Available online at http://scik.org
Adv. Inequal. Appl. 2014, 2014:23
ISSN: 2050-7461

GEOMETRICAL PROOF OF NEW STEFFENSEN'S INEQUALITY AND APPLICATIONS

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Abstract. In this paper, we give a geometrical proof of a new Steffensen's inequality for convex functions. In addition, we present applications of the Steffensen's inequality leading to the determination of Fourier coefficients.

Keywords: Steffensen's inequality, geometrical proof, Fourier coefficient.

2010 AMS Subject Classification: 26D15.

1. Introduction

The the following inequality was discovered in 1918 by Steffensen [9]

$$\int_{b-\lambda}^{b} g(s)ds \leq \int_{a}^{b} g(s)f(s)ds \leq \int_{a}^{a+\lambda} g(s)ds, \tag{1}$$

where $\lambda = \int_a^b f(s)ds$, f and g are integrable functions defined on (a,b), g is monotone decreasing and for each $s \in (a,b)$, $0 \leq f(s) \leq 1$; see also [5], [8], [7] and [6] and the references therein. Godunova and Levin in [3] noted that the generalisation of (1) by Bellman in [2] was incorrect.

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Received February 9, 2014

Pecaric [8] corrected the Bellman generalisation with a narrow subclass. The corrected result is

$$\left(\int_0^1 f(s)g(s)ds\right)^p \leq \int_0^{\lambda} g(s)^p ds, \tag{2}$$

where $\lambda = \left(\int_0^1 f(s)ds\right)^p$, $g : [0,1] \longrightarrow \Re$ is a nonnegative and nonincreasing function, $f : [0,1] \longrightarrow \Re$ is an integrable function with $0 \leq f(s) \leq 1$ $(\forall s \in [0,1])$ and $p \geq 1$, for the proof; see [8] and the references therein.

The purpose of this paper is to present a refinement of inequality (2) with proofs consisting of both analytical and geometrical.

2. Preliminaries

We begin with convex functions.

Definition 2.1. (Convex functions) Let I be an interval in $\Re$. Then $\psi : I \longrightarrow \Re$ is said to be convex if for all $t_1, t_2 \in I$ and for all positive λ and μ satisfying $\lambda + \mu = 1$, we have

$$\psi(\lambda t_1 + \mu t_2) \leq \lambda \psi(t_1) + \mu \psi(t_2). \tag{3}$$

A convex function necessarily is continuous for $t_1, t_2 \in I$.

A function ψ is said to be strictly convex if for all $t_1 \neq t_2$, ψ is said to be strictly convex.

Remark 2.1. The convexity of a function $\psi : I \longrightarrow \Re$ means geometrically that, the function ψ falls below (or lies on and not above) the chord joining the endpoints $(t_1, \psi(t_1))$ and $(t_2, \psi(t_2))$, for every $t_1, t_2 \in I$.

Intuitively, a convex function has a tangent line at each point and lies above of its tangent lines. That is, for each $t \in I$ there exists a slope C_t such that

$$\psi(s) \geq \psi(t) + C_t(s - t), \quad \forall x \in I.$$

We remark here that if ψ is differentiable at t then $C_t = \psi'(t)$.

Definition 2.2. A function ψ is said to be concave if $-\psi$ is convex (i.e. if the inequality (3) is reversed). If it is strict for all $t_1 \neq t_2$, ψ is said to be strictly concave.

Remark 2.2. If $\psi''(t)$ exists at each point of the interval I, then a necessary and sufficient condition that $\psi(t)$ is convex is that $\psi''(t) \geq 0$ for all $t \in I$.

For the above discussion, we refer authors to [5] and [1]. Some examples of convex functions are: $|t|$, t^k for $k > 1$ and $-t^k$ for $0 < k < 1$, e^t, $t \log t$, $-\log t$ and concave functions are: t^k for $0 < k < 1$, $\log t$, $\sqrt{t}$ for $t \geq 0$ and so on.

3. Main results

We first present a refinement of inequality (2) here.

Theorem 3.1. *Let the function $f : [0,1] \longrightarrow \Re$ be continuous such that $0 \leq f(s) \leq 1$. If $\psi : [0,1] \longrightarrow \Re$ is a convex, differentiable function with $\psi(0) = 0$, then*

$$\psi\left(\int_0^1 f(s)ds\right) \leq \int_0^1 f(s)\psi'(s)ds \tag{4}$$

for all $s \in [0,1]$.

Proof. Let $p = 1$. Since the differential of $\psi(s)$ denoted $\psi'(s)$ is increasing and $-\psi'(s)$ is nonincreasing for all $s \in [0,1]$, substitution of $g(s) = -\psi'(s)$ in (2) gives

$$-\int_0^1 f(s)\psi'(s)ds \leq \int_0^\lambda -\psi'(s)ds.$$

This simplifies to

$$\int_0^\lambda \psi'(s)ds \leq \int_0^1 f(s)\psi'(s)ds,$$

$$\psi(\lambda) - \psi(0) \leq \int_0^1 f(s)\psi'(s)ds. \tag{5}$$

Since $\lambda = \int_0^1 f(s)ds$ and $\psi(0) = 0$, thus (5) becomes

$$\psi\left(\int_0^1 f(s)ds\right) \leq \int_0^1 f(s)\psi'(s)ds$$

This completes the proof.

Let us consider a case of a simple function f on an interval $[s_0, s_2]$ such that $0 \leq s_0 < s_2 \leq 1$. We give some definitions

Definition 3.1. Let a_1 and a_2 be real numbers. Define a function $f : [s_0, s_2] \to \Re$ by

$$f(s) = \begin{cases} a_1 & \text{if} \quad s_0 \leq s < s_1, \\ \\ a_2 & \text{if} \quad s_1 \leq s \leq s_2. \end{cases}$$

Then f is called a simple function since for every $s \in [s_0, s_2]$, we have $f(s) = a_j$ for $j = 1, 2$.

Let us obtain a continuous function f_ε from f. Let $\varepsilon > 0$, we have the partition $\{[s_0, s_1 - \varepsilon), [s_1 - \varepsilon, s_1 + \varepsilon), [s_1 + \varepsilon, s_2]\}$ of $[s_0, s_2]$.

Definition 3.2. Let a_1 and a_2 be real numbers. Define a function $f_\varepsilon : [s_0, s_2] \to \Re$ by

$$f_\varepsilon(s) = \begin{cases} a_1 & \text{if} \quad s_0 \leq s < s_1 - \varepsilon, \\ \\ \frac{a_2 - a_1}{2\varepsilon}(s - s_1 + \varepsilon) + a_1 & \text{if} \quad s_1 - \varepsilon \leq s < s_1 + \varepsilon, \\ \\ a_2 & \text{if} \quad s_1 + \varepsilon \leq s \leq s_2. \end{cases}$$

Remark 3.1. Let us remark that f_ε is continuous in $[s_0, s_2]$ since $\lim_{s \to s^*} f_\varepsilon(s) = f_\varepsilon(s^*)$ for every $s^* \in [s_0, s_2]$.

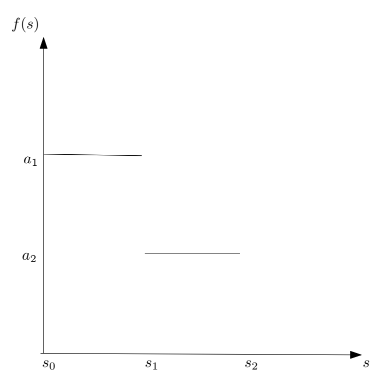

*Figure*1.

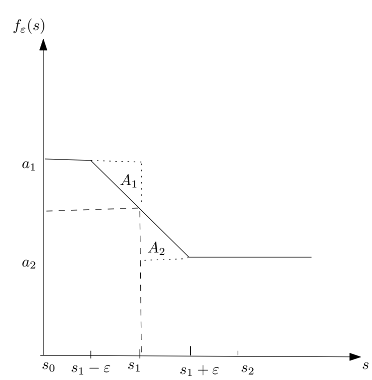

*Figure*2.

Lemma 3.1. *Let $f(s)$ and $f_\varepsilon(s)$ be functions as in Definitions 3.1 and 3.2 respectively. Then*

(6) $$\int_{s_0}^{s_2} f(s)ds = \int_{s_0}^{s_2} f_\varepsilon(s)ds.$$

Proof. The midpoint of the line

$$f_\varepsilon(s) = \frac{a_2 - a_1}{2\varepsilon}(s - s_1 + \varepsilon) + a_1 \ \ for \ \ s_1 - \varepsilon \le s < s_1 + \varepsilon$$

is $P = \left(s_1, \frac{a_1+a_2}{2}\right)$. (See Figure 2). Therefore, the areas

$$A_1 = \frac{\varepsilon}{2}\left[a_1 - \left(\frac{a_1+a_2}{2}\right)\right] = \frac{\varepsilon}{4}(a_1 - a_2),$$

$$A_2 = \frac{\varepsilon}{2}\left[\left(\frac{a_1+a_2}{2}\right) - a_2\right] = \frac{\varepsilon}{4}(a_1 - a_2).$$

Therefore, we have

$$A_1 = A_2.$$

Lemma 3.2. *Let $f(s)$ and $f_\varepsilon(s)$ be functions as in Definitions 3.1 and 3.2 respectively. If $\psi(s)$ is a convex, differentiable function with $\psi(0) = 0$, then*

$$\int_{s_0}^{s_2} [f_\varepsilon(s) - f(s)]\psi'(s)ds = \frac{a_1 - a_2}{2\varepsilon}\int_{s_1-\varepsilon}^{s_1+\varepsilon} [\psi(s) - \psi(s_1)]ds.$$

Proof. Write

$$\int_{s_0}^{s_2} [f_\varepsilon(s) - f(s)]\psi'(s)ds = \int_{s_0}^{s_2} f_\varepsilon(s)\psi'(s)ds - \int_{s_0}^{s_2} f(s)\psi'(s)ds. \tag{7}$$

The second term on the right side of (7) gives

$$\int_{s_0}^{s_2} f(s)\psi'(s)ds = \int_{s_0}^{s_1} a_1\psi'(s)ds + \int_{s_1}^{s_2} a_2\psi'(s)ds,$$

$$\int_{s_0}^{s_2} f(s)\psi'(s)ds = (a_1 - a_2)\psi(s_1) + a_2\psi(s_2) - a_1\psi(s_0). \tag{8}$$

Also, the first term on the right side of (7) is expressed as

$$\begin{aligned}\int_{s_0}^{s_2} f_\varepsilon(s)\psi'(s)ds = &\int_{s_0}^{s_1-\varepsilon} a_1\psi'(s)ds \\ &+ \int_{s_1-\varepsilon}^{s_1+\varepsilon}\left[\frac{a_2-a_1}{2\varepsilon}(s - s_1 + \varepsilon) + a_1\right]\psi'(x)ds \\ &+ \int_{s_1+\varepsilon}^{s_2} a_2\psi'(s)ds.\end{aligned}$$

Applying integration by parts, we obtain

$$\int_{s_0}^{s_2} f_\varepsilon(s)\psi'(s)ds = a_1[\psi(s_1-\varepsilon)-\psi(s_0)] + \frac{a_2-a_1}{2\varepsilon}\left[2\varepsilon\psi(s_1+\varepsilon) - \int_{s_1-\varepsilon}^{s_1+\varepsilon}\psi(s)ds\right]$$
$$+a_1\psi(s_1+\varepsilon)-a_1\psi(s_1-\varepsilon)+a_2\psi(s_2)-a_2\psi(s_1+\varepsilon),$$

which simplifies to

$$\int_{s_0}^{s_2} f_\varepsilon(s)\psi'(s)ds = \frac{a_1-a_2}{2\varepsilon}\int_{s_1-\varepsilon}^{s_1+\varepsilon}\psi(s)ds + a_2\psi(s_2)-a_1\psi(s_0). \tag{9}$$

Thus, the difference between inequalities (8) and (9) gives

$$\int_{s_0}^{s_2}[f_\varepsilon(s)-f(s)]\psi'(s)ds = \frac{(a_1-a_2)}{2\varepsilon}\int_{s_1-\varepsilon}^{s_1+\varepsilon}[\psi(s)-\psi(s_1)]ds$$

as required.

Lemma 3.3. *Let $g(s)$ be a continuous function on the interval $[s_0,s_2]$. Then*

$$\lim_{\eta\to 0}\frac{1}{2\eta}\int_{s_1-\eta}^{s_1+\eta} g(s)ds = g(s_1).$$

Proof. Let $\eta>0$ and set

$$I(\eta) = \frac{1}{2\eta}\int_{s_1-\eta}^{s_1+\eta} g(s)ds.$$

Continuity of g at s_1. Let $\varepsilon>0$, there exists $\delta>0$ such that $|g(s)-g(s_1)|<\varepsilon$ whenever $|s-s_1|<\delta$. Since

$$|I(\eta)-g(s_1)| \le \frac{1}{2\eta}\int_{s_1-\eta}^{s_1+\eta}|g(s)-g(s_1)|ds$$

for $\eta<\delta$, we have

$$s_1-\eta \in (s_1-\delta, s_1+\delta)$$

and

$$s_1+\eta \in (s_1-\delta, s_1+\delta).$$

Thus, $|s-s_1|<\eta$ and hence $|I(\eta)-g(s_1)|<\varepsilon$. Therefore $I(\eta)\to g(s_1)$ as $\eta\to 0$.

Lemma 3.4. *Let f be a simple function defined as in Definition 3.1 such that $0\le f\le 1$. If ψ is a convex, differentiable function with $\psi(0)=0$, then*

$$\psi\left(\int_{s_0}^{s_2} f(s)ds\right) \le \int_{s_0}^{s_2} f(s)\psi'(s)ds.$$

Proof. Let $f_\varepsilon(s)$ be continuous as in Definition 3.2. Then by Lemma 3.1, Theorem 3.1 and Lemma 3.2 respectively, we have

$$\begin{aligned}\psi\left(\int_{s_0}^{s_2} f(s)ds\right) &= \psi\left(\int_{s_0}^{s_2} f_\varepsilon(s)ds\right)\\ &\le \int_{s_0}^{s_2} f_\varepsilon(s)\psi'(s)ds\\ &\le \int_{s_0}^{s_2} f(s)\psi'(s)ds + \int_{s_0}^{s_2}[f_\varepsilon(s)-f(s)]\psi'(s)ds\\ &\le \int_{s_0}^{s_2} f(s)\psi'(s)ds + \frac{(a_1-a_2)}{2\varepsilon}\int_{s_1-\varepsilon}^{s_1+\varepsilon}[\psi(s)-\psi(s_1)]ds.\end{aligned}$$

Thus, by Lemma 3.3, when $\varepsilon \to 0$, we obtain

$$\psi\left(\int_{s_0}^{s_2} f(s)ds\right) \le \int_{s_0}^{s_2} f(s)\psi'(s)ds$$

as required.

Theorem 3.1. *Let f be a simple function on $[0,1]$ such that $0 \le f(s) \le 1$ for all $s \in [0,1]$. If ψ is a convex, differentiable function with $\psi(0) = 0$, then*

$$\psi\left(\int_0^1 f(s)ds\right) \le \int_0^1 f(s)\psi'(s)ds.$$

Proof. Let f be a simple function. There exists $\{0 = s_0, s_1, \cdots, s_n = 1\}$ and $\{a_1, a_2, \cdots, a_n\}$ such that $f(s) = a_j$ on $[s_j, s_{j+1})$ for $0 \le j \le n-1$. Let $0 < \varepsilon < \min|s_{j+1}-s_j|$ and define

$$f_\varepsilon(s) = f(s)$$

if

$$s \in [0, s_1-\varepsilon) \cup [s_1+\varepsilon, s_2-\varepsilon) \cup \cdots \cup [s_j+\varepsilon, s_{j+1}-\varepsilon) \cup \cdots \cup [s_{n-1}+\varepsilon, 1).$$

And

$$f_\varepsilon(s) = \frac{a_{(j+1)}-a_j}{2\varepsilon}(s-s_j+\varepsilon)+a_j$$

if

$$s \in [s_j-\varepsilon, s_j+\varepsilon)$$

where $j = 1, \cdots, n-1$. (See Figure 3 and Figure 4). Then, following Lemma 3.4, we have

$$\int_0^1 f(s)ds = \int_0^1 f_\varepsilon(s)ds$$

and

$$\begin{aligned}\psi\left(\int_0^1 f(s)ds\right) &= \psi\left(\int_0^1 f_\varepsilon(s)ds\right)\\ &\leq \int_0^1 f(s)\psi'(s)ds + \sum_{j=1}^{n-1}\frac{a_j - a_{j+1}}{2\varepsilon}\int_{s_j-\varepsilon}^{s_j+\varepsilon}[\psi(s) - \psi(s_j)]ds.\end{aligned}$$

Therefore

$$\psi\left(\int_0^1 f(s)ds\right) \leq \int_0^1 f(s)\psi'(s)ds$$

as required.

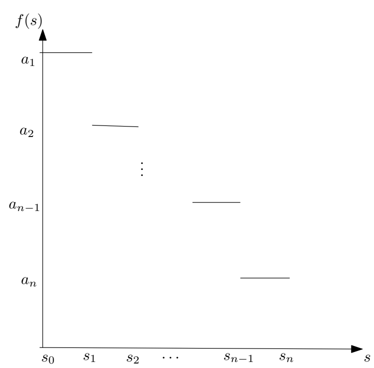

Figure3

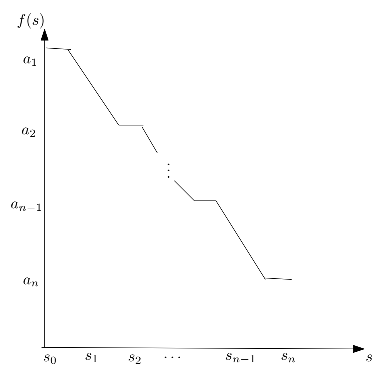

Figure4

4. Applications

In Theorem 3.1, replace 1 by $a > 0$. Thus

$$\psi\left(\int_0^{2\pi} f(s)ds\right) \leq \int_0^{2\pi} f(s)\psi'(s)ds.$$

We estimate the Fourier coefficients of ψ :

$$a_n = \frac{1}{2\pi}\int_0^{2\pi}\psi(s)\cos(ns)ds$$

and

$$b_n = \frac{1}{2\pi}\int_0^{2\pi}\psi(s)\sin(ns)ds$$

for $n \geq 1$. For the estimate of b_n, let $f(s) = \frac{1}{2}(1+\varepsilon\cos ns)$ for $\varepsilon = 1$ or -1. Thus

$$\frac{1}{2}\int_0^{2\pi}(1+\varepsilon\cos ns)ds = \pi$$

and

$$\frac{1}{2}\int_0^{2\pi}(1+\varepsilon\cos ns)\psi'(s)ds = \frac{\psi(2\pi)}{2}(1+\varepsilon)+\frac{n\varepsilon}{2}\int_0^{2\pi}\psi(s)\sin(ns)ds$$

Hence

$$\psi(\pi) \leq \frac{\psi(2\pi)}{2}(1+\varepsilon)+\frac{n\varepsilon}{2}\int_0^{2\pi}\psi(s)\sin(ns)ds$$

Take $\varepsilon = 1$ or -1 and we obtain

$$\frac{\psi(\pi)-\psi(2\pi)}{n\pi} \leq b_n \leq \frac{-\psi(\pi)}{n\pi}.$$

Also, for the estimate of a_n, let $f(s) = \frac{1}{2}(1+\varepsilon\sin ns)$ for $\varepsilon = 1$ or -1. Thus

$$\frac{1}{2}\int_0^{2\pi}(1+\varepsilon\sin ns)ds = \pi$$

and

$$\frac{1}{2}\int_0^{2\pi}(1+\varepsilon\sin ns)\psi'(s)ds = \frac{\psi(2\pi)}{2}-\frac{n\varepsilon}{2}\int_0^{2\pi}\psi(s)\cos(ns)ds.$$

Hence

$$\psi(\pi) \leq a_n \leq \frac{\psi(2\pi)}{2}-\frac{n\varepsilon}{2}\int_0^{2\pi}\psi(s)\cos(ns)ds.$$

Take $\varepsilon = 1$ or -1 and we obtain

$$\frac{\psi(\pi)-\psi(2\pi)/2}{n\pi} \leq a_n \leq \frac{\psi(2\pi)/2-\psi(\pi)}{n\pi}.$$

Example 3.1. If

- $\psi(s) = s$, then $a_n = 0$ and $b_n = -\frac{1}{n}$.
- $\psi(s) = s^2$, then $\frac{-\pi}{n} \leq a_n \leq \frac{\pi}{n}$ and $\frac{-3\pi}{n} \leq b_n \leq \frac{-\pi}{n}$.

4. Conclusion

The new Steffensen's inequality (4) is thus proved for continuous functions as well as simple (discontinuous) functions and also valid for all functions $f \in L^1([0,1])$. An application of the inequality has also been established for the determination of Fourier coefficients.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgement

The first author wish to thank Frederic SYMESAK for his attention and care and also deeply thank the French Government for the financial assistance and cooperation during his research period at the University of Angers, France.

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