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# NEW CLASSES OF A-I$_2$ CONVERGENCE DOUBLE SEQUENCE SPACES OF FUZZY NUMBERS DEFINED BY SEQUENCE OF ORLICZ FUNCTIONS

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**Abstract.** In this article we introduce the spaces $2^{{W_0}^{I(F)}}(A,\boldsymbol{M},p)$ , $2^{W^{I(F)}}(A,\boldsymbol{M},p)$ and $2^{{W_\infty}^{I(F)}}(A,\boldsymbol{M},p)$ and under special cases several other spaces. We study some properties relevant to these spaces.

**Keywords:** Sequence of fuzzy numbers; A-I$_2$ convergent; Orlicz fuction; double lacunary sequence; $P$-convergent; sequence of Orlicz functions.

**2010 AMS Subject Classification**: 40A05, 40D25, 46A45, 46E30.

## 1. Introduction

The notion of $I$-convergence of real valued sequence was studied at the initial stage by Kostyrko, Šalát and Wilczyński [4] which generalizes and unifies different notions of convergence of sequences. The notion was further studied by Šalát, Tripathy and Ziman [11].

The notion of fuzzy sets was introduced by Zadeh [25]. After that many authors have studied and generalized this notion in many ways, due to the potential of the introduced notion. Also it has wide range of applications in almost all the branches of studied in particular science, where mathematics is used. It attracted many workers to introduce different types of fuzzy sequence spaces.

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Bounded and convergent sequences of fuzzy numbers were studied by Matloka [7]. Later on sequences of fuzzy numbers have been studied by Kaleva and Seikkala [1], Tripathy and Sarma ([20] and many others.

An *Orlicz function* is a function $M$: $[0, \infty) \rightarrow [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. If the convexity of $M$ is replaced by

$$M(x + y) \leq M(x) + M(y)$$

then this function is called the *modulus function*.

**Remark 1.** It is well known if $M$ is an Orlicz function, then $M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0<\lambda <1$.

Lindenstrauss and Tzafriri [6] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left( \frac{|x_k|}{r} \right) < \infty, \text{for some } r > 0 \right\}$$

The space $\ell_M$ becomes a Banach space, with the norm

$$\|x\| = \inf \left\{ r > 0 : \sum_{k=1}^{\infty} M\left( \frac{|x_k|}{r} \right) \leq 1 \right\}.$$

## 2. Definitions and Background

Let $X$ be a non-empty set, then a non-void class $I \subseteq 2^X$ (power set of $X$ ) is called an *ideal* if $I$ is additive (i.e. $A, B \in I \Rightarrow A \cup B \in I$) and hereditary (i.e. $A \in I$ and $B \subseteq A \Rightarrow B \in I$). An ideal $I \subseteq 2^X$ is said to be non-trivial if $I \neq 2^X$. A non-trivial ideal $I$ is said to be *admissible* if $I$ contains every finite subset of $N$. A non-trivial ideal $I$ is said to be *maximal* if there does not exist any non-trivial ideal $J \neq I$ containing $I$ as a subset.

Let $X$ be a non-empty set, then a non-void class $F \subseteq 2^X$ is said to be a *filter* in $X$ if $\phi \notin F$ ; $A, B \in F \Rightarrow A \cap B \in F$ and $A \in F$, $A \subseteq B \Rightarrow B \in F$ . For any ideal $I$, there is a filter $\Psi(I)$ corresponding to $I$, given by

$$\Psi(I) = \{K \subseteq N : N \setminus K \in I\}.$$

**Example.** (*a*) Let $I = I_f$ , the class of all finite subsets of $N$. Then $I_f$ is a non-trivial admissible ideal.

(*b*) Let $A \subset N$. If $\delta(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k)$ exists, then the class $I_\delta$ of all $A \subset N$ with $\delta(A) = 0$ forms a non-trivial admissible ideal.

($c$) Let $A \subset N$ and $s_n = \sum_{k=1}^{n} \frac{1}{k}$ , for all $n \in N$. If $d(A) = \lim_{n\to\infty} \frac{1}{s_n} \sum_{k=1}^{n} \frac{\chi_A(k)}{k}$ exists, then the class $I_d$ of all $A \subset N$ with $d(A) = 0$ forms a non-trivial admissible ideal.

($d$) The *uniform density* of a set $A \subset N$ is defined as follows. For integers $t \geq 0$ and $s \geq 1$,

let $A(t+1, t+s) = \text{card}\{ n \in A : t+1 \leq n \leq t+s \}$. Put $\beta_s = \liminf_{t\to\infty} A(t+1, t+s)$, $\beta^s = \limsup_{t\to\infty} A(t+1, t+s)$. If $\lim_{s\to\infty} \frac{\beta_s}{s}$ and $\lim_{s\to\infty} \frac{\beta^s}{s}$ both exist and $\lim_{s\to\infty} \frac{\beta_s}{s} = \lim_{s\to\infty} \frac{\beta^s}{s}$ (= $u(A)$, say), then $u(A)$ is called the uniform density of $A$. The class $I_u$ of all $A \subset N$ with $u(A) = 0$ forms a non-trivial ideal.

Let $D$ denote the set of all closed and bounded intervals $X = [a_1, b_1]$ on the real line $R$. For $X = [a_1, b_1] \in D$ and $Y = [a_2, b_2] \in D$, define $d(X, Y)$ by

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|).$$

It is known that $(D, d)$ is a complete metric space.

A fuzzy real number $X$ is a fuzzy set on $R$ *i.e.* a mapping $X : R \to L(= [0,1])$ associating each real number $t$ with its grade of membership $X(t)$.

The $\alpha$- level set $[X]^\alpha$ set of a fuzzy real number $X$ for $0 < \alpha \leq 1$, defined as $X^\alpha = \{ t \in R : X(t) \geq \alpha \}$.

A fuzzy real number $X$ is called *convex*, if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number $X$ is called *normal*.

A fuzzy real number $X$ is said to be *upper semi- continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in L$ is open in the usual topology of $R$.

The set of all upper semi-continuous, normal, convex fuzzy number is denoted by $L(R)$.

The absolute value $|X|$ of $X \in L(R)$ is defined as (see for instance Kaleva and Seikkala [1] )

$$|X|(t) = \begin{cases} \max\{ X(t), X(-t) \}, & \text{if } t > 0 \\ 0, & \text{if } t < 0. \end{cases}$$

Let $\bar{d} : L(R) \times L(R) \to R$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

Then $\bar{d}$ defines a metric on $L(R)$.

A sequence $(X_k)$ of fuzzy real numbers is said to be convergent to the fuzzy real number $X_0$, if for every $\varepsilon>0$, there exists $n_0 \in N$ such that $\bar{d}\,(X_k, X_0) < \varepsilon$ for all $k \geq k_0$.

A fuzzy real valued sequence space $E^F$ is said to be *solid* if $(Y_k) \in E^F$ whenever $(X_k) \in E^F$ and $|Y_k| \leq |X_k|$, for all $k \in N$.

A sequence $X = (X_k)$ of fuzzy numbers is said to be $I$- convergent if there exists a fuzzy number $X_0$ such that for all $\varepsilon>0$, the set $\{n \in N : \bar{d}\,(X_k, X_0) \geq \varepsilon\} \in I$. We write $I$-$\lim X_k = X_0$.

A sequence $(X_k)$ of fuzzy numbers is said to be $I^*$- convergent to $X_0$ ($I^*$-$\lim X_k = X_0$) if there is a set $\{k_1 < k_2 < \text{-----}\} \in \Psi(I)$ such that $\lim\limits_{i\to\infty} X_{k_i} = X_0$.

A sequence $(X_k)$ of fuzzy numbers is said to be $I$- bounded if there exists a real number $\mu$ such that the set $\{k \in N : \bar{d}\,(X_k, \bar{0}) > \mu\} \in I$.

If $I = I_f$, then $I_f$ convergence coincides with the usual convergence of fuzzy sequences. If $I = I_d\,(I_\delta)$, then $I_d\,(I_\delta)$ convergence coincides with statistical convergence (logarithmic convergence) of fuzzy sequences. If $I = I_u$, $I_u$ convergence is said to be uniform convergence of fuzzy sequences.

Throughout $c^{I(F)}$, $c_0^{I(F)}$ and $\ell_\infty^{I(F)}$ denote the spaces of fuzzy real-valued *I- convergent*, *I-null* and *I- bounded* sequences respectively.

It is clear from the definitions that $c_0^{I(F)} \subset c^{I(F)} \subset \ell_\infty^{I(F)}$ and the inclusions are proper.

It can be easily shown that $\ell_\infty^{I(F)}$ is complete with respect to the metric $\rho$ defined by $f\,(X, Y) = \sup\limits_n \bar{d}\,(X_k, Y_k)$, where $X = (X_k)$, $Y = (Y_k) \in \ell_\infty^{I(F)}$.

**Lemma 1.** *A sequence space $E^F$ is solid implies $E^F$ is monotone.*

**Lemma 2.** *If $I \subset 2^N$ is a maximal ideal, then for each $A \subset N$ we have either $A \in I$ or $N \setminus A \in I$.*

The notion of I-convergent of double sequence A was introduced by Tripathy and Tripathy [24] . In this section, we shall denote the ideals of $2^N$ by I and that of $2^{N\times N}$ by $I_2$ .

A double sequence $(X_{k,l})$ of fuzzy numbers is said to be convergent in Pringsheim sence or P- convergent to a fuzzy real number $X_0$ if for each $\varepsilon>0$ there exist $k_0, l_0 \in N$ such that

$\bar{d}(X_{k,l}, X_0) > \varepsilon \quad for\ all\ \ k \geq k_0\,, l \geq l_0.$ We write $P - lim X_{k,l} = X_0.$

A double sequence $(X_{k,l})$ of fuzzy numbers is said to be null in Pringsheim sence or P- null if $P - limX_{k,l} = \bar{0}$ .

A double sequence $(X_{k,l})$ of fuzzy numbers is said to be bounded in Pringsheim sence or P- bounded if $sup_{k,l}\bar{d}(X_{k,l}, X_0) < \infty$ .

Let $I_2$ be an ideal of $2^{N\times N}$ .A double *sequence* $(X_{k,l})$ of fuzzy numbers is said to be I-convergent in Pringsheim sence if for each $\varepsilon$>0 such that

$$\{(k,l) \in N \times N: \bar{d}(X_{k,l}, X_0) \geq \varepsilon\} \in I_2$$

For $X_0 = \bar{0}$ , it is called I-null in Pringsheim sence.

Let $I_2$ be an ideal of $2^{N\times N}$ and $I$ be an ideal of $2^N$. A double sequence $(X_{k,l})$ of fuzzy numbers is said to be regularly $I$- convergent to a fuzzy number $X_0$ if it is $I$-convergent in Pringsheim sence and for each $\varepsilon$>0 the followings hold:

For each $l \in N$ there exists $L_l \in L(R)$ such that $\{k \in N : \bar{d}(X_{k,l}, L_l) \geq \varepsilon\} \in I$, and for each k$\in N$ there exists $M_k \in L(R)$ such that $\{l \in N : \bar{d}(X_{k,l}, M_k) \geq \varepsilon\} \in I$.

If $L_l = M_k = \bar{0}$ for all $, l \in N$ , the sequence $(X_{k,l})$ is said to be regularly I-null.

A double sequence $(X_{k,l})$ of fuzzy numbers is said to be I- Cauchy if for each $\varepsilon$>0 there exists $s = s(\varepsilon), \mathrm{t} = \mathrm{t}(\varepsilon) \in N$ such that $\{(k,l) \in N \times N: \bar{d}(X_{k,l}, X_{s,t}) \geq \varepsilon\} \in I_2$ .

A double sequence $(X_{k,l})$ of fuzzy numbers is said to be I- bounded if there exists a real number M>0 such that $\{(k,l) \in N \times N: \bar{d}(X_{k,l}, \bar{0}) \geq M\} \in I_2$.

Let $A$ denote a four dimensional summability method that maps the complex double sequences $x$ into the double sequence $Ax$ where the $mn$-th term of $Ax$ is as follows:

$$(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$$

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. Robison, in 1926 presented a four dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is $P$-convergent is not necessarily bounded. The definition of the regularity for four dimensional matrices will be stated next, followed by the Robison-Hamilton characterization of the regularity of four dimensional matrices.

The four dimensional matrix $A$ is said to be $RH$-regular if it maps every bounded $P$-convergent sequence into a $P$-convergent sequence with the same $P$-limit.

Theorem 2.1: The four dimensional matrix $A$ is $RH$-regular if and only if

RH$_1$: $P-\lim_{k,l} a_{m,n,k,l} = 0$ for each $m,n$.

RH$_2$: $P- \lim_{k,l} \sum_{m,n=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$

RH$_3$: $P- \lim_{k,l} \sum_{m=1}^{\infty} |a_{m,n,k,l}| = 0$ for each $n$

RH$_4$: $P- \lim_{k,l} \sum_{n=1}^{\infty} |a_{m,n,k,l}| = 0$ for each $m$

RH$_5$: $\sum_{m,n=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ *is* $P$-convergent

RH$_6$: there exist positive number X and Y such that $\sum_{k,l>Y} |a_{m,n,k,l}| < X$ .

Let $\boldsymbol{M} = (M_{k,l})$ be a double sequence of Orlicz functions and $A = (a_{m,n,k,l})$ be infinite matrix. We now present the following set of spaces:

$${}_2w^{I(F)}(A,\boldsymbol{M},p)$$

$$= \left\{ X = (X_{k,l}) \in {}_2w^F : \forall\varepsilon > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ M_{k,l}\left( \frac{\bar{d}(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2 \right\}$$

$${}_2w_0^{I(F)}(A,\boldsymbol{M},p)$$

$$= \left\{ X = (X_{k,l}) \in {}_2w^F : \forall\varepsilon > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ M_{k,l}\left( \frac{\bar{d}(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2 \right\}$$

$${}_2w_\infty^{I(F)}(A,\boldsymbol{M},p)$$

$$= \left\{ X = (X_{k,l}) \in {}_2w^F : \forall K > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ M_{k,l}\left( \frac{\bar{d}(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \geq K \right\} \in I_2 \right\}$$

**Lemma2.1**: If $\bar{d}$ is translation invariant then

(a) $\bar{d}(X_{k,l} + Y_{k,l}, 0) \leq \bar{d}(X_{k,l}. 0) + \bar{d}(Y_{k,l}, 0)$

(b) $\bar{d}(\alpha X_{k,l}, 0) \leq |\alpha| \bar{d}(X_{k,l}. 0)$ , $|\alpha| > 1$ .

**Lemma2.2**: Let $(\alpha_k)$ and $(\beta_k)$ be sequences of real or complex numbers and $(p_k)$ be a bounded sequence of positive real numbers , then

$$|\alpha_k + \beta_k|^{p_k} \leq C(|\alpha_k|^{p_k} + |\beta_k|^{p_k})$$

and $|\lambda|^{p_k} \leq max(1, |\lambda|^G)$

where C = $max(1, |\lambda|^{G-1}), G = supp_k$ , $\lambda$ is any real or complex number .

## Some special cases:

**a.** If we take $A = (C, 1,1)$ , the above spaces becomes ,

$_2w^{I(F)}(\boldsymbol{M}, p)$

$$= \left\{ X = (X_{k,l}) \in {}_2w^F : \forall \varepsilon > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} \left[ M_{k,l}\left( \frac{\bar{d}(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2 \right\}$$

$_2w_0^{I(F)}(\boldsymbol{M}, p)$

$$= \left\{ X = (X_{k,l}) \in {}_2w^F : \forall \varepsilon > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} \left[ M_{k,l}\left( \frac{\bar{d}(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2 \right\}$$

$_2w_\infty^{I(F)}(\boldsymbol{M}, p)$

$$= \left\{ X = (X_{k,l}) \in {}_2w^F : \forall K > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} \left[ M_{k,l}\left( \frac{\bar{d}(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \geq K \right\} \in I_2 \right\}$$

**b.** If $M_{k,l}(x) = x$ for $k,l \in N$ , then we can obtain,

$_2w^{I(F)}(A, p)$

$$= \left\{ X = (X_{k,l}) \in {}_2w^F : \forall \varepsilon > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ \bar{d}(X_{k,l}, X_0) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2 \right\}$$

$_2w_0^{I(F)}(A, p)$

$$= \left\{ X = (X_{k,l}) \in {}_2w^F : \forall \varepsilon > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ \bar{d}(X_{k,l}, \bar{0}) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2 \right\}$$

$_2w_\infty^{I(F)}(A, p)$

$$= \left\{ X = (X_{k,l}) \in {}_2w^F : \forall K > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ \bar{d}(X_{k,l}, \bar{0}) \right]^{p_{k,l}} \geq K \right\} \in I_2 \right\}$$

**c.** If $p_{k,l} = 1 \; for \; all \; k, l \in N \;, wehave,$

$_2w^{I(F)}(A, \boldsymbol{M})$

$$=\left\{X=\left(X_{k,l}\right)\in {}_2w^F:\forall\varepsilon>0,\left\{\left(m,n\right)\in N\times N:\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},X_0)}{\rho}\right)\right]\geq\varepsilon\right\}\in I_2\right\}$$

${}_2w_0^{I(F)}\left(A,\boldsymbol{M}\right)$

$$=\left\{X=\left(X_{k,l}\right)\in {}_2w^F:\forall\varepsilon>0,\left\{\left(m,n\right)\in N\times N:\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right]\geq\varepsilon\right\}\in I_2\right\}$$

${}_2w_\infty^{I(F)}\left(A,\boldsymbol{M}\right)$

$$=\left\{X=\left(X_{k,l}\right)\in {}_2w^F:\forall K>0,\left\{\left(m,n\right)\in N\times N:\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right]\geq K\right\}\in I_2\right\}$$

**d**. If we take

$a_{i,j,k,l}=\frac{1}{\lambda_{i,j}}$ if $k\in I_i=[i-\lambda_i+1,i\,]$ $and\ l\in I_j=\left[j-\lambda_j+1,j\,\right]$

= 0 otherwise.

where $\lambda_{i,j}\ by\ \lambda_i\mu_j$ . Let λ=($\lambda_i$) and μ = ($\mu_j$) be two non decreasing sequences of positive real numbers such that each tends to ∞ and $\lambda_{i+1}\leq\lambda_i+1\,,\lambda_1=1$

and $\mu_{j+1}\leq\mu_j+1\,,\mu_1=1$ .

Then our spaces become:

${}_2w^{I(F)}\left(\lambda,\boldsymbol{M},p\right)$

$$=\left\{X=\left(X_{k,l}\right)\in {}_2w^F:\forall\varepsilon>0,\left\{\left(i,j\right)\in N\times N:\frac{1}{\lambda_{i,j}}\sum_{(k,l)\in I_{i,j}}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},X_0)}{\rho}\right)\right]^{p_{k,l}}\geq\varepsilon\right\}\in I_2\right\}$$

${}_2w_0^{I(F)}\left(\lambda,\boldsymbol{M},p\right)$

$$=\left\{X=\left(X_{k,l}\right)\in {}_2w^F:\forall\varepsilon>0,\left\{\left(i,j\right)\in N\times N:\frac{1}{\lambda_{i,j}}\sum_{(k,l)\in I_{i,j}}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right]^{p_{k,l}}\geq\varepsilon\right\}\in I_2\right\}$$

${}_2w_\infty^{I(F)}\left(\lambda,\boldsymbol{M},p\right)$

$$= \left\{ X = \left( X_{k,l} \right) \in {}_2w^F : \forall K > 0, \left\{ (i,j) \in N \times N : \frac{1}{\lambda_{i,j}} \sum_{(k,l) \in I_{i,j}} \left[ M_{k,l} \left( \frac{\bar{d}(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \geq K \right\} \in I_2 \right\}$$

$${}_2w^F(\lambda, \boldsymbol{M}, p) = \left\{ X = \left( X_{k,l} \right) \in {}_2w^F : P - \lim_{i,j \to \infty,\infty} \frac{1}{\lambda_{i,j}} \sum_{(k,l) \in I_{i,j}} \left[ M_{k,l} \left( \frac{\bar{d}(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} = 0 \right\}$$

$${}_2w_0^F(\lambda, \boldsymbol{M}, p) = \left\{ X = \left( X_{k,l} \right) \in {}_2w^F : P - \lim_{i,j \to \infty,\infty} \frac{1}{\lambda_{i,j}} \sum_{(k,l) \in I_{i,j}} \left[ M_{k,l} \left( \frac{\bar{d}(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} = 0 \right\}$$

$${}_2w_\infty^F(\lambda, \boldsymbol{M}, p) = \left\{ X = \left( X_{k,l} \right) \in {}_2w^F : \sup_{i,j,k,l} \frac{1}{\lambda_{i,j}} \sum_{(k,l) \in I_{i,j}} \left[ M_{k,l} \left( \frac{\bar{d}(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} < \infty \right\}$$

**e**. A double sequence $\theta_{r,s} = (\alpha_r, \beta_s)$ is said to be double lacunary if there exists sequences $(\alpha_r)\ and\ (\beta_s)$ of integers such that

$$v_r = \alpha_r - \alpha_{r-1} \to \infty \ \ as \ \ r \to \infty, \ \ \alpha_0 = 0$$
$$v_s = \beta_s - \beta_{s-1} \to \infty \ \ as \ \ s \to \infty, \ \ \beta_0 = 0$$

Let $v_{r,s} = v_r v_s$ , $\theta_{r,s}$ is obtain by $I_{r,s} = \{ (x,y) : \alpha_{r-1} < x \leq \alpha_r \ and\ \beta_{s-1} < y \leq \beta_s \}$

If we take,

$$a_{r,s,k,l} = \frac{1}{\bar{v}_{r,s}}, if\ (k,l) \in I_{r,s}$$
$$= 0, \text{ otherwise}$$

Then we have,

${}_2w^{I(F)}(\theta, \boldsymbol{M}, p)$

$$= \left\{ X = \left( X_{k,l} \right) \in {}_2w^F : \forall \varepsilon > 0, \left\{ (r,s) \in N \times N : \frac{1}{\bar{v}_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M_{k,l} \left( \frac{\bar{d}(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2 \right\}$$

${}_2w_0^{I(F)}(\theta, \boldsymbol{M}, p)$

$$= \left\{ X = \left( X_{k,l} \right) \in {}_2w^F : \forall \varepsilon > 0, \left\{ (r,s) \in N \times N : \frac{1}{\bar{v}_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M_{k,l} \left( \frac{\bar{d}(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2 \right\}$$

${}_2w_\infty^{I(F)}(\theta, \boldsymbol{M}, p)$

$$=\left\{X=\left(X_{k,l}\right)\in {}_2w^F : \forall K>0, \left\{\left(r,s\right)\in N\times N : \frac{1}{\bar{v}_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right]^{p_{k,l}}\geq K\right\}\in I_2\right\}$$

$${}_2w^F\left(\theta,\boldsymbol{M},p\right)=\left\{X=\left(X_{k,l}\right)\in {}_2w^F : P-\lim_{i,j\to\infty,\infty}\frac{1}{\bar{v}_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},X_0)}{\rho}\right)\right]^{p_{k,l}}=0\right\}$$

$${}_2w_0^F\left(\theta,\boldsymbol{M},p\right)=\left\{X=\left(X_{k,l}\right)\in {}_2w^F : P-\lim_{i,j\to\infty,\infty}\frac{1}{\bar{v}_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right]^{p_{k,l}}=0\right\}$$

$${}_2w_\infty^F\left(\theta,\boldsymbol{M},p\right)=\left\{X=\left(X_{k,l}\right)\in {}_2w^F : \sup_{r,s,k,l}\frac{1}{\bar{v}_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right]^{p_{k,l}}<\infty\right\}$$

**f**. If $I=I_f$, then we can obtain,

$${}_2w^F\left(A,\boldsymbol{M},p\right)=\left\{X=\left(X_{k,l}\right)\in {}_2w^F : P-\lim_{m,n\to\infty,\infty}\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},X_0)}{\rho}\right)\right]^{p_{k,l}}=0\right\}$$

$${}_2w_0^F\left(A,\boldsymbol{M},p\right)=\left\{X=\left(X_{k,l}\right)\in {}_2w^F : P-\lim_{m,n\to\infty,\infty}\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right]^{p_{k,l}}=0\right\}$$

$${}_2w_\infty^{I(F)}\left(A,\boldsymbol{M},p\right)=\left\{X=\left(X_{k,l}\right)\in {}_2w^F : \sup_{m,n}\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right]^{p_{k,l}}<\infty\right\}$$

**g**. If $I=I_\delta$ an admissible ideal, then we can obtain

$${}_2w^{I(F)}\left(A,\boldsymbol{M},p\right)$$

$$=\left\{X=\left(X_{k,l}\right)\in {}_2w^F : \forall\varepsilon>0, \left\{\left(m,n\right)\in N\times N : \sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},X_0)}{\rho}\right)\right]^{p_{k,l}}\geq\varepsilon\right\}\in I_\delta\right\}$$

$${}_2w_0^{I(F)}\left(A,\boldsymbol{M},p\right)$$

$$=\left\{X=\left(X_{k,l}\right)\in {}_2w^F : \forall\varepsilon>0, \left\{\left(m,n\right)\in N\times N : \sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right]^{p_{k,l}}\geq\varepsilon\right\}\in I_\delta\right\}$$

$${}_2w_\infty^{I(F)}\left(A,\boldsymbol{M},p\right)$$

$$=\left\{X=\left(X_{k,l}\right)\in {}_2w^F:\forall K>0,\left\{(m,n)\in N\times N:\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right]^{p_{k,l}}\geq K\right\}\in I_\delta\right\}$$

## 3. Main results

**Theorem 3.1 :** Let $(p_{k,l})$ be a bounded sequence . Then the classes of sequence spaces

$2^{W_0{}^{I(F)}}(A,\boldsymbol{M},p)$ , $2^{W^{I(F)}}(A,\boldsymbol{M},p)$ and $2^{W_\infty{}^{I(F)}}(A,\boldsymbol{M},p)$ are linear spaces.

**Proof:** We shall give the prove for the space $2^{W_0{}^{I(F)}}(A,\boldsymbol{M},p)$ . The others are similar.

Let $X=(X_{k,l})$ and $Y=(Y_{k,l})$ be two elements in $2^{W_0{}^{I(F)}}(A,\boldsymbol{M},p)$ . Then there exists $\rho_1>0$ and $\rho_2>0$ such that –

$$A_1=\left\{(m,n)\in N\times N:\ \sum_{k,l=1,1}^{\infty,\infty}a_{m,n,l,k}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l}\,,\bar{0})}{\rho_1}\right)\right]^{p_{k,l}}\geq\ {}^{\varepsilon}/_{2}\right\}\in I_1$$

and

$$B_1=\left\{(m,n)\in N\times N:\ \sum_{k,l=1,1}^{\infty,\infty}a_{m,n,l,k}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l}\,,\bar{0})}{\rho_2}\right)\right]^{p_{k,l}}\geq\ {}^{\varepsilon}/_{2}\right\}\in I_2$$

By continuity of $\boldsymbol{M}=(M_{k,l})$ and for scalars $a$ and b, we have ,

$$\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}\left(aX_{k,l}+bY_{k,l},\bar{0}\right)}{\rho_1|a|+\rho_2|b|}\right)\right]^{p_{k,l}}$$

$$\leq C\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[\frac{|a|}{\rho_1|a|+\rho_2|b|}M_{k,l}\left(\frac{\bar{d}\left(X_{k,l},\bar{0}\right)}{\rho_1}\right)\right]^{p_{k,l}}$$

$$+C\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[\frac{|b|}{\rho_1|a|+\rho_2|b|}M_{k,l}\left(\frac{\bar{d}\left(Y_{k,l},\bar{0}\right)}{\rho_2}\right)\right]^{p_{k,l}}$$

$$\leq CD\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}\left(X_{k,l},\bar{0}\right)}{\rho_1}\right)\right]^{p_{k,l}}$$

$$+CD\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}\left(Y_{k,l},\bar{0}\right)}{\rho_2}\right)\right]^{p_{k,l}}$$

Where $D = \max\left\{1, \dfrac{|a|}{\rho_1|a| + \rho_2|b|}, \dfrac{|b|}{\rho_1|a| + \rho_2|b|}\right\}$

This relation implies that:

$$\left\{(m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}\left(aX_{k,l} + bY_{k,l}, \bar{0}\right)}{\rho_1|a| + \rho_2|b|}\right)\right]^{p_{k,l}} \geq \varepsilon\right\}$$

$$\subseteq \left\{(m,n) \in N \times N : CD\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}\left(X_{k,l}, \bar{0}\right)}{\rho_1}\right)\right]^{p_{k,l}} \geq \frac{\varepsilon}{2}\right\}$$

$$\cup \left\{(m,n) \in N \times N : CD\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}\left(Y_{k,l}, \bar{0}\right)}{\rho_2}\right)\right]^{p_{k,l}} \geq \frac{\varepsilon}{2}\right\}$$

This completes the proof.

**Theorem 3.2 :** If $0 < inf p_{k,l} \leq p_{k,l} \leq 1$ then ,

(a) $2^{W^{I(F)}}(A, \boldsymbol{M}, p) \subset 2^{W^{I(F)}}(A, \boldsymbol{M})$

(b) $2^{W_0{}^{I(F)}}(A, \boldsymbol{M}, p) \subset 2^{W_0{}^{I(F)}}(A, \boldsymbol{M})$

**Proof :** (a) Let $X = (X_{k,l}) \in 2^{W^{I(F)}}(A, \boldsymbol{M}, p)$ . Since $0 < inf p_{k,l} \leq p_{k,l} \leq 1$, we have,

$$\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l}, X_0)}{\rho}\right)\right] \leq \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}}$$

So that

$$\left\{(m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l}, X_0)}{\rho}\right)\right] \geq \varepsilon\right\}$$

$$\subset \left\{(m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}} \geq \varepsilon\right\} \in I_2$$

This follows that,

$$2^{W^{I(F)}}(A, \boldsymbol{M}, p) \subset 2^{W^{I(F)}}(A, \boldsymbol{M})$$

(b) Similar proof as part (a)

**Theorem 3.3 :** If $1 \leq p_{k,l} \leq supp_{k,l} < \infty$ , then

(a) $2^{W^{I(F)}}(A, \boldsymbol{M}) \subset 2^{W^{I(F)}}(A, \boldsymbol{M}, p)$

(b) $2^{{W_0}^{I(F)}}(A,\boldsymbol{M}) \subset 2^{{W_0}^{I(F)}}(A,\boldsymbol{M},p)$

**Proof : (a)** Let $X = (X_{k,l}) \in 2^{W^{I(F)}}(A,\boldsymbol{M})$ . Since $1 \le p_{k,l} \le supp_{k,l} < \infty$, then for each $0 < \varepsilon < 1$ there exist a positive integer $n_0$ such that,

$$\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},X_0)}{\rho}\right)\right] \le \varepsilon < 1 \quad for\ all\ \ m,n \ge n_0.$$

This implies,

$$\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},X_0)}{\rho}\right)\right]^{p_{k,l}} \le \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},X_0)}{\rho}\right)\right]$$

Therefore,

$$\left\{(m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},X_0)}{\rho}\right)\right]^{p_{k,l}} \ge \varepsilon\right\}$$

$$\subset \left\{(m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},X_0)}{\rho}\right)\right] \ge \varepsilon\right\} \in I_2$$

Hence,

$$2^{W^{I(F)}}(A,\boldsymbol{M}) \subset 2^{W^{I(F)}}(A,\boldsymbol{M},p) .$$

(b) Similar proof as part (a)

**Corollary 3.1 :** Let A= (C ,1 ,1) i.e the Cesaro matrix $\boldsymbol{M} = (M_{k,l})$ be sequence of Orlicz fuctions,

(a) If $0 < inf p_{k,l} \le p_{k,l} \le 1$ then,

1. $2^{W^{I(F)}}(\boldsymbol{M},p) \subset 2^{W^{I(F)}}(\boldsymbol{M})$
2. $2^{{W_0}^{I(F)}}(M,p) \subset 2^{{W_0}^{I(F)}}(M)$

(b) If $1 \le p_{k,l} \le supp_{k,l} < \infty$ , then

1. $2^{W^{I(F)}}(\boldsymbol{M}) \subset 2^{W^{I(F)}}(\boldsymbol{M},p)$
2. $2^{{W_0}^{I(F)}}(\boldsymbol{M}) \subset 2^{{W_0}^{I(F)}}(\boldsymbol{M},p)$

**Theorem 3.4 :** The spaces $2^{{W_0}^{I(F)}}(A,\boldsymbol{M},p)$ and $2^{W^{I(F)}}(A,\boldsymbol{M},p)$ are solid .

**Proof :** We shall give the prove for the space $2^{{W_0}^{I(F)}}(A,\boldsymbol{M},p)$.

Let $X = (X_{k,l}) \in 2^{{W_0}^{I(F)}}(A, \boldsymbol{M}, p)$ and $Y = (Y_{k,l})$ such that $|Y_{k,l}| \leq |X_{k,l}|$ $for\ all\ k, l \in N$.

Then for each $\varepsilon > 0$,

$$A = \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \frac{\bar{d}(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2; \text{ for some } \rho > 0.$$

Since M is non decreasing, we have:

$$B = \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \frac{\bar{d}(Y_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \subset A; \text{ for some } \rho > 0.$$

Thus B $\subset$ A and so Y $\in 2^{{W_0}^{I(F)}}(A, \boldsymbol{M}, p)$ . This completes the proof.

**Theorem 3.5 :** The spaces $2^{W^{I(F)}}(A, \boldsymbol{M}, p)$ , $2^{{W_0}^{I(F)}}(A, \boldsymbol{M}, p)$ , and $2^{{W_\infty}^{I(F)}}(A, \boldsymbol{M}, p)$ are linear topological spaces under the paranorm ‘ $h$’ defined by

$$h(X) = \inf \left\{ \rho^{\frac{p_{m,n}}{P}} : \left( \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \frac{\bar{d}(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \right)^{1/P} \leq 1 \right\}$$

where $\rho > 0$ ; $m,n \in N$ and $P = \max\left\{1, \sup_{k,l} p_{k,l}\right\}$.

**Proof :** Obviously $h(X) = h(-X)$ $and$ $h(\theta) = 0$ $where$ $\theta = (\theta_{r,s})$ a double lacunary sequence.

Let $X = (X_{k,l}) \in 2^{{W_\infty}^{I(F)}}(A, \boldsymbol{M}, p)$ and Y = $(Y_{k,l}) \in 2^{{W_\infty}^{I(F)}}(A, \boldsymbol{M}, p)$.

Then,

$$A_1 = \left\{ \rho > 0 : \left( \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \frac{\bar{d}(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \right)^{1/P} \leq 1 \right\}$$

$$A_2 = \left\{ \rho > 0 : \left( \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \frac{\bar{d}(Y_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \right)^{1/P} \leq 1 \right\}$$

Take, $\rho_1 \in A_1$ , $\rho_2 \in A_2$ $and$ $\rho = \rho_1 + \rho_2$, we can obtain:

$$\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right] \le \frac{\rho_1}{\rho_1+\rho_2}\left(\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right]\right)$$

$$+\frac{\rho_2}{\rho_1+\rho_2}\left(\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(Y_{k,l},\bar{0})}{\rho}\right)\right]\right)$$

Hence,

$$\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}(X_{k,l},\bar{0})}{\rho}\right)\right] \le 1$$

Also,

$$h(X+Y) = inf\left\{(\rho_1+\rho_2)^{\frac{p_{m,n}}{P}} \;:\; \rho_1 \in A_1\;, \rho_2 \in A_2\right\}$$

$$\le inf\left\{(\rho_1)^{\frac{p_{m,n}}{P}} \;:\; \rho_1 \in A_1\right\} + \; inf\left\{(\rho_2)^{\frac{p_{m,n}}{P}} \;:\; \rho_2 \in A_2\right\}$$

$$= h(X) + h(Y)$$

Let $\gamma_{k,l}^i \to \gamma \quad and \;\; h\left(X_{k,l}^i - X_{k,l}\right) \to 0 \;\; as \;\; i \to \infty$ . To prove $h\left(\gamma_{k,l}^i X_{k,l}^i - \gamma X_{k,l}\right) \to 0 \; as \;\; i \to \infty$.

We have,

$$A_3 = \left\{\rho_{k,l} > 0 : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}\left(X_{k,l},\bar{0}\right)}{\rho_{k,l}}\right)\right]^{p_{k,l}} \le 1\right\}$$

and

$$A_4 = \left\{\rho'_{k,l} > 0 : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}\left(X_{k,l},\bar{0}\right)}{\rho'_{k,l}}\right)\right]^{p_{k,l}} \le 1\right\}$$

By continuity of $\boldsymbol{M} = (M_{k,l})$ , we have

$$M_{k,l}\left(\frac{\bar{d}\left(\gamma^i X_{k,l}^i - \gamma X,\bar{0}\right)}{|\gamma^i-\gamma|\rho_{k,l}+|\gamma|\rho'_{k,l}}\right) \le M_{k,l}\left(\frac{\bar{d}\left(\gamma^i X_{k,l}^i - \gamma X_{k,l},\bar{0}\right)}{|\gamma^i-\gamma|\rho_{k,l}+|\gamma|\rho'_{k,l}}\right) + M_{k,l}\left(\frac{\bar{d}\left(\gamma^i X_{k,l} - \gamma X,\bar{0}\right)}{|\gamma^i-\gamma|\rho_{k,l}+|\gamma|\rho'_{k,l}}\right)$$

$$\le \frac{|\gamma^i-\gamma|\rho_{k,l}}{|\gamma^i-\gamma|\rho_{k,l}+|\gamma|\rho'_{k,l}} M_{k,l}\left(\frac{\bar{d}\left(X_{k,l}^i,\bar{0}\right)}{\rho_{k,l}}\right)$$

$$+\frac{|\gamma|}{|\gamma^i-\gamma|\rho_{k,l}+|\gamma|\rho'_{k,l}} M_{k,l}\left(\frac{\bar{d}\left(X_{k,l}^i - X_{k,l},\bar{0}\right)}{\rho'_{k,l}}\right)$$

It follows that,

$$\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}\left[M_{k,l}\left(\frac{\bar{d}\left(\gamma^i X_{k,l}^i - \gamma X, \bar{0}\right)}{\left|\gamma^i - \gamma\right|\rho_{k,l} + \left|\gamma\right|\rho'_{k,l}}\right)\right]^{p_{k,l}} \leq 1$$

and consequently,

$$h(\gamma_{k,l}^i X_{k,l} - \gamma X_{k,l}) = inf\left\{(|\gamma_{k,l}^i - \gamma|\rho_{k,l} + |\gamma|\rho_{k,l}{}^{/})^{\frac{p_{m,n}}{P}} : \ \rho_{k,l} \in A_3 \ ; \ \rho_{k,l}{}^{/} \in A_4\right\}$$

$$\leq max\left\{|\gamma|, |\gamma|^{\frac{p_{m,n}}{P}}\right\} h(X_{k,l}^i - X_{k,l})$$

It is noted that $h(X^i) \leq h(X) + h(X^i - X) \ for \ all \ \ i \in N$

Therefore by our assumption, $h(\gamma_{k,l}^i X_{k,l} - \gamma X_{k,l}) \to 0 \ \ as \ \ i \to \infty$ . This completes the proof.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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