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FIXED POINT THEOREMS IN CONE RANDOM METRIC SPACES

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Abstract. We define Cone random metric space and find some fixed point results for weak contraction condition we also illustrate an example in support of our result.

Keywords: random operator; cone random metric space; Cauchy sequence; random fixed point.
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1. INTRODUCTION

Random fixed point theorem for contraction mappings in polish spaces and random fixed point theorems are of fundamental importance in probabilistic functional analysis. Beg and Shahzad[2] studied the structure of common random fixed points and random coincidence points of a pair of compatible random operators. Recently Dhagat et. al.[3] given some results for random operators. In [4] Huang and Zhang generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space, hence they have defined the cone metric spaces. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality of a cone. Subsequently, various authors [1], [6] have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cones. Recently Sumitra, V.R., Uthariaraj,R. Hemavathy[6] gave some results for normal cones. There exist a lot of works

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involving points used the Banach contraction principle. Mehta, Singh, Sanodia and Dhagat [5] has given some results in random cone metric space.

2. PRELIMINARY

Definition 2.1: Let (E, τ) be a topological vector space and P a subset of E, P is called a cone if

1. P is non-empty and closed, $P \neq \{0\}$,

2. For x, y $\in$ P and a, b $\in$ R $\Longrightarrow$ ax + by $\in$ P where a,b≥ 0

1. If $x \in P$ and $-x \in P \Longrightarrow x = 0$
   For a given cone $P \subseteq E$, a partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$, $x < y$ if $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int } P$, int P denotes the interior of P.

A cone $P \subset E$ is called normal if there is a number K>0 such that for all x,y $\in$E $0 \leq x \leq y$ implies $||x|| \leq K||y||$.

The least positive number satisfying the above inequality is called the normal constant of P. It is clear that K$\geq$1. We know that that there exists an ordered Banach Space E with cone P which is not normal but int $P \neq \emptyset$.

Definition 2.2: Measurable function: Let (Ω, $\sum$) be a measurable space with $\sum$ a sigma algebra of subsets of Ω and M a non-empty subset of a metric space X = (X, d). Let 2^M be the family of all non-empty subsets of M and C(M) the family of all nonempty closed subsets of M. A mapping G: $\Omega \rightarrow 2^M$ is called measurable if, for each open subset U of M,

Definition 2.3: Measurable selector: A mapping $\xi : \Omega \rightarrow M$ is called a measurable selector of a measurable mapping G: $\Omega \rightarrow 2^M$ if ξ is measurable and $\xi(\omega) \in G(\omega)$ for each $\omega \in \Omega$.

Definition 2.4: Random operator: Mapping T : $\Omega \times M \rightarrow X$ is said to be a random operator if, for each fixed x $\in$ M, T(., x) : $\Omega \rightarrow X$ is measurable.

Definition 2.5:Continuous Random operator: A random operator $T : \Omega \times M \rightarrow X$ is said to be continuous random operator if, for each fixed $x \in M$, $T(., x) : \Omega \rightarrow X$ is continuous.

Definition 2.6: Random fixed point: A measurable mapping $\xi : \Omega \rightarrow M$ is a random fixed point of a random operator $T: \Omega \times M \rightarrow X$ if $\xi(\omega) = T\left(\omega, \xi(\omega)\right)$ for each $\omega \in \Omega$.

Definition 2.7: Let M be a nonempty set and the mapping $d: \Omega \times M \rightarrow X$ and $P \subset X$ be a cone, $\omega \in \Omega$ be a selector, satisfies the following conditions:

2.7.1) $d(x(\omega), y(\omega)) > 0 \; \forall \; x(\omega), y(\omega) \in \Omega \times X \Leftrightarrow x(\omega) = y(\omega)$
2.7.2) $d(x(\omega), y(\omega)) = d(y(\omega), x(\omega)) \;\; \forall x, y \in X, \omega \in \Omega \; and \; x(\omega), y(\omega) \in \Omega \times X$
2.7.3) $d(x(\omega), y(\omega)) = d(x(\omega), z(\omega)) \; + d(z(\omega), y(\omega)) \, \forall x, y \in X \; and \; \omega \in \Omega \, be \, a \, selector.$
2.7.4) $For \; any \; x, y \in X, \omega \in \Omega, \; d(x(\omega), y(\omega)) \quad is \, non-increa\sin g \; and \; left \; continuous.$

Then d is called cone random metric on M and (M, d) is called a cone random metric space.

Definition 2.8: Implicit Relation

Let Φ be the class of all real-valued continuous functions $\varphi : (R^+)^5 \rightarrow R^+$ non -decreasing in the first argument and satisfying the following conditions:

For $x, y \geq 0$, $x \leq \varphi(y,0,x,y, x+y)$ or $x \leq \varphi(y,y, x, y,x)$ or $x \leq \varphi(x, y,0,x, y)$
there exists a real number $0< h <1$ such that $x \leq h y$

3. MAIN RESULTS

Theorem 3.1 : Let (X, d) be a complete cone metric space and P a normal cone with normal constant K. Suppose M be a nonempty separable closed subset of cone metric space X and let T and S be commuting random operators defined on M such that for $w \in \Omega, T(\omega,.), S(\omega,.): \Omega \times M \rightarrow M$ satisfying contraction

$\|d(T(x(\omega), Ty(\omega)) \| \leq \lambda \| (d(Sx(\omega), Sy(\omega)) \|$ for all $x, y \in X$, $\omega \in \Omega$ ……………(1)

And range of S contains range of T and if S is continuous, then T have unique common fixed point in X.

Proof: For each $x_0(\omega) \in \Omega \times X$ and $x_1(\omega) \in \Omega \times X$ considered such that

$y_0(\omega)=T x_0 (\omega) = Sx_1(\omega)$. Therefore in general, $y_n(\omega)=Tx_n (\omega) = Sx_{n+1}(\omega)$

$\|d(y_n(\omega), y_{n-1}(\omega)) \| =\| (d(Tx_n(\omega), Tx_{n-1}(\omega)) \|$

$\leq \lambda \| (d(Sx_n(\omega), Sx_{n-1}(\omega)) \| = \lambda \| d(y_{n-1}(\omega), y_{n-2}(\omega)) \|$

$\Rightarrow \|d(y_n(\omega), y_{n-1}(\omega)) \| \leq \lambda \| d(y_{n-1}(\omega), y_{n-2}(\omega)) \|$

$\leq \lambda^2 \| d(y_{n-2}(\omega), y_{n-3}(\omega)) \| \leq \lambda^3 \| d(y_{n-3}(\omega), y_{n-4}(\omega)) \|$

…

$\leq \lambda^{n-1} \| d(y_1(\omega), y_0(\omega)) \|$

Now for n>m

$d(y_n(\omega), y_m(\omega)) \leq d(y_n(\omega), y_{n-1}(\omega)) + d(y_{n-1}(\omega), y_{n-2}(\omega)) + d(y_{n-2}(\omega), y_{n-3}(\omega)) + \ldots\ldots\ldots$

$+ \ldots\ldots. \| d(y_{m+1}(\omega), y_m(\omega)) \|$

Since P is normal cone

$\|d(y_n(\omega), y_m(\omega)) \| \leq K[\|d(y_n(\omega), y_{n-1}(\omega)) + d(y_{n-1}(\omega), y_{n-2}(\omega)) + d(y_{n-2}(\omega), y_{n-3}(\omega)) + \ldots$

$+ \ldots\ldots. d(y_{m+1}(\omega), y_m(\omega)) \|]$

$\|d(y_n(\omega), y_m(\omega)) \| \leq K[\|d(y_n(\omega), y_{n-1}(\omega)) \| + \| d(y_{n-1}(\omega), y_{n-2}(\omega)) \| + \| d(y_{n-2}(\omega), y_{n-3}(\omega)) \|$

$+ \ldots\ldots. \| d(y_{m+1}(\omega), y_m(\omega)) \|]$

$\|d(y_n(\omega), y_m(\omega)) \| \leq K[\lambda^{n-1} + \lambda^{n-1} + \lambda^{n-1} + \ldots\ldots.. \lambda^m] \| d(y_1(\omega), y_0(\omega)) \|$

$\|d(y_n(\omega), y_m(\omega)) \| \leq \frac{K\lambda^m}{1-\lambda} \|d(y_1(\omega), y_0(\omega)) \|$

$\Rightarrow \|d(y_n(\omega), y_m(\omega)) \| \to 0$ as $m \to \infty$

Therefore sequences $\{y_n(\omega)\} = \{Tx_n(\omega)\} = \{Sx_{n+1}(\omega)\}$ is Cauchy sequence and X in complete therefore there exist $p(\omega)$ in X such that

$$\lim_{n\to\infty} Tx_n(\omega) = \lim_{n\to\infty} Sx_{n+1}(\omega) = p(\omega)$$

Now S is continuous and T and S are commuting mappings, we get

$$Sp(\omega) = S \lim_{n\to\infty} Sx_n(\omega) = \lim_{n\to\infty} S^2x_n(\omega)$$

$$Sp(\omega) = S \lim_{n\to\infty} Tx_n(\omega) = \lim_{n\to\infty} STx_n(\omega) = \lim_{n\to\infty} TSx_n(\omega)$$

Now from (1) we have

$\|d(TSx_n(\omega), Sp(\omega))\| \leq \lambda \| (d(S^2x_n(\omega), Sp(\omega)) \|$

On taking $n \to \infty$, *we get*

$\|d(Sp(\omega), Tp(\omega))\| \leq \lambda \| (d(Sp(\omega), Sp(\omega)) \|$

Since $0 < \lambda < 1$, $\|d(Sp(\omega), Tp(\omega))\| = 0 \Rightarrow Sp(\omega) = Tp(\omega))$

Again from (1) we have

$\|d(Tx_n(\omega), Tp(\omega)\| \leq \lambda \| (d(Sx_n(\omega), Sp(\omega)) \|$

$\|d(p(\omega), Tp(\omega)\| \leq \lambda \| (d(p(\omega), Sp(\omega)) \| = \lambda \| (d(p(\omega), Tp(\omega)) \|$

$\Rightarrow Tp(\omega) = p(\omega)$.

$\Rightarrow Sp(\omega) = Tp(\omega)) = p(\omega)$.

For uniqueness let there exists another fixed point $q(\omega)$ in X such that from (1)

$\|d(p(\omega), q(\omega)) \| = \|d(T(p(\omega), Tq(\omega)) \| \leq \lambda \| (d(Sp(\omega), Sq(\omega)) \| = \|d(p(\omega), q(\omega)) \|$

Hence for all $0 < \lambda < 1$ we have $p(\omega) = q(\omega)$.

Theorem 3.2: Let (X, d) be a complete cone metric space and P a normal cone with normal constant K. Suppose M be a nonempty separable closed subset of cone metric space X and let T and S be commuting random operators defined on M such that for $w \in \Omega, T(\omega,.), S(\omega,.): \Omega \times M \to M$ satisfying contraction

$\|d(T(x(\omega), Ty(\omega)) \| \leq \lambda \| (d(Sx(\omega), Sy(\omega)) \|$

for all $x, y \in X$, $\omega \in \Omega$ and $0 < .\lambda < ½$……………(3.2.1)

$\|d(T(x(\omega), Ty(\omega)) \| \leq \lambda \| (d(Tx(\omega), Sx(\omega)) \| + \| (d(Ty(\omega), Sy(\omega)) \|$

for all $x, y \in X$, $\omega \in \Omega$ and $0 < .\lambda < ½$……………(3.2.2)

$\|d(T(x(\omega), Ty(\omega)) \| \leq \lambda \| (d(Tx(\omega), Sy(\omega)) \| + \| (d(Ty(\omega), Sx(\omega)) \|$

for all $x, y \in X$, $\omega \in \Omega$ and $0 < .\lambda < ½$……………(3.2.3)

And range of S contains range of T and if SX is continuous, then T and S have unique point of coincidence . If T and S weakly compatible, S and T have unique common fixed point in X.

Proof:- For each $x_0(\omega) \in \Omega \times X$ and $x_1(\omega) \in \Omega \times X$ considered such that

$y_0(\omega) = T x_0(\omega) = Sx_1(\omega)$. Therefore in general, $y_n(\omega) = Tx_n(\omega) = Sx_{n+1}(\omega)$

As per theorem 1 and for all the cases (3.1),(3.2),(3.3) we have

$\Rightarrow$ $\|d(y_n(\omega), y_{n-1}(\omega))\| \leq \lambda \| d(y_{n-1}(\omega), y_{n-2}(\omega))\|$……..(3.2.4)

Indeed by (3.2.1) it follows that

$\|d(y_n(\omega), y_{n-1}(\omega))\| = \|(d(Sx_{n+1}(\omega), Sx_n(\omega))\| = \|(d(Tx_n(\omega), Tx_{n-1}(\omega))\|$

$\leq \lambda \|(d(Sx_n(\omega), Sx_{n-1}(\omega))\| = \lambda \| d(y_{n-1}(\omega), y_{n-2}(\omega))\|.$

Indeed by (3.2.2) it follows that

$\|d(y_n(\omega), y_{n-1}(\omega))\| = \|(d(Sx_{n+1}(\omega), Sx_n(\omega))\| = (\|(d(Tx_n(\omega), Tx_{n-1}(\omega))\|$

$\leq \lambda[\|(d(Tx_n(\omega), Sx_n(\omega))\| + \| d(Tx_{n-1}(\omega), gy_{n-1}(\omega))\|]$

$\leq \lambda[\|(d(y_n(\omega), y_{n-1}(\omega))\| + (d(y_{n-1}(\omega), y_{n-2}(\omega))]$

$\|d(y_n(\omega), y_{n-1}(\omega))\| \leq h(d(y_{n-1}(\omega), y_{n-2}(\omega))$ where $h = \frac{\lambda}{\lambda-1} \in (0,1)$.

Indeed by (3.2.3) it follows that

$\|d(y_n(\omega), y_{n-1}(\omega))\| = \|(d(Tx_n(\omega), Tx_{n-1}(\omega))\|$

$\leq \lambda[\|(d(Tx_n(\omega), Sx_{n-1}(\omega))\| + \| d(Tx_{n-1}(\omega), gy_n(\omega))\|]$

$\leq \lambda[\|(d(y_n(\omega), y_{n-2}(\omega))\| + (d(y_{n-1}(\omega), y_{n-1}(\omega))]$

$\leq \lambda[\|(d(y_n(\omega), y_{n-1}(\omega))\| + (d(y_{n-1}(\omega), y_{n-2}(\omega))]$

$\|d(y_n(\omega), y_{n-1}(\omega))\| \leq h(d(y_{n-1}(\omega), y_{n-2}(\omega))$ where $h = \frac{\lambda}{\lambda-1} \in (0,1)$.

Now, by (3.4) for all cases we get

$\|d(y_n(\omega), y_{n-1}(\omega))\| \leq \lambda \| d(y_{n-1}(\omega), y_{n-2}(\omega))\|$

$\leq \lambda^2 \| d(y_{n-2}(\omega), y_{n-3}(\omega))\| \leq \lambda^3 \| d(y_{n-3}(\omega), y_{n-4}(\omega))\|$

…………………………………………………………………………………..

$\leq \lambda^{n-1} \| d(y_1(\omega), y_0(\omega))\|$

Now for n>m

$d(y_n(\omega), y_m(\omega)) \leq d(y_n(\omega), y_{n-1}(\omega)) + d(y_{n-1}(\omega), y_{n-2}(\omega)) + d(y_{n-2}(\omega), y_{n-3}(\omega)) + \ldots\ldots\ldots$

$+\ldots\ldots\| d(y_{m+1}(\omega), y_m(\omega))\|$

Since P is normal cone

$\|d(y_n(\omega), y_m(\omega))\| \leq K[\|d(y_n(\omega), y_{n-1}(\omega)) + d(y_{n-1}(\omega), y_{n-2}(\omega)) + d(y_{n-2}(\omega), y_{n-3}(\omega)) + \ldots$

$+\ldots\ldots d(y_{m+1}(\omega), y_m(\omega))\|]$

$\|d(y_n(\omega), y_m(\omega))\| \leq K[\|d(y_n(\omega), y_{n-1}(\omega))\| + \|d(y_{n-1}(\omega), y_{n-2}(\omega))\| + \|d(y_{n-2}(\omega), y_{n-3}(\omega))\|$

$+\ldots\ldots\| d(y_{m+1}(\omega), y_m(\omega))\|]$

$\|d(y_n(\omega), y_m(\omega))\| \leq K[\lambda^{n-1} + \lambda^{n-1} + \lambda^{n-1} + \ldots\ldots\lambda^{m}] \; \| d(y_1(\omega), y_0(\omega))\|$

$\|d(y_n(\omega), y_m(\omega))\| \leq \frac{K\lambda^m}{1-\lambda} \|d(y_1(\omega), y_0(\omega))\|$

$\Rightarrow \|d(y_n(\omega), y_m(\omega))\| \to 0$ as $m \to \infty$

Therefore sequences $\{y_n(\omega)\}=\{Tx_n(\omega)\} = \{Sx_{n+1}(\omega)\}$ is Cauchy sequence and S(X) is complete therefore there exist $p(\omega)$ in $X\text{x}\Omega$ such that $Sp(\omega) = q(\omega)$. Now we will show that for all cases $T(p) = q$.

From 3.2.1

$\|d(Sx_n(\omega),Tp(\omega))\| = \|d(Tx_{n-1}(\omega),Tp(\omega))\| \leq \lambda \| d(Sx_{n-1}(\omega), Sp(\omega))\|$

By taking $n\to\infty$, we get

$\Rightarrow \|d(Sp(\omega),Tp(\omega))\| \leq \lambda \| d(Sp(\omega), Sp(\omega))\| = 0.$

$\Rightarrow \|d(Sp(\omega),Tp(\omega))\| = 0$. Hence $Sp = Tp$.

Now for unique coincidence let us consider another point of coincidence $p_1(\omega)$ in $X\text{x}\Omega$ such that $Tp_1(\omega) = Sp_1(\omega) = q_1(\omega)$. Now ,

$\|d(Sp_1(\omega),Sp(\omega))\| = \|d(Tp_1(\omega),Tp(\omega))\| \leq \lambda \| d(Sp_1(\omega), Sp(\omega))\|.$

$\Rightarrow \|d(Sp_1(\omega),Sp(\omega))\| = 0$. Hence $Sp_1 = Sp = Tp = Tp_1$.

Now, from 3.2.2 it follows

$\|d(Sx_n(\omega),Tp(\omega))\| = \|d(Tx_{n-1}(\omega),Tp(\omega))\|$

$\leq \lambda [\| d(Tx_{n-1}(\omega), Sx_{n-1}(\omega)\| + \|d(Sp(\omega),Tp(\omega))\|.$

$\Rightarrow \|d(Sp(\omega),Tp(\omega))\| \leq \lambda \|d(Sp(\omega),Sp(\omega))\| + \|d(Tp(\omega),Sp(\omega))\| = \|d(Tp(\omega),Sp(\omega))\|.$

$\Rightarrow Tp=Sp.$

Again for uniqueness let us consider another point of coincidence $p_1(\omega)$ in $X\text{x}\Omega$ such that $Tp_1(\omega) = Sp_1(\omega) = q_1(\omega)$. Now

$\|d(Sp_1(\omega),Sp(\omega))\| = \|d(Tp_1(\omega),Tp(\omega))\|$

$$\leq \lambda\,[\,\|d(Tp_1(\omega), Sp_1(\omega))\| + \|d(Tp(\omega), Sp(\omega))\|\,].$$

$\Rightarrow \|d(Sp_1(\omega),Sp(\omega))\| = 0$. Hence $Sp_1 = Sp = Tp = Tp_1$.

Again from (3.2.3)

$\|d(Sx_n(\omega),Tp(\omega))\| = \|d(Tx_{n-1}(\omega),Tp(\omega))\|$

$$\leq \lambda\,[\,\|d(Tx_{n-1}(\omega), Sp(\omega)\| + \|d(Tp(\omega), Sx_{n-1}(\omega))\|\,].$$

By taking $n\to\infty$, we get

$\|d(Sp(\omega),Tp(\omega))\| \leq \lambda\,[\,\|d(Tp(\omega), Sp(\omega)\| + \|d(Tp(\omega), Sp(\omega))\|\,]$.

$\|d(Sp(\omega),Tp(\omega))\| \leq \lambda .2. \|d(Tp(\omega), Sp(\omega)\|$

Since $0<\lambda<1/2$ therefore $\|d(Sp(\omega),Tp(\omega))\|=0$. Hence $Sp(\omega) = Tp(\omega)$.

For uniqueness let us consider another point of coincidence $p_1(\omega)$ in $X x \Omega$ such that $Tp_1(\omega) = Sp_1(\omega) = q_1(\omega)$. Now

$\|d(Sp_1(\omega),Sp(\omega))\| = \|d(Tp_1(\omega),Tp(\omega))\|$

$$\leq \lambda\,[\,\|d(Tp_1(\omega), Sp(\omega))\| + \|d(Tp(\omega), Sp_1(\omega))\|\,]$$

$$= \lambda\,[\,\|d(Sp_1(\omega), Sp(\omega))\| + \|d(Sp(\omega), Sp_1(\omega))\|\,]$$

$\Rightarrow \|d(Sp_1(\omega),Sp(\omega))\| \leq \lambda .2. \|d(Sp_1(\omega),Sp(\omega))\|$.

Since $0<\lambda<1/2$ therefore $\|d(Sp_1(\omega),Sp(\omega))\|=0$. Hence $Sp_1(\omega)= Sp(\omega) = Tp(\omega)= Tp_1(\omega)$.

By the use of preposition 1.4 of [1] in all above cases we can find that $p(\omega)$ is unique common fixed point of T and S.

Example: Let $M=R$ and $P=\{x\in M:x\geq 0\}$, also $\Omega=[0,1]$ and Σ be the sigma algebra of Lebesgue's measurable subset of $[0,1]$. Let $X = [0,\infty)$ and define mapping as $d:(\Omega x X)x(\Omega x X)\to M$ by$d(x(\omega),y(\omega)) = |x(\omega)-y(\omega)|$. Then (X,d) is a cone random metric space. Define random operator T from $\Omega x X$ to X as $T(x(\omega)) = x(\omega)/2$. Also sequence of mapping $x_n : \Omega \to X$ is defined by $x_n(\omega) = \{(1-\omega^2)^{1+1/n}\}$for every $\omega \in \Omega$ and $n \in N$. Define measurable mapping $x : \Omega \to X$ as $x(\omega) = \{1-\omega^2\}$for every $\omega \in \Omega$. T Satisfies all condition of the theorem 3.1 and hence $(1-\omega^2)$ is fixed point of the space.

Conflict of Interests

The authors declare that there is no conflict of interests.

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