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NOOR ITERATION FOR FIXED POINT AND VARIATIONAL INCLUSION PROBLEMS

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Abstract. In this article, Noor iteration is considered for finding a common element in the set of fixed points of a non-expansive mapping and in the set of solutions of a variational inclusion problem. Strong convergence theorems are established in the framework of Hilbert spaces.

Keywords: monotone operator; nonexpansive mapping; fixed point; Noor iteration.

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1. Introduction-Preliminaries

Variational inclusion problems are being used as mathematical programming models to study a large number of optimization problems arising in finance, economics, network, transportation, and engineering sciences; see [1-21] and the references therein.

Let H be a real Hilbert space H and A a mapping on H. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in H;$$

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A is said to be α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in H;$$

A is said to be α-strongly anti-monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \leq (-\alpha)\|x - y\|^2, \quad \forall x, y \in H;$$

A is said to be L-Lipschitz continuous if there exits a constant such that $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in H;$$

A is said to be nonexpansive if

$$\|Ax - Ay\| \leq \|x - y\|, \quad \forall x, y \in H.$$

A is said to be strictly pseudocontractive if

$$\|Ax - Ay\|^2 \leq \|x - y\|^2 + \kappa\|(I - A)x - (I - A)y\|^2, \quad \forall x, y \in H.$$

Let C be a nonempty, closed and convex subset of H. Recall that the classical variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

One can see that the variational inequality problem (1.1) is equivalent to a fixed point problem. $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where I is the identity mapping and $\lambda > 0$ is a constant.

Recently, Noor and Huang [15] consider a three-step iterative method for finding a common element in the set of fixed points of a non-expansive mapping and in the set of solutions of the variational inequality problem (1.1) in a real Hilbert space. To be more precise, they introduced the following algorithm:

$$\begin{cases} x_0 \in C, \\ z_n = (1 - c_n)x_n + c_n SP_C(x_n - \rho T x_n), \\ y_n = (1 - b_n)x_n + b_n SP_C(y_n - \rho T y_n), \\ x_{n+1} = (1 - a_n)x_n + a_n SP_C(y_n - \rho T y_n), \quad \forall n \geq 0 \end{cases}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$ for all $n \geq 0$, S is a non-expansive mapping and T is a monotone-type operator. They showed that the sequence $\{x_n\}$ generated by the above iterative sequence converges strongly to a common element in the set of fixed points of a non-expansive mapping S and in the set of solutions of the variational inequality problem (1.1); see [15] for details.

In [16], Noor and Huang considered the following variational inclusion problem. Find an $u \in H$ such that

$$0 \in Au + Tu, \tag{1.2}$$

where T and A are monotone operators. They also consider the following three-step iterative algorithm:

$$\begin{cases} x_0 \in H, \\ z_n = (1-c_n)x_n + c_n SJ_A(x_n - \rho T x_n), \\ y_n = (1-b_n)x_n + b_n SJ_A(y_n - \rho T y_n), \\ x_{n+1} = (1-a_n)x_n + a_n SJ_A(y_n - \rho T y_n), \quad \forall n \geq 0 \end{cases}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$ for all $n \geq 0$, S is a non-expansive mapping, $J_A = (I + \rho A)^{-1}$. They showed that the sequence $\{x_n\}$ generated by the above iterative sequence converges strongly to a common element in the set of fixed points of a non-expansive mapping S and in the set of solutions of the variational inclusion problem (1.2); see [16] for details.

Motivated by the recent research work, we continue to study the problem of finding a solution of the problem by a Noor iteration.

Lemma 1.1 [22] *Suppose that $\{\delta_n\}$ is a nonnegative sequence satisfying the following inequality*

$$\delta_{n+1} \leq (1-\lambda_n)\delta_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ is a sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} \lambda_n = \infty$. Then $\lim_{n\to\infty} \delta_n = 0$.

Lemma 1.2 [21] *Let H be a Hilbert space. An element $u \in H$ is a solution of the problem (1.3) if and only if $u \in H$ is a fixed point of the mapping $J_A(I + \rho T)$, where $J_A = (I + \rho A)^{-1}$, I is the identity mapping and T is a strongly anti-monotone mapping.*

Lemma 1.3. *Let H be a Hilbert space and $S : H \to H$ a nonexpansive mapping with a fixed point. Assume that $F(S) \cap S(A,T) \neq \emptyset$. If $u \in F(S) \cap S(A,T)$, then $u = SJ_A(I+\rho T)u$.*

Proof. Fix $u \in F(S) \cap S(A,T)$. From Lemma 1.2, we see that $u = J_A(I+\rho T)u$. We also have $u = Su$. It follows that $u = J_A(I+\rho T) = Su = SJ_A(I+\rho T)$. This completes the proof.

2. Main results

Theorem 2.1. *Let H be a Hilbert space, A a maximal monotone mapping on H and T an α-strongly anti-monotone and β-Lipschitz continuous mapping on H. Let $R : H \to H$ be a strictly pseudocontractive mapping with a fixed point and let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in H, \\ z_n = (1-c_n)x_n + c_n\big(\alpha I + (1-\alpha)R\big)J_A(x_n + \rho T x_n), \\ y_n = (1-b_n)x_n + b_n\big(\alpha I + (1-\alpha)R\big)J_A(z_n + \rho T z_n), \\ x_{n+1} = (1-a_n)x_n + a_n\big(\alpha I + (1-\alpha)R\big)J_A(y_n + \rho T y_n), \quad \forall n \geq 0 \end{cases}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$ for all $n \geq 0$, $J_A = (I+\rho A)^{-1}$ and ρ is a constant satisfying the restriction $0 < \rho < \frac{2\alpha}{\beta^2}$. Assume that $\kappa \in [\alpha, 1)$ $F(R) \cap S(A,T) \neq \emptyset$ and $\sum_{n=0}^{\infty} a_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a point in $F(R) \cap S(A,T)$.

Proof. Put $S := \alpha I + (1-\alpha)R$. From Zhou [23], we see that S is nonexpansive with $F(R) = F(S)$. Let $x^* \in F(S) \cap S(A,T)$. It follows from (2.1) that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-a_n)(x_n - x^*) + a_n\big(SJ_A(y_n + \rho T y_n) - SJ_A(x^* + \rho T x^*)\big)\| \\ &\leq (1-a_n)\|x_n - x^*\| + a_n\|J_A(y_n + \rho T y_n) - J_A(x^* + \rho T x^*)\| \\ &\leq (1-a_n)\|x_n - x^*\| + a_n\|y_n - x^* + \rho(T y_n - T x^*)\|. \end{aligned}$$

From the α-strongly anti-monotone and β-Lipschitz assumptions on T, we have

$$\begin{aligned}&\|y_n - x^* + \rho_n(Ty_n - Tx^*)\|^2\\ &\le \|y_n - x^*\|^2 - 2\rho\alpha\|y_n - x^*\|^2 + \rho^2\beta^2\|y_n - x^*\|^2\\ &= (1 - 2\rho\alpha + \rho^2\beta^2)\|y_n - x^*\|^2.\end{aligned}$$

That is, $\|y_n - x^* + \rho(Ty_n - Tx^*)\| \le \theta_n\|y_n - x^*\|$, where $\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}$. From the assumption $0 < \rho < \frac{2\alpha}{\beta^2}$, we see that $\theta < 1$.

Next, we estimate $\|y_n - x^*\|$. It follows that

$$\begin{aligned}\|y_n - x^*\| &= \|(1 - b_n)(x_n - x^*) + b_n\big(SJ_A(z_n + \rho Tz_n) - SJ_A(x^* + \rho Tx^*)\big)\|\\ &\le (1 - b_n)\|x_n - x^*\| + b_n\|J_A(z_n + \rho Tz_n) - J_A(x^* + \rho Tx^*)\|\\ &\le (1 - b_n)\|x_n - x^*\| + b_n\|z_n - x^* + \rho(Tz_n - Tx^*)\|.\end{aligned}$$

From the α-strongly anti-monotone and β-Lipschitz assumptions on T, we have

$$\begin{aligned}&\|z_n - x^* + \rho(Tz_n - Tx^*)\|^2\\ &\le \|z_n - x^*\|^2 - 2\rho\alpha\|z_n - x^*\|^2 + \rho^2\beta^2\|z_n - x^*\|^2\\ &= (1 - 2\rho\alpha + \rho^2\beta^2)\|z_n - x^*\|^2.\end{aligned}$$

That is, $\|z_n - x^* + \rho(Tz_n - Tx^*)\| \le \theta\|z_n - x^*\|$.

Finally, we estimate $\|z_n - x^*\|$. It follows that

$$\begin{aligned}\|z_n - x^*\| &\le (1 - c_n)\|x_n - x^*\| + c_n\|J_A(x_n + \rho Tx_n) - J_A(x^* + \rho Tx^*)\|\\ &\le (1 - c_n)\|x_n - x^*\| + c_n\|x_n - x^* + \rho(Tx_n - Tx^*)\|.\end{aligned}$$

In a similar way, we can obtain that $\|x_n - x^* + \rho(Tx_n - Tx^*)\| \le \theta\|x_n - x^*\|$. Notice that $\|z_n - x^*\| \le [1 - c_n(1 - \theta)]\|x_n - x^*\|$, It follows that that

$$\|y_n - x^*\| \le \Big(1 - b_n\big(1 - \theta(1 - c_n(1 - \theta))\big)\Big)\|x_n - x^*\| \le \|x_n - x^*\|.$$

It follows that

$$\begin{aligned}\|x_{n+1} - x^*\| &\le (1 - a_n)\|x_n - x^*\| + a_n\theta\|y_n - x^*\|\\ &\le [1 - a_n(1 - \theta)]\|x_n - x^*\|.\end{aligned}$$

Applying Lemma 1.1, we can conclude the desired conclusion immediately.

Conflict of Interests

The author declares that there is no conflict of interests.

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