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AN EXTENSION TO FUGLEDE-PUTNAM'S THEOREM AND ORTHOGONALITY

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Abstract. An asymmetric Fuglede-Putnam's theorem for $w-$hyponormal operators and $(p,k)-$quasihyponormal operator is proved. As a consequence of this result, we obtain that the range of the generalized derivation induced by the above classes of operators is orthogonal to its kernel.

Keywords: Fuglede-Putnam theorem; w-hyponormal operator; $(p,k)-$quasihyponormal operator; derivation.

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1. Introduction

For complex Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, $B(\mathscr{H}),B(\mathscr{K})$ and $B(\mathscr{H},\mathscr{K})$ denote the set of all bounded linear operators on $\mathscr{H}$, the set of all bounded linear operators on $\mathscr{K}$ and the set of all bounded linear transformations from $\mathscr{H}$ to $\mathscr{K}$ respectively. A bounded operator $A \in B(\mathscr{H})$ is called normal if $A^*A = AA^*$. According to [10] a bounded operator $A \in B(\mathscr{H})$ is called $(p,k)-$quasihyponormal if

$$A^{*k}(A^*A) - (AA^*)A^k \geq 0,\ 0 < p \leq 1,\ k \in \mathbb{N}.$$

If $p = 1,\ k = 1$ and $p = k = 1$, then A is $k-$quasihyponormal, $p-$quasihyponormal and quasi-hyponormal respectively.

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If $\{N\}$, $\{HN\}$, $\{Q(p)\}$, and $\{Q(p,k)\}$ denote the classes of normal, hyponormal, $p-$quasihyponormal and $(p,k)-$quasihyponormal operators. These classes are related by proper inclusion

$$\{N\} \subseteq \{HN\} \subseteq \{Q(p)\} \subseteq \{Q(p,k)\}.$$

Also A is called $p-$hyponormal [1, 5, 7, 8, 16, 17], if $(A^*A)^p \geq (AA^*)^p$ for some $0 < p \leq 1$, semi-hyponormal if $p = 1/2$, $\log-$hyponormal [14] if A is invertible operator and satisfies $\log(A^*A) \geq \log(AA^*)$, and $w-$hyponormal if $|\widetilde{A}| \geq |A| \geq |(\widetilde{A})^*|$, where $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ is the Aluthge transformation. It was shown in [2] and [3] that the class of $w-$hyponormal operators contains both the $p-$hyponormal and $\log-$hyponormal operators. Let $p-H$ and $w-H$ denote the class of $p-$hyponormal and $w-$hyponormal operators respectively. We have

$$\{N\} \subset \{HN\} \subset \{p-H\} \subset \{w-H\}.$$

These classes are interesting and have similar properties to those of hyponormal operators.

The familiar Fuglede-Putnam's theorem asserts that if $A \in B(\mathscr{H})$ and $B \in B(\mathscr{K})$ are normal operators and $AX = XB$ for some operators $X \in B(\mathscr{K},\mathscr{H})$, then $A^*X = XB^*$ [12]. Many authors have extended this theorem for several classes of operators. H.I. Kim [11] proved that Fuglede-Putnam's theorem holds for injective $(p,k)-$quasihyponormal and $p-$hyponormal operators.

Let $A \in B(\mathscr{H})$, $B \in B(\mathscr{K})$. We say that the pair (A,B) satisfies the Fuglede-Putnam's theorem if $AX = XB$ for some $X \in B(\mathscr{H},\mathscr{K})$ implies $A^*X = XB^*$.

The organization of this paper is as follows, in Section 2, we recall some well known results which will be used in the sequel. In Section 3, our aim is to extend the Fuglede-Putnam theorem [12], we prove that if either

(1) A is injective $(p.k)-$quasihyponormal and B^* is $w-$hyponormal such that $\ker B^* \subset \ker B$ or

(2) A is $w-$hyponormal such that $\ker A^* \subset \ker A$ and B^* is $(p,k)-$quasihyponormal, then the pair (A,B) satisfies the Fuglede-Putnam's theorem.

Let $A, B \in L(\mathscr{H})$, we define the generalized derivation $\delta_{A,B}$ induced by A and B by

$$\delta_{A,B}(X) = AX - XB, \text{ for all } X \in B(\mathscr{H}).$$

Definition 1.1 [4] Given subspaces $\mathscr{M}$ and $\mathscr{N}$ of a Banach space $\mathscr{V}$ with norm $\|\cdot\|$. $\mathscr{M}$ is said to be orthogonal to $\mathscr{N}$ if $\|m+n\| \geq \|n\|$ for all $m \in \mathscr{M}$ and $n \in \mathscr{N}$.

Anderson and Foias [4] proved that if A and B are normal, S is an operator such that $AS = SB$, then

$$\| \delta_{A,B}(X) - S \| \geq \| S \|, \text{ for all } X \in B(\mathscr{H}),$$

where $\|\cdot\|$ is the usual operator norm. Hence the range of $\delta_{A,B}$ is orthogonal to the null space of $\delta_{A,B}$. The orthogonality here is understood to be in the sense of definition [4].

2. Preliminaries

We begin by the following known results which will be used in the sequel.

Definition 2.1. [1] Let $A \in B(\mathscr{H})$ and $A = U|A|$ be the polar decomposition of A, the Aluthge transformation of A is $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$.

Theorem 2.2. [9] An operator $A \in B(\mathscr{H})$ is $w-$hyponormal if and only if

$$(|A^*|^{\frac{1}{2}}|A||A^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |A^*|.$$

Lemma 2.3. [17] *Let $A \in B(\mathscr{H})$ be $p-$hyponormal operator and $\mathscr{M} \subset \mathscr{H}$ be an invariant subspace for A, then the restriction of A to $\mathscr{M}$ is $p-$hyponormal.*

Lemma 2.4. [15] *Let $A \in B(\mathscr{H})$ be $(p,k)-$quasihyponormal operator and $\mathscr{M} \subset \mathscr{H}$ be an invariant subspace for A, then the restriction of A to $\mathscr{M}$ is $(p,k)-$quasihypo-normal operator.*

Lemma 2.5. [10] *Let $A \in B(\mathscr{H})$ be $(p,k)-$quasihyponormal operator and $\mathscr{M} \subset \mathscr{H}$ be an invariant subspace for A. If $A|_M$ is an injective normal operator, then $\mathscr{M}$ reduces A.*

Lemma 2.6. ([6], [12]) *Let $A \in B(\mathscr{H})$ be $w-$hyponormal and $\mathscr{M} \subset \mathscr{H}$ be an invariant subspace for A, then the restriction of A to $\mathscr{M}$ is $w-$hyponormal.*

Lemma 2.7. [16] *Let $A \in B(\mathscr{H})$ be $w-$hyponormal operator, then its Aluthge transform*

$$\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$$

is semi-hyponormal.

Theorem 2.8. [11] Let $A \in B(\mathscr{H})$ be an injective $(p,k)-$quasihyponormal and $B^* \in B(\mathscr{K})$ be $p-$hyponormal on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ respectively. If $C \in B(\mathscr{K},\mathscr{H})$ and $AC = CB$, then $A^*C = CB^*$.

Theorem 2.9. [14]*Let $A \in B(\mathscr{H})$ and $B \in B(\mathscr{K})$. Then the following assertions are equivalent*

(1) *The pair (A,B) satisfy Fuglede-Putnam's theorem.*

(2) *If $AX = XB$ for some $X \in B(\mathscr{K},\mathscr{H})$, then $\overline{R(X)}$ reduces A, $\ker(X)^{\perp}$ reduces B, and $A\mid_{\overline{R(X)}}$, $B\mid_{(\ker X)^{\perp}}$ are normal operators.*

3. Main results

Our goal is to investigate the orthogonality of $R(\delta_{A,B})$ (the range of $\delta_{A,B}$) and $\ker(\delta_{A,B})$ (the kernel of $\delta_{A,B}$) for some operators. We prove that $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$ when either

(i) A is an injective $(p,k)-$hyponormal operator and B^* is $w-$hyponormal such that $\ker B^* \subset \ker B$ or

(ii) A is $w-$hyponormal such that $\ker A \subset \ker A^*$ and B^* is an injective $(p,k)-$hyponormal operator.

Theorem 3.1 *Let $A \in B(\mathscr{H})$ be is an injective $(p,k)-$hyponormal operator and $B^* \in B(\mathscr{K})$ be $w-$hyponormal such that* $\ker B^* \subset \ker B$. *If $AC = CB$ for some $C \in B(\mathscr{H},\mathscr{K})$, then $A^*C = CB^*$.*

Proof. Case 1. If B^* is injective. Assume that $AC = CB$ for some $C \in B(\mathscr{K},\mathscr{H})$.
Since $\overline{R(C)}$ is invariant by A and $(\ker C)^{\perp}$ is invariant by B^*, we consider the following decompositions:

$$\mathscr{H} = \overline{R(C)} \oplus (R(C))^{\perp}, \ \ \mathscr{K} = (\ker C)^{\perp} \oplus (\ker C),$$

then it yields

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \ \ B = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}$$

and

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} : \ (\ker C)^{\perp} \oplus (\ker C) \longrightarrow \overline{R(C)} \oplus (R(C))^{\perp}.$$

From $AC = CB$ we get

$$A_1 C_1 = C_1 B_1. \tag{1}$$

Let $B_1^* = U^*|B_1^*|$ be the polar decomposition of B_1^*. Multiply the both members of (1) by $|B_1^*|^{1/2}$, we obtain

$$A_1 C_1 |B_1^*|^{1/2} = C_1 B_1 |B_1^*|^{1/2}.$$

Hence

$$A_1 C_1 |B_1^*|^{1/2} = C_1 |B_1^*|^{1/2} (\widetilde{B_1^*})^*. \tag{2}$$

In equation (2), A_1 is is an injective $(p,k)-$hyponormal operator by Lemma and B_1^* is $w-$hyponormal by Lemma , and $(\widetilde{B_1^*})^*$ is semi-hyponormal by Lemma . By applying Theorem , the pair $(A_1, \widetilde{B_1^*})$ satisfies the Fuglede-Putnam's theorem. Therefore $A_1|_{R(C_1|B_1^*|^{1/2})}$ and $\widetilde{B_1^*}|_{(\ker(C_1|B_1^*|^{1/2})^{\perp}}$ are normal operators. Since C_1 is injective with dense range and $|B_1^*|^{1/2}$ is injective, then

$$\overline{R(C_1|B_1^*|^{1/2})} = \overline{R(C_1)} = \overline{R(C)},$$

and

$$\ker(C_1|B_1^*|^{1/2}) = \{0\}.$$

It follows that $\widetilde{B_1^*}$ is normal and $(\ker C)^{\perp}$ reduces B^*. Therefore $\overline{R(C)}$ reduces A and $(\ker C)^{\perp}$ reduces B. Thus, $A_2 = 0$ and $B_2 = 0$. Hence $A_1 C_1 = C_1 B_1$ with A_1, B_1 normal operators, then $A_1^* C_1 = C_1 B_1^*$. Consequently $A^* C = C B^*$.

Case 2. If B^* is not injective, the condition $\ker B^* \subset \ker B$ implies $\ker B^*$ reduces B^*, since $\ker A$ reduces A, the operators A and B can be written on the following decompositions

$$\mathscr{H} = (\ker A)^{\perp} \oplus \ker A, \ \mathscr{K} = (\ker B^*)^{\perp} \oplus \ker B^*,$$

as follows

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since A_1 is an injective $(p,k)-$hyponormal operator and B_1^* is injective $w-$hyponormal operator. Let

$$C:(\ker B^*)^{\perp}\oplus\ker B^*\to(\ker A)^{\perp}\oplus\ker A,$$

and let $C=[C_{ij}]_{i,j=1}^{2}$ be the matrix representation of C. Then $AC=CB$ implies $A_1C_{11}=C_{11}B_1$ and $C_{12}=0,C_{21}=0$. From case 1, we deduce that $A_1^*C_{11}=C_{11}B_1^*$. Thus $A^*C=CB^*$. Since $AC=CB$ implies $C^*A^*=B^*C^*$, then $C^*A=BC^*$ and $A^*C=CB^*$ by theorem . This completes the proof.

Corollary 3.2. *Let $A\in B(\mathscr{H})$ be $w-$hyponormal operator such that* $\ker A\subseteq\ker A^*$ *and $B^*\in B(\mathscr{K})$ be an injective $(p,k)-$hyponormal operator. If $AC=CB$ for some $C\in B(\mathscr{K},\mathscr{H})$, then $A^*C=CB^*$.*

Corollary 3.3. *$A\in B(\mathscr{H})$ is normal if and only if A is an injective $(p,k)-$hyponormal and A^* is $w-$hyponormal such that* $\ker A^*\subset\ker A$.

Theorem 3.4. *Let $A,B\in B(\mathscr{H})$. If one of the following assertions*

(1) *A is an injective $(p,k)-$hyponormal operator and B^* is w-hyponormal such that* $\ker B^*\subset\ker B$.
(2) *A is $w-$hyponormal such that* $\ker A\subset\ker A^*$ *and B^* is an injective $(p,k)-$hyponormal operator.*

holds, then $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$.

Proof. The pair (A,B) verify the Fuglede-Putman's theorem by Theorem and Corollary respectively. Let $C\in B(\mathscr{H})$ be such that $AC=CB$. According to the following decompositions of $\mathscr{H}$:

$$\mathscr{H}=\mathscr{H}_1=\overline{R(C)}\oplus\overline{R(C)}^{\perp},\ \mathscr{H}=\mathscr{H}_2=(\ker C)^{\perp}\oplus\ker C,$$

we can write A,B,C and X

$$A=\begin{pmatrix}A_1&0\\0&A_2\end{pmatrix},\ B=\begin{pmatrix}B_1&0\\0&B_2\end{pmatrix},\ C=\begin{pmatrix}C_1&0\\0&0\end{pmatrix},\ X=\begin{pmatrix}X_1&X_2\\X_3&X_4\end{pmatrix},$$

where A_1 and B_1 are normal operators and X is an operator from $\mathscr{H}_1$ to $\mathscr{H}_2$. Since $AC = CB$, then $A_1C_1 = C_1B_1$. Hence

$$AX - XB - C = \begin{pmatrix} A_1X_1 - X_1B_1 - C_1 & A_2X_2 - X_2B_2 \\ A_1X_3 - X_3B_1 & A_2X_4 - X_4B_2 \end{pmatrix}.$$

Since $C_1 \in \ker(\delta_{A_1,B_1})$ and $A_1,\ B_1$ are normal, it follows by [4]

$$\| AX - XB - C \| \geq \| A_1X_1 - X_1B_1 - C_1 \| \geq \| C_1 \| = \| C \|,\ \forall X \in B(\mathscr{H}).$$

This implies that $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$.

Conflict of Interests

The author declares that there is no conflict of interests.

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