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INEQUALITIES INCLUDING FUNCTIONAL AFFINE COMBINATIONS

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Abstract. In this paper, we combine discrete and functional forms of Jensen's inequality for convex functions of several variables. We apply convex and affine combinations to inequalities. Using this approach, we consider the functional form of Jensen's inequality for affine combination of mappings. We also offer simple generalizations of the known results.

Keywords: positive linear functional; affine combination; Jensen's inequality.

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1. Introduction

We use the introduction to highlight three basic types of sets in a linear space, as well as their associated functions. For this purpose we use combinations of points (vectors) and coefficients (scalars). Let $\mathbb{X}$ be a real linear space. We can start with the binomial combinations

$$\alpha a + \beta b \tag{1}$$

of points $a, b \in \mathbb{X}$ and coefficients $\alpha, \beta \in \mathbb{R}$.

The combination in equation (1) is called linear. A set $\mathbb{L} \subseteq \mathbb{X}$ is linear (usually called a linear subspace) if it contains all binomial linear combinations of its points. A function $L : \mathbb{L} \to \mathbb{R}$ is

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linear (usually called a linear functional) if the equality

$$L(\alpha a+\beta b)=\alpha L(a)+\beta L(b) \tag{2}$$

holds for all binomial linear combinations $\alpha a+\beta b$ of the linear subspace $\mathbb{L}$.

If $\alpha+\beta=1$, the term linearity becomes affinity using the adjective affine for combinations, sets and functions.

The combination in equation (1) is called convex if $\alpha+\beta=1$ and $\alpha,\beta\geq 0$. A set $\mathscr{C}\subseteq\mathbb{X}$ is convex if it contains all binomial convex combinations of its points. A function $f:\mathscr{C}\to\mathbb{R}$ is convex if the inequality

$$f(\alpha a+\beta b)\leq \alpha f(a)+\beta f(b) \tag{3}$$

holds for all binomial convex combinations $\alpha a+\beta b$ of the convex set $\mathscr{C}$. A function f is concave if the function $-f$ is convex. Thus, a function is affine if, and only if, it is convex and concave.

The concept of linear, affine and convex hull of a vector set can also be described using binomial combinations. For example, the convex hull of a set $\mathscr{S}$, denoted by conv$\mathscr{S}$, consists of all binomial convex combinations of its vectors.

Using the mathematical induction, the above binomial combination properties can be extended to all finite combinations

$$\sum_{i=1}^{n}\alpha_i a_i, \tag{4}$$

of points $a_i\in\mathbb{X}$ and coefficients $\alpha_i\in\mathbb{R}$, assuming that $\sum_{i=1}^{n}\alpha_i=1$ in the affine case, and additionally assuming that all $\alpha_i\geq 0$ in the convex case.

2. Discrete and functional form of Jensen's inequality

Using the inductive method, Jensen presented in [2] the discrete form of the inequality relating to convex combinations and convex functions. Adapted to our needs, this inequality reads as follows.

Theorem A. *Let $\mathscr{C}$ be a convex set of a real linear space, let $a_1,\dots,a_n \in \mathscr{C}$ be points, and let $\sum_{i=1}^n \alpha_i a_i$ be a convex combination.*

Then every convex function $f:\mathscr{C}\to\mathbb{R}$ satisfies the inequality

$$f\left(\sum_{i=1}^n \alpha_i a_i\right) \le \sum_{i=1}^n \alpha_i f(a_i). \tag{5}$$

If f is concave, then the reverse inequality is valid in equation (5). *If f is affine, then the equality is valid in equation* (5).

In more than one hundred years of its existence, the famous Jensen's inequality in equation (5) even today occupies the attention of mathematicians who deal with inequalities. The following is a description of the functional form.

Let $\mathscr{X}$ be a non-empty set. Let $\mathbb{X}$ be a subspace of the linear space $\mathbb{R}^{\mathscr{X}}$ of all real functions on the domain $\mathscr{X}$. We use the space $\mathbb{X}$ that contains the unit function e_0, defined by $e_0(x)=1$ for every $x\in\mathscr{X}$.

A linear functional $L:\mathbb{X}\to\mathbb{R}$ is said to be positive (non-negative) or monotone if $L(g)\ge 0$ for every non-negative function $g\in\mathbb{X}$. We use the positive functional L satisfying $L(e_0)=1$. Such functional is called unital or normalized. For any function $g\in\mathbb{X}$, the number $L(g)$ is located in the closed convex hull of the set $\{g(x): x\in\mathscr{X}\}$, that is, in the closed interval of real numbers which contains the image of g.

Cite two examples of unital positive linear functionals. Given the n-tuple of non-negative coefficients $\alpha_i\in\mathbb{R}$ satisfying $\sum_{i=1}^n \alpha_i = 1$, and the n-tuple of points $x_i\in\mathscr{X}$, we define the summarizing linear functional L on the space $\mathbb{X}$ by

$$L^{\text{sum}}(g) = \sum_{i=1}^n \alpha_i g(x_i). \tag{6}$$

Obviously, the functional L^{sum} is positive, and since $L^{\text{sum}}(e_0)=1$, it is unital. If $\mathscr{X}$ is the measurable set respecting some positive measure μ so that $\mu(\mathscr{X})>0$, then the integrating linear functional L^{int} on the space $\mathbb{X}$ of all μ-integrable functions on $\mathscr{X}$, can be defined by

$$L^{\text{int}}(g) = \frac{1}{\mu(\mathscr{X})}\int_{\mathscr{X}} g\,d\mu. \tag{7}$$

The integrating functional is also positive and unital.

The functional form of Jensen's inequality for convex functions of one variable is due to Jessen, see [3]. McShane extended the functional form of Jensen's inequality to convex functions of several variables, see [4, Theorem 1 and Theorem 2]. He has covered the generalization in two steps, calling them the geometric formulation [4, Theorem 1] and analytic formulation [4, Theorem 2] of Jensen's inequality. We present these formulations in the theorem that follows.

Theorem B. *Let $\mathscr{C} \subseteq \mathbb{R}^m$ be a closed convex set. Let $g_1, \ldots, g_m \in \mathbb{X}$ be functions such that $\big(g_1(x), \ldots, g_m(x)\big) \in \mathscr{C}$ for every $x \in \mathscr{X}$. Let $f : \mathscr{C} \to \mathbb{R}$ be a continuous convex function such that $f(g_1, \ldots, g_m) \in \mathbb{X}$.*

Then every unital positive linear functional $L : \mathbb{X} \to \mathbb{R}$ satisfies the inclusion

$$\big(L(g_1), \ldots, L(g_m)\big) \in \mathscr{C}, \tag{8}$$

and the inequality

$$f\big(L(g_1), \ldots, L(g_m)\big) \leq L\big(f(g_1, \ldots, g_m)\big). \tag{9}$$

If f is concave, then the reverse inequality is valid in equation (9)*. If f is affine, then the equality is valid in equation* (9)*.*

The hyperplanes that contain the set $\mathscr{C}$ were used in the proof of the inclusion in equation (8). The epigraph of the convex function f,

$$\operatorname{epi}(f) = \{(x_1, \ldots, x_m, x_{m+1}) \in \mathscr{C} \times \mathbb{R} \,|\, x_{m+1} \geq f(x_1, \ldots, x_m)\}, \tag{10}$$

and the inclusion in equation (8) were applied in the proof of the inequality in equation (9).

3. Inequality for convex functions and affine combinations in the plane

If $A, B, C \in \mathbb{R}^2$ are planar points that do not belong to the same line, then every point $P \in \mathbb{R}^2$ can be presented by the unique affine combination

$$P = \alpha A + \beta B + \gamma C. \tag{11}$$

The above trinomial combination is convex if, and only if, the point P belongs to the triangle $\operatorname{conv}\{A,B,C\}$. If we use the point coordinates, the affine combination coefficients α,β,γ in equation (11) can be calculated using determinants, see [8, equations (24) and (25)].

Given the function $f:\mathbb{R}^2\to\mathbb{R}$, let $f^{\text{plane}}_{\{A,B,C\}}:\mathbb{R}^2\to\mathbb{R}$ be the function of the plane passing through the points $(A,f(A))$, $(B,f(B))$ and $(C,f(C))$ of the graph of the function f. Because of the affinity of $f^{\text{plane}}_{\{A,B,C\}}$, it follows that

$$f^{\text{plane}}_{\{A,B,C\}}(P)=\alpha f(A)+\beta f(B)+\gamma f(C). \tag{12}$$

In the case that the function f is convex, the inequality

$$f(P)\le \alpha f(A)+\beta f(B)+\gamma f(C)=f^{\text{plane}}_{\{A,B,C\}}(P) \tag{13}$$

holds for all points P belonging to the triangle $\operatorname{conv}\{A,B,C\}$.

Our main results are based on the discrete variant of the inequality relating to the planar affine combination $\alpha A+\beta B+\gamma C-\delta D$, where $A,B,C,D\in\mathbb{R}^2$ are points satisfying $D\in\operatorname{conv}\{A,B,C\}$, and $\alpha,\beta,\gamma,\delta\in[0,1]$ are coefficients satisfying $\alpha,\beta,\gamma\ge\delta$ and $\alpha+\beta+\gamma-\delta=1$. Including the convex combination of $D=\alpha_1A+\beta_1B+\gamma_1C$, it follows that

$$\alpha A+\beta B+\gamma C-\delta D=(\alpha-\delta\alpha_1)A+(\beta-\delta\beta_1)B+(\gamma-\delta\gamma_1)C. \tag{14}$$

The trinomial combination on the right-hand side of equation (14) is convex because its coefficients are non-negative, and their sum is equal to 1.

Relying on the equations in (14) and (13), and applying Jensen's inequality, we attain the following result for four-membered affine combinations and convex functions on the planar domain.

Theorem C. *Let $A,B,C,D\in\mathbb{R}^2$ be points such that $D\in\operatorname{conv}\{A,B,C\}$. Let $\alpha,\beta,\gamma,\delta\in[0,1]$ be coefficients such that $\alpha,\beta,\gamma\ge\delta$ and $\alpha+\beta+\gamma-\delta=1$.*

Then the affine combination

$$P=\alpha A+\beta B+\gamma C-\delta D \tag{15}$$

is in the set $\operatorname{conv}\{A,B,C\}$*, and every convex function* $f:\operatorname{conv}\{A,B,C\}\to\mathbb{R}$ *satisfies the inequality*

$$f(P)\le \alpha f(A)+\beta f(B)+\gamma f(C)-\delta f(D). \tag{16}$$

If f *is concave, then the reverse inequality is valid in equation* (16)*. If* f *is affine, then the equality is valid in equation* (16)*.*

The proof of Theorem C can be found in [7, Lemma 3.1]. The proof of the first part of Theorem C can also be found in [8, Lemma 13]. The affine combination of the line segment was applied in Jensen-Mercer's inequality for convex functions of one variable, see [5].

Remark 3.1. *To prove the inequality in equation* (16) *if* $\operatorname{conv}\{A,B,C\}$ *is the triangle, we can apply the function* $f^{plane}_{\{A,B,C\}}$ *and the inequality in equation* (13)*. Thus, it follows that*

$$\begin{aligned} f(P) &\le f^{plane}_{\{A,B,C\}}(\alpha A+\beta B+\gamma C-\delta D) && (17)\\ &= \alpha f(A)+\beta f(B)+\gamma f(C)-\delta f^{plane}_{\{A,B,C\}}(D) && (18)\\ &\le \alpha f(A)+\beta f(B)+\gamma f(C)-\delta f(D). && (19)\end{aligned}$$

If $\operatorname{conv}\{A,B,C\}$ *is the line segment* $\operatorname{conv}\{A,B\}$*, we use the function* $f^{line}_{\{A,B\}}$*. If* $\operatorname{conv}\{A,B,C\}$ *is the point* A*, the inequality in equation* (16) *is trivially reduced to* $f(A)\le f(A)$*.*

4. Main results

We focus on the formulation of the functional variant of Theorem C. First we will analyze the special case of the equality in equation (9) for functions $g_1,g_2\in\mathbb{X}$, and the affine function $f:\mathbb{R}^2\to\mathbb{R}$. Using the equation $f(x_1,x_2)=\lambda_1x_1+\lambda_2x_2+\lambda_3$ with real constants λ_1, λ_2 and λ_3, we have the equality

$$\begin{aligned} f\big(L(g_1),L(g_2)\big) &= \lambda_1L(g_1)+\lambda_2L(g_2)+\lambda_3\\ &= \lambda_1L(g_1)+\lambda_2L(g_2)+\lambda_3L(e_0)\\ &= L(\lambda_1g_1+\lambda_2g_2+\lambda_3e_0)\\ &= L\big(f(g_1,g_2)\big). \end{aligned} \tag{20}$$

The above equality, using f as the triangle plane function, will be applied in the proof of the functional variant of Theorem C.

Let $\mathscr{C} \subseteq \mathbb{R}^2$ be a convex set, and let $\mathbb{X} \subseteq \mathbb{R}^{\mathscr{X}}$ be a real linear space. The subset $\mathbb{X}^2_{\mathscr{C}}$ of the product space $\mathbb{X}^2$ which contains all mappings with the image in $\mathscr{C}$ is convex. Indeed, if $\alpha a + \beta b$ is the convex combination of mappings

$$a = (a_1, a_2)\,,\, b = (b_1, b_2) \tag{21}$$

belonging to $\mathbb{X}^2_{\mathscr{C}}$, then the planar convex combination $\alpha a(x) + \beta b(x)$ is in $\mathscr{C}$ for every $x \in \mathscr{X}$, that is, $\alpha a + \beta b$ is in $\mathbb{X}^2_{\mathscr{C}}$.

In the following we use the closed convex set $\mathscr{C} \subseteq \mathbb{R}^2$. Working with mappings $a = (a_1, a_2) \in \mathbb{X}^2_{\mathscr{C}}$, and functionals $L : \mathbb{X} \to \mathbb{R}$, we will use the abbreviation

$$L(a) = \big(L(a_1), L(a_2)\big). \tag{22}$$

Theorem 4.1. *Let $\mathscr{C} \subseteq \mathbb{R}^2$ be a closed convex set. Let $a,b,c,d \in \mathbb{X}^2_{\mathscr{C}}$ be mappings, and let $L : \mathbb{X} \to \mathbb{R}$ be a unital positive linear functional so that $d \in \mathbb{X}^2_{\triangle}$ where $\triangle = \operatorname{conv}\{L(a), L(b), L(c)\}$. Let $f : \mathscr{C} \to \mathbb{R}$ be a continuous convex function such that compositions $f(a), f(b), f(c), f(d) \in \mathbb{X}$. Let $\alpha, \beta, \gamma, \delta \in [0,1]$ be coefficients such that $\alpha, \beta, \gamma \geq \delta$ and $\alpha + \beta + \gamma - \delta = 1$.*

Then the affine combination

$$l = \alpha L(a) + \beta L(b) + \gamma L(c) - \delta L(d) \tag{23}$$

is in the set $\triangle$, and we have the inequalities

$$f(l) \leq \alpha f(L(a)) + \beta f(L(b)) + \gamma f(L(c)) - \delta f(L(d)) \tag{24}$$

and

$$f(l) \leq \alpha L(f(a)) + \beta L(f(b)) + \gamma L(f(c)) - \delta L(f(d)). \tag{25}$$

Proof. Since $L(d) \in \triangle$ by Theorem B, then also $l \in \triangle$ by Theorem C. Thus it is clear that the inequality in equation (24) follows from the inequality in equation (16). Prove the inequality in equation (25).

If $\triangle$ is the triangle, we can use the triangle plane function $f^{\text{plane}}_{\{L(a),L(b),L(c)\}}$. Since the number inequality $f^{\text{plane}}_{\{L(a),L(b),L(c)\}}(d(x)) \geq f(d(x))$ holds for every $x \in \mathscr{X}$, we have the composition inequality

$$f^{\text{plane}}_{\{L(a),L(b),L(c)\}}(d) \geq f(d). \tag{26}$$

Applying the convexity of f, the affinity of $f^{\text{plane}}_{\{L(a),L(b),L(c)\}}$, as well as the formulae in equations (13), (20) and (26), we get the series of inequalities

$$\begin{aligned} f(l) &\leq f^{\text{plane}}_{\{L(a),L(b),L(c)\}}\Big(\alpha L(f(a)) + \beta L(f(b)) + \gamma L(f(c)) - \delta L(f(d))\Big) \\ &= \alpha f(L(a)) + \beta f(L(b)) + \gamma f(L(c)) - \delta f^{\text{plane}}_{\{L(a),L(b),L(c)\}}(L(d)) \\ &= \alpha f(L(a)) + \beta f(L(b)) + \gamma f(L(c)) - \delta L\big(f^{\text{plane}}_{\{L(a),L(b),L(c)\}}(d)\big) \\ &\leq \alpha L(f(a)) + \beta L(f(b)) + \gamma L(f(c)) - \delta L(f(d)) \end{aligned} \tag{27}$$

concluding the proof of the inequality in equation (25) for this case.

If $\triangle$ is the line segment, to prove the inequality in equation (25) we can apply the series of inequalities in equation (27) using the chord line instead of the triangle plane.

If $\triangle$ is the plane point, the mapping d is constant. Using the representation $d = L(a)e_0 = L(d)e_0$, we get that the composition

$$f(d) = f(L(d))e_0. \tag{28}$$

In this case, (25) follows from (24) because $f(L(a)) \leq L(f(a))$, $f(L(b)) \leq L(f(b))$ and $f(L(c)) \leq L(f(c))$ by (9), and $L(f(d)) = f(L(d))$ by (28). $\square$

Remark 4.2. *If the mappings $a,b,c,d \in \mathbb{X}^2_{\mathscr{C}}$ satisfy the conditions of Theorem 4.1, then the affine combination $\alpha a(x) + \beta b(x) + \gamma c(x) - \delta d(x)$ does not necessarily belong to $\triangle$ for every $x \in \mathscr{X}$.*

Theorem 4.1 can be extended including the convex combination of mappings d_i satisfying the same condition as the mapping d. It will be affirmed in the following theorem.

Corollary 4.3. *Let $\mathscr{C} \subseteq \mathbb{R}^2$ be a closed convex set. Let $a,b,c,d_1,\ldots,d_n \in \mathbb{X}^2_{\mathscr{C}}$ be mappings, and let $L : \mathbb{X} \to \mathbb{R}$ be a unital positive linear functional so that all $d_i \in \mathbb{X}^2_{\triangle}$ where $\triangle =$*

$\operatorname{conv}\{L(a),L(b),L(c)\}$. *Let* $f:\mathscr{C}\to\mathbb{R}$ *be a continuous convex function such that all compositions* $f(a),f(b),f(c),f(d_i)\in\mathbb{X}$. *Let* $\alpha,\beta,\gamma,\delta,\delta_i\in[0,1]$ *be coefficients such that* $\alpha+\beta+\gamma-\delta=\sum_{i=1}^n\delta_i=1$.

Then the affine combination

$$l=\alpha L(a)+\beta L(b)+\gamma L(c)-\delta\sum_{i=1}^{n}\delta_i L(d_i) \tag{29}$$

is in the set $\triangle$, *and we have the inequalities*

$$f(l)\le\alpha f(L(a))+\beta f(L(b))+\gamma f(L(c))-\delta\sum_{i=1}^{n}\delta_i f(L(d_i)) \tag{30}$$

and

$$f(l)\le\alpha L(f(a))+\beta L(f(b))+\gamma L(f(c))-\delta\sum_{i=1}^{n}\delta_i L(f(d_i)). \tag{31}$$

Proof. The convex combination $d(x)=\sum_{i=1}^n\delta_i d_i(x)$ belongs to $\triangle$ for every $x\in\mathscr{X}$. Therefore, respecting Theorem 4.1 the affine combination l in equation (29) also belongs to $\triangle$.

The series of inequalities in equation (27) can be applied to the proof of the inequality in equation (31) because the inequality in equation (26) holds for every mapping d_i. $\square$

5. Application to functional quasi-arithmetic means in the plane

The Jensen inequality in equation (5) contains two means. The first mean is the convex combination in parentheses on the left-hand side, and the second mean is the convex combination on the right-hand side. Basic facts on means and their associated inequalities can be found in [1] and [9]. Among other means, the quasi-arithmetic functional means were investigated in [6].

The notion of quasi-arithmetic mean is usually associated to convex combinations and injective mappings that preserve the convexity of domain. Let $\mathscr{C}\subseteq\mathbb{R}^m$ be a convex set, let $a=\sum_{i=1}^n\alpha_i a_i$ be a convex combination where $a_i\in\mathscr{C}$, and let $\varphi:\mathscr{C}\to\mathbb{R}^m$ be an injective mapping so that the image set $\varphi(\mathscr{C})$ is convex. We define the φ-quasi-arithmetic mean of the observed combination a as the point

$$M_\varphi(a) = \varphi^{-1}\left(\sum_{i=1}^{n} \alpha_i \varphi(a_i)\right). \tag{32}$$

The convexity of $\varphi(\mathscr{C})$ ensures that the convex combination $a_\varphi = \sum_{i=1}^{n} \alpha_i \varphi(a_i)$ be contained in $\varphi(\mathscr{C})$, and so the quasi-arithmetic mean $M_\varphi(a) = \varphi^{-1}(a_\varphi)$ is contained in the set $\mathscr{C}$.

The concept of quasi-arithmetic mean can be extended to affine combinations having the above properties. For this purpose, let us take an affine combination as in Corollary 4.3, where $\triangle = \operatorname{conv}\{L(a), L(b), L(c)\}$ is the triangle,

$$l = \alpha L(a) + \beta L(b) + \gamma L(c) - \delta \sum_{i=1}^{n} \delta_i L(d_i), \tag{33}$$

and take an injective mapping $\varphi : \operatorname{conv}\{L(a), L(b), L(c)\} \to \mathbb{R}^2$ satisfying the convexity condition

$$\varphi\big(\operatorname{conv}\{L(a), L(b), L(c)\}\big) = \operatorname{conv}\{\varphi(L(a)), \varphi(L(b)), \varphi(L(c))\}. \tag{34}$$

For example, the mapping φ can be the composition of translations, homotheties and rotations in the plane. We define the φ-quasi-arithmetic mean of the combination l as the point

$$M_\varphi(l) = \varphi^{-1}\left(\alpha\varphi\big(L(a)\big) + \beta\varphi\big(L(b)\big) + \gamma\varphi\big(L(C)\big) - \delta \sum_{i=1}^{n} \delta_i \varphi\big(L(d_i)\big)\right). \tag{35}$$

The assumptions ensure that the quasi-arithmetic mean point $M_\varphi(l)$ is located in the triangle $\operatorname{conv}\{L(a), L(b), L(c)\}$, and the point

$$l_\varphi = \alpha\varphi\big(L(a)\big) + \beta\varphi\big(L(b)\big) + \gamma\varphi\big(L(C)\big) - \delta \sum_{i=1}^{n} \delta_i \varphi\big(L(d_i)\big) \tag{36}$$

is located in the triangle $\operatorname{conv}\{\varphi(L(a)), \varphi(L(b)), \varphi(L(c))\}$. The quasi-arithmetic means defined in (35) are invariant with respect to affine mappings of $\mathbb{R}^2$, that is, the equality

$$M_{\lambda\varphi(x,y)+(x_0,y_0)}(l) = M_{\varphi(x,y)}(l) \tag{37}$$

holds for all pairs of the nonnegative number λ in $\mathbb{R}$ and the point $(x_0, y_0) \in \mathbb{R}^2$. Indeed, if

$$\psi(x,y) = \lambda\varphi(x,y) + (x_0, y_0), \tag{38}$$

then

$$l_\psi = \lambda l_\varphi + (x_0, y_0) \tag{39}$$

and

$$\psi^{-1}(x,y) = \varphi^{-1}\left(\frac{(x,y)-(x_0,y_0)}{\lambda}\right), \tag{40}$$

and therefore, it follows that

$$M_\psi(l) = \psi^{-1}(l_\psi) = \varphi^{-1}\left(\frac{l_\psi-(x_0,y_0)}{\lambda}\right) = \varphi^{-1}(l_\varphi) = M_\varphi(l).$$

6. Generalizations

We first present two simple generalizations of Theorem B. Given the linear functional $L : \mathbb{X} \to \mathbb{R}$, and mappings

$$g_i = \big(g_{i1}, \dots, g_{im}\big) \tag{41}$$

belonging to the product space $\mathbb{X}^m$, we use the abbreviations

$$L(g_i) = \big(L(g_{i1}), \dots, L(g_{im})\big). \tag{42}$$

Theorem B can be presented with several mappings $g_i \in \mathbb{X}^m_{\mathscr{C}}$. The following corollary is based on the discrete and functional form of Jensen's inequality.

Corollary 6.1. *Let $\mathscr{C} \subseteq \mathbb{R}^m$ be a closed convex set. Let $g_1, \dots, g_n \in \mathbb{X}^m_{\mathscr{C}}$ be mappings, and let $\sum_{i=1}^n \alpha_i g_i$ be a convex combination. Let $f : \mathscr{C} \to \mathbb{R}$ be a continuous convex function such that all compositions $f(g_i) \in \mathbb{X}$.*

Then every unital positive linear functional $L : \mathbb{X} \to \mathbb{R}$ satisfies the inclusion

$$\sum_{i=1}^n \alpha_i L(g_i) \in \mathscr{C}, \tag{43}$$

and the double inequality

$$f\left(\sum_{i=1}^n \alpha_i L(g_i)\right) \le \sum_{i=1}^n \alpha_i f(L(g_i)) \le \sum_{i=1}^n \alpha_i L(f(g_i)). \tag{44}$$

Theorem B can also be presented with several linear functionals L_i. The following is such a generalization.

Corollary 6.2. *Let $\mathscr{C} \subseteq \mathbb{R}^m$ be a closed convex set. Let $g_1, \dots, g_n \in \mathbb{X}^m_{\mathscr{C}}$ be mappings. Let $f : \mathscr{C} \to \mathbb{R}$ be a continuous convex function such that all compositions $f(g_i) \in \mathbb{X}$. Let $L_1, \dots, L_n : \mathbb{X} \to \mathbb{R}$ be positive linear functionals such that all $L_i(e_0) > 0$ and $\sum_{i=1}^n L_i(e_0) = 1$.*

Then we have the inclusion

$$\sum_{i=1}^n L_i(g_i) \in \mathscr{C}, \tag{45}$$

and the double inequality

$$f\left(\sum_{i=1}^n L_i(g_i)\right) \leq \sum_{i=1}^n f\big(L_i(g_i)\big) \leq \sum_{i=1}^n L_i\big(f(g_i)\big). \tag{46}$$

Proof. Taking positive coefficients $\alpha_i = L_i(e_0)$, and unital positive linear functionals

$$M_i = \frac{1}{\alpha_i} L_i, \tag{47}$$

we have the convex combination

$$\sum_{i=1}^n L_i(g_i) = \sum_{i=1}^n \alpha_i M_i(g_i) \tag{48}$$

to which we can apply the inequality in equation (44). □

The following is the generalization of Corollary 4.3. We conclude the inequality for convex functions and functional affine combinations in the space $\mathbb{R}^m$ as follows.

Corollary 6.3. *Let $\mathscr{C} \subseteq \mathbb{R}^m$ be a closed convex set. Let $a_1, \dots, a_{m+1}, b_1, \dots, b_n \in \mathbb{X}^m_{\mathscr{C}}$ be mappings, and let $L : \mathbb{X} \to \mathbb{R}$ be a unital positive linear functional so that all $b_i \in \mathbb{X}^m_{\triangle}$ where $\triangle = \operatorname{conv}\{L(a_1), \dots, L(a_{m+1})\}$ is the simplex. Let $f : \mathscr{C} \to \mathbb{R}$ be a continuous convex function such that all compositions $f(a_j), f(b_i) \in \mathbb{X}$. Let $\alpha_j, \beta, \beta_i \in [0,1]$ be coefficients such that $\alpha_j \geq \beta$ and $\sum_{j=1}^{m+1} \alpha_j - \beta = \sum_{i=1}^n \beta_i = 1$.*

Then the affine combination

$$l = \sum_{j=1}^{m+1} \alpha_j L(a_j) - \beta \sum_{i=1}^n \beta_i L(b_i) \tag{49}$$

is in the set $\triangle$, and we have the inequalities

$$f(l) \leq \sum_{j=1}^{m+1} \alpha_j f(L(a_j)) - \beta \sum_{i=1}^{n} \beta_i f(L(b_i)) \tag{50}$$

and

$$f(l) \leq \sum_{j=1}^{m+1} \alpha_j L(f(a_j)) - \beta \sum_{i=1}^{n} \beta_i L(f(b_i)). \tag{51}$$

If $\triangle$ is the m-simplex, in which case the points $L(a_1), \ldots, L(a_{m+1})$ are its vertices, to prove the inequalities in equations (50) and (51) we use the hyperplane function $f^{\text{hyperplane}}_{\{L(a_1),\ldots,L(a_{m+1})\}}$.

Continuing the previous corollary, we will give the formula for the functional quasi-arithmetic means in the space $\mathbb{R}^m$. Take an affine combination l as in equation (49) satisfying the assumptions of Corollary 6.3, where $\triangle$ is the m-simplex, and take an injective mapping

$$\varphi : \operatorname{conv}\{L(a_1), \ldots, L(a_{m+1})\} \to \mathbb{R}^m \tag{52}$$

satisfying the convexity condition

$$\varphi\big(\operatorname{conv}\{L(a_1), \ldots, L(a_{m+1})\}\big) = \operatorname{conv}\{\varphi(L(a_1)), \ldots, \varphi(L(a_{m+1}))\}. \tag{53}$$

We define the φ-quasi-arithmetic mean of the combination l as the point

$$M_\varphi(l) = \varphi^{-1}\left(\sum_{j=1}^{m+1} \alpha_j \varphi\big(L(a_j)\big) - \beta \sum_{i=1}^{n} \beta_i \varphi\big(L(b_i)\big)\right) \tag{54}$$

which belongs to the m-simplex $\operatorname{conv}\{L(a_1), \ldots, L(a_{m+1})\}$.

Conflict of Interests

The author declares that there is no conflict of interests.

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