Available online at http://scik.org
Adv. Inequal. Appl. 2014, 2014:8
ISSN: 2050-7461

VISCOSITY ITERATIVE ALGORITHMS FOR VARIATIONAL INEQUALITY

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Abstract. In this paper, we consider a viscosity approximation algorithm for variational inequality problems and fixed point set of nonexpansive mappings. Strong convergence theorems are established in the framework of Hilbert spaces.

Keywords: viscosity approximation; variational inequality; Hilbert space; nonexpansive mapping

2000 AMS Subject Classification: 47H05, 47H09, 47H10

1. Introduction-Preliminaries

In this paper, we always assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$. Let C be a nonempty closed and convex subset of H. Recall that a mapping A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

Recall that a mapping A is said to be α-inverse-strongly monotone if there exists a real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

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Received August 18, 2013

Recall that the classical variational inequality problem, denoted by $VI(C,A)$, is to find $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

Given $z \in H, u \in C$, the following inequality holds

$$\langle u-z, v-u\rangle \geq 0, \quad \forall v \in C,$$

if and only if $u = P_C z$. It is known that the projection P_C is firmly nonexpansive. That is,

$$\|P_C x - P_C y\|^2 \leq \langle x-y, P_C x - P_C y\rangle, \quad \forall x, y \in C.$$

One can see that the variational inequality problem (1.2) is equivalent to a fixed point problem. That is, an element $u \in C$ is a solution of the variational inequality (1.2) if and only if $u \in C$ is a fixed point of the mapping $P_C(I-\lambda A)$, where $\lambda > 0$ is a constant and I is the identity mapping.

Let $S : C \to C$ be a mapping. In this paper, we use $F(S)$ to stand for the set of fixed points of the mapping S. Recall that the mapping S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x-y\|, \quad \forall x, y \in C.$$

Recently, the classical variational inequality (1.2) and fixed point problem of nonexpansive mappings have received rapid development, see, for example, [1-17] and the references therein.

In this paper, we consider a viscosity approximation algorithm for variational inequality problems and fixed point set of nonexpansive mappings. Strong convergence theorems are established in the framework of Hilbert spaces.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.2 *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero.*

Lemma 1.3 *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_i : 1 \leq i \leq r\}$ be a sequence of nonexpansive mappings on C. Suppose $\cap_{i=1}^{r} F(T_i)$ is nonempty. Let $\{\mu_i\}$ be a sequence of positive numbers with $\sum_{i=1}^{r} \mu_i = 1$. Then a mapping S on C defined by $Sx = \sum_{i=1}^{r} \mu_i T_i x$ for $x \in C$ is well defined, nonexpansive and $F(S) = \cap_{i=1}^{\infty} F(T_i)$ holds.*

Lemma 1.4 *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main results

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A_i : C \to H$ be a μ_i-inverse-strongly monotone mapping for each $1 \leq i \leq r$, where r is some positive integer. Let $S : C \to C$ be a nonexpansive mapping with a fixed point. Assume that $\mathscr{F} := \cap_{i=1}^{r} VI(C, A_i) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by the following manner:*

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S \sum_{i=1}^{r} \eta_i P_C(x_n - \lambda_i A_i x_n), \quad n \geq 1, \tag{2.1}$$

where $f : C \to C$ is a fixed contractive mapping, $\lambda_1, \lambda_2, \ldots$ and λ_r are real numbers such that $\lambda_i \in (0, 2\mu_i)$ for each $1 \leq i \leq r$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. Assume that the above control sequences satisfies the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = \sum_{i=1}^{r} \eta_i = 1$, $\forall n \geq 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *generated by the iterative algorithm* (2.1) *converges strongly to* $p = P_{\mathscr{F}} f(p)$.

Proof. For any $x, y \in C$, we see that

$$\begin{aligned}
&\|(I-\lambda_i A_i)x-(I-\lambda_i A_i)y\|^2 \\
&= \|x-y\|^2 - 2\lambda_i\langle A_i x - A_i y, x-y\rangle + \lambda_i^2\|A_i x - A_i y\|^2 \\
&\le \|x-y\|^2 - \lambda_i(2\mu_i - \lambda_i)\|A_i x - A_i y\|^2.
\end{aligned}$$

Since, for each $1 \le i \le r$, $\lambda_i \in (0, 2\mu_i)$, we see that $I - \lambda_i A_i$ is nonexpansive. Put $y_n = \sum_{i=1}^r \eta_i P_C(x_n - \lambda_i A_i x_n)$ for each $n \ge 1$. For any $x^* \in \mathscr{F}$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S y_n - x^*\| \\
&\le \alpha_n \|f(x_n) - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|Sy_n - x^*\| \\
&\le (1-\alpha_n(1-\alpha))\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\|.
\end{aligned}$$

By mathematical inductions, we find that the sequence $\{x_n\}$ is bounded. Note that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\sum_{i=1}^r \eta_i P_C(x_{n+1} - \lambda_i A_i x_{n+1}) - \sum_{i=1}^r \eta_i P_C(x_n - \lambda_i A_i x_n)\| \\
&\le \|x_{n+1} - x_n\|.
\end{aligned} \tag{2.2}$$

Put $l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, for all $n \ge 1$. That is,

$$x_{n+1} = (1-\beta_n) l_n + \beta_n x_n, \quad \forall n \ge 1. \tag{2.3}$$

Note that

$$\begin{aligned}
&l_{n+1} - l_n \\
&= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} S y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n S y_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}} f(x_{n+1}) + \frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}} S y_{n+1} - \frac{\alpha_n}{1-\beta_n} f(x_n) - \frac{1-\beta_n-\alpha_n}{1-\beta_n} S y_n \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(f(x_{n+1}) - S y_{n+1}\big) + \frac{\alpha_n}{1-\beta_n}\big(S y_n - f(x_n)\big) + S y_{n+1} - S y_n.
\end{aligned}$$

It follows that

$$\begin{aligned}\|l_{n+1}-l_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-Sy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|Sy_n-f(x_n)\|+\|Sy_{n+1}-Sy_n\|\\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-Sy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|Sy_n-f(x_n)\|+\|y_{n+1}-y_n\|.\end{aligned}$$

By virtue of (2.2), we arrive at

$$\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-Sy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|Sy_n-f(x_n)\|.$$

It follows from the conditions (ii) and (iii) that

$$\limsup_{n\to\infty}(\|l_{n+1}-l_n\|-\|x_{n+1}-x_{n+1}\|)<0.$$

It follows from Lemma 1.1 that $\lim_{n\to\infty}\|l_n-x_n\|=0$. Therefore, we find that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.4}$$

Note that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|S\sum_{i=1}^r\eta_iP_C(x_n-\lambda_iA_ix_n)-x^*\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\sum_{i=1}^r\eta_i\|P_C(x_n-\lambda_iA_ix_n)-x^*\|^2.\end{aligned} \tag{2.5}$$

This implies that

$$\begin{aligned}&\|x_{n+1}-x^*\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2\\ &\quad+\gamma_n\sum_{i=1}^r\eta_i\|x_n-x^*-\lambda_i(A_ix_n-A_ix^*)\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\sum_{i=1}^r\eta_i(\|x_n-x^*\|^2\\ &\quad-2\lambda_i\langle A_ix_n-A_ix^*,x_n-x^*\rangle+\lambda_i^2\|A_ix_n-A_ix^*\|^2)\\ &\le \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\gamma_n\sum_{i=1}^r\eta_i\lambda_i(2\mu_i-\lambda_i)\|A_ix_n-A_ix^*\|^2.\end{aligned}$$

So, we have

$$\begin{aligned}
&\gamma_n \sum_{i=1}^{r} \eta_i \lambda_i (2\mu_i - \lambda_i) \|A_i x_n - A_i x^*\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_n - x_{n+1}\|.
\end{aligned}$$

In view of hte conditions (ii) and (iii), one obtains that

$$\lim_{n\to\infty} \|A_i x_n - A_i x^*\| = 0, \quad \forall 1 \leq i \leq r. \tag{2.6}$$

On the other hand, one has

$$\begin{aligned}
&\|P_C(I - \lambda_i A_i) x_n - x^*\|^2 \\
&= \|P_C(I - \lambda_i A_i) x_n - P_C(I - \lambda_i A_i) x^*\|^2 \\
&\leq \frac{1}{2} \big(\|x_n - x^*\|^2 + \|P_C(I - \lambda_i A_i) x_n - x^*\|^2 \\
&\quad - \|x_n - P_C(I - \lambda_i A_i) x_n - \lambda_i (A_i x_n - A_i x^*)\|^2 \big) \\
&= \frac{1}{2} \big(\|x_n - x^*\|^2 + \|P_C(I - \lambda_i A_i) x_n - x^*\|^2 - \|x_n - P_C(I - \lambda_i A_i) x_n\|^2 \\
&\quad + 2\lambda_i \langle A_i x_n - A_i x^*, x_n - P_C(I - \lambda_i A_i) x_n \rangle - \lambda_i^2 \|A_i x_n - A_i x^*\|^2 \big).
\end{aligned}$$

It follows that

$$\|P_C(I - \lambda_i A_i) x_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - P_C(I - \lambda_i A_i) x_n\|^2 + M_i \|A_i x_n - A_i x^*\|, \tag{2.7}$$

where M_i is an appropriate constant such that $M_i = \max\{2\lambda_i \|x_n - P_C(I - \lambda_i A_i) x_n\| : \forall n \geq 1\}$. On the other hand, we have

$$\|y_n - x_n\| = \|\sum_{i=1}^{r} \eta_i P_C(I - \lambda_i A_i) x_n - x_n\|^2 \leq \sum_{i=1}^{r} \eta_i \|P_C(I - \lambda_i A_i) x_n - x_n\|^2,$$

which combines with (2.7) yields that

$$\sum_{i=1}^{r} \eta_i \|P_C(I - \lambda_i A_i) x_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|y_n - x_n\|^2 + \sum_{i=1}^{r} \eta_i M_i \|A_i x_n - A_i x^*\|.$$

From (2.5), we see that

$$\|x_{n+1} - x^*\|^2 \leq \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 + \gamma_n \sum_{i=1}^{r} \eta_i M_i \|A_i x_n - A_i x^*\| - \gamma_n \|y_n - x_n\|^2,$$

from which it follows that

$$\begin{aligned}\gamma_n\|y_n-x_n\|^2 &\le \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\quad+\gamma_n\sum_{i=1}^{r}\eta_iM_i\|A_ix_n-A_ix^*\|\\ &\le \alpha_n\|f(x_n)-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|\\ &\quad+\gamma_n\sum_{i=1}^{r}\eta_iM_i\|A_ix_n-A_ix^*\|.\end{aligned}$$

It follows from (2.4), (2.6) and the conditions (ii) and (iii) that

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{2.8}$$

It follows that

$$\lim_{n\to\infty}\|Sy_n-x_n\|=0. \tag{2.9}$$

Observe that

$$\begin{aligned}\|Sx_n-x_n\| &\le \|x_n-Sy_n\|+\|Sy_n-Sx_n\|\\ &\le \|x_n-Sy_n\|+\|y_n-x_n\|.\end{aligned}$$

It follows from (2.8) and (2.9) that

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0. \tag{2.10}$$

Now, we are in a position to show that

$$\limsup_{n\to\infty}\langle f(p)-p,x_n-p\rangle\le 0,$$

where $p=P_{\mathscr{F}}f(p)$. To show it, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(p)-p,x_n-p\rangle=\lim_{i\to\infty}\langle u-p,x_{n_i}-p\rangle. \tag{2.11}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to f. Without loss of generality, we can assume that $x_{n_i}\rightharpoonup f$. Define a mapping $W:C\to C$ by

$$Wx=\sum_{i=1}^{r}\eta_iP_C(I-\lambda_iA_i)x,\quad \forall x\in C.$$

From Lemma 1.2, we see that W is nonexpansive such that

$$F(W)=\cap_{i=1}^{r}F(P_C(I-\lambda_iA_i))=\cap_{i=1}^{r}VI(C,A_i).$$

From (2.8), we see that

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{2.12}$$

From Lemma 1.2, we can obtain that $f \in F(W)$. In view of (2.10) and Lemma 1.2, we see that $f \in F(S)$. This proves that

$$f \in F(W) \cap F(S) = \cap_{i=1}^{r} VI(C, A_i) \cap F(S).$$

It follows from (2.11) that

$$\limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle \le 0.$$

Notice that

$$\|x_{n+1} - p\|^2 \le (1 - \alpha_n(1-\alpha))\|x_n - p\|^2 + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle.$$

By Lemma 1.4, we draw the decision immediately. This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

Acknowledgements

The author is grateful to the editor and the reviewer's suggestions which improved the contents of the article.

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