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A FIXED POINT LIKE THEOREM IN A T_0-ULTRA-QUASI-METRIC SPACE

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Abstract. In this paper, we prove a fixed point like theorem for a generalized nonexpansive mapping in q-spherically complete T_0-ultra-quasi-metric spaces.

Keywords: fixed point; q-spherically complete; T_0-ultra-quasi-metric space.

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1. Introduction

In [1], Agyingi proved that every generalized contractive mapping defined in a q-spherically complete T_0-ultra-quasi-metric space has a unique fixed point. This work is based on a previous result established by Petalas et al. in [3] where it was proved that every contractive mapping on a spherically complete non-Archimedean normed space has a unique fixed point. This existence result, as observed by Petalas et al., fails when the map in nonexpansive. In this paper, we shall prove a fixed point like theorem for a generalized nonexpansive mapping in q-spherically complete T_0-ultra-quasi-metric space. The concept of q-spherically completeness has been introduced by Isbell and studied for T_0-ultra-quasi-metric spaces by Künzi and Otafudu in [3].

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2. Preliminaries

In this section, we recall some elementary definitions from the asymmetric topology which are necessary for a good understanding of the work below.

Definition 2.1. Let X be a non empty set. A function $d : X \times X \to [0,\infty)$ is called an *quasi-pseudometric* on X if:

i) $d(x,x) = 0 \quad \forall\, x \in X$, and

ii) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in X$.

Moreover, if $d(x,y) = 0 = d(y,x) \Longrightarrow x = y$, then d is said to be a T_0-*quasi-pseudometric*. The latter condition is referred as the T_0 condition.

Definition 2.2. (Compare[2]) Let (X,d) be a quasi-pseudometric space. We say that X is an *ultra-quasi-pseudometric* space if d satifies the strong triangular inequality

$$d(x,z) \leq \max\{d(x,y),d(y,z)\} \quad \forall x,y,z \in X.$$

Moreover, if d satisfies the T_0 condition, then X is said to be a T_0-*ultra-quasi-pseudometric space*.

Remark 2.1.

- Since the strong triangular inequality implies the classical triangular inequality, in the definition on ultra-quasi-pseudometric, we don't really need a quasi-pseudometric s-pace. Hence an equivalent definition is:

$$d \text{ is an ultra-quasi-pseudometric } \Longleftrightarrow \begin{cases} d(x,x) = 0 \quad \forall\, x \in X, \\ d(x,z) \leq \max\{d(x,y),d(y,z)\} \\ \forall x,y,z \in X. \end{cases}$$

- Let d be an *ultra-quasi-pseudometric* on X, then the map d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ whenever $x,y \in X$ is also a an *ultra-quasi-pseudometric* on X, called the conju-gate of d.

- It is easy to verify that the function d^s defined by $d^s(x,y) = \max\{d(x,y), d(y,x)\}$, i.e. $d^s := d \vee d^{-1}$ defines an *ultra metric* on X whenever d is a T_0-ultra-quasi- pseudometric.

Definition 2.3. (Compare[1]) A map $f : X \to X$ where (X,d) is an (ultra-)quasi-pseudometric space is called is called *nonexpansive* if

$$d(f(x), f(y)) \leq d(x,y),$$

whenever $x,y \in X$.

Definition 2.4. (Compare[1]) A map $f : X \to X$ where (X,d) is an (ultra-)quasi-pseudometric space is called *generalized nonexpansive* if for each $x,y \in X$ with $d(x,y) > 0$, we have that

$$d(f(x), f(y)) \leq \max\{d(x,y), d(f(x),x), d(y, f(y))\}.$$

3. q-Spherically Complete Spaces

In this section, we recall some results about q-spherical completeness, which we take from [1].

Let (X,d) be an ultra-quasi-pseudometric space. For $x \in X$ and $\varepsilon \geq 0$,

$$C_d(x,\varepsilon) = \{y \in X : d(x,y) \leq \varepsilon\}$$

denotes the closed ε-ball at x.

Definition 3.1. (Compare[1]) Let (X,d) be an ultra-quasi-pseudometric space. Let $(x_i)_{i\in I}$ be a family of points of X and let $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be families of non-negative real numbers. We say that the family $(C_d(x_i,r_i), C_{d^{-1}}(x_i,s_i))_{i\in I}$ has the *mixed binary intersection property* provided that

$$d(x_i,x_j) \max\{r_i,s_j\},$$

for all $i,j \in I$.

Definition 3.2. Let (X,d) be an ultra-quasi-pseudometric space. We say that (X,d) is *q-spherically complete* provided that each family $(C_d(x_i,r_i),C_{d^{-1}}(x_i,s_i))_{i\in I}$ that has the mixed binary intersection property is such that

$$\bigcap_{i\in I}(C_d(x_i,r_i)\cap C_{d^{-1}}(x_i,s_i))\neq\emptyset.$$

Examples of such spaces can be found in [2].

Proposition 3.1. [[1]] *Let (X,d) be an ultra-quasi-pseudometric space. Then (X,d) is q-spherically complete if and only if (X,d^{-1}) is q-spherically complete.*

Proposition 3.2. [[1]] *Let (X,d) be an T_0-ultra-quasi-pseudometric space. If (X,d) is q-spherically complete, then (X,d) is spherically complete.*

4. Main results

The terminology *fixed point like* comes from the fact that for nonexpansive maps, the existence of fixed point is not guaranteed. Nevertheless, such maps leave invariant a specific ball, say B. In other words if $T:X\to X$ is a nonexpansive map on X, then there exists a ball B such that $T(B)=B$.

Theorem 4.1. *Suppose (X,d) is q-spherically complete T_0-ultra-quasi-pseudometric space and $T:X\to X$ is a nonexpansive map. Then either T has at least one fixed point or there exists a closed ball B radius r such that $T:B\to B$. Moreover, $d(a,Ta)=d(Ta,a)=r$ for each $a\in B$.*

Proof. Let $a\in X$. Let us denote by

$$C_d^a=C_d(a,d(Ta,a))\text{ and }C_{d^{-1}}^a=C_{d^{-1}}(a,d(a,Ta)),$$

with $d(Ta,a)=d(a,Ta)$. Set

$$C^a=C_d^a\cap C_{d^{-1}}^a$$

and $\mathscr{A}:=\{C^a,a\in X\}$. Define the relation $C^a\preccurlyeq C^b$ on $\mathscr{A}$ by

$$C^a\preccurlyeq C^b\text{ if and only if }C^b\subseteq C^a.$$

Then $(\mathscr{A}, \preccurlyeq)$ is a partially ordered set. With this relation and the Zorn's lemma, Agyingi [1] proved that $\mathscr{A}$ has a maximal element C^z.

Consider now such maximal element C^z. For any $b \in C^z$, we have

$$d(b,Tb) \leq \max\{d(b,z), d(z,Tz), d(Tz,Tb)\} = d(z,Tz),$$

and

$$d(Tb,b) \leq \max\{d(Tb,z), d(z,Tz), d(Tz,b)\} = d(Tz,z).$$

Therefore, we conclude that for any $h \in C^b$, $d(z,h) \leq d(z,Tz)$ and $d(h,z) \leq d(Tz,z)$, which entails that $C^b \subseteq C^z$ and then $Tb \in C^z$.

Now, if we assume that $d(b,Tb) < d(z,Tz)$ then

$$d(b,z) = d(z,Tz) > d(b,Tb).$$

This implies that $z \in C^z_d$ but $z \notin C^b_d$, which is impossible from the maximality of C^z. Thus

$$d(b,Tb) = d(z,Tz) =: r \text{ for any } b \in C^z.$$

Similarly, if we assume that $d(Tb,b) < d(Tz,z)$ then

$$d(z,b) = d(Tz,z) > d(Tb,b).$$

This implies that $z \in C^z_{d^{-1}}$ but $z \notin C^b_{d^{-1}}$, which is impossible from the maximality of C^z. Thus

$$d(Tb,b) = d(Tz,z) \text{ for any } b \in C^z.$$

Hence

$$d(b,Tb) = d(z,Tz) = d(Tz,z) = d(Tb,b) = r \text{ for any } b \in C^z.$$

This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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