any positive integer n:

$$\big(\sum_{k=1}^{n} r^k\big)^2 = \sum_{k=0}^{n}(k+1)r^k + r^{n+1}\sum_{k=1}^{n-1}(n-k)r^k.$$

The case when n is an odd integer is treated similarly.

Conflict of Interests

The author declares that there is no conflict of interests.

References

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