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GENERALIZED HYERS-ULAM STABILITY OF SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATION WITH INITIAL CONDITION

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Abstract. The stability of ordinary differential equation has been investigated and this investigation is ongoing. In this work, we investigate the stability of second-order linear ordinary differential nonhomogeneous equation with initial condition in the Hyers-Ulam sense.

Keywords: Hyers-Ulam stability; differential equation, initial value problem; integral equation; differential inequality.

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1. Introduction

The stability problem of functional equation started with the question concerning stability of group homomorphisms proposed by Ulam [18] during a talk before a mathematical colloquium of the University of Wisconsin as follows:

> "Give condition in order for a linear mapping near an approximately linear mapping to exist."

This problem was also put in this sense:

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> "For what metric group G is it true that an ε automorphism of G is necessarily near to a strict automorphism?"

In 1941, Hyers [3] gave an answer to the problem as follows:

> "Let E_1 and E_2 be two real Banach spaces and $f : E_1 \to E_2$ be a mapping. If there exist an $\varepsilon \geq 0$ such that
> $$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$
> for all $x, y \in E_1$, then there exist a unique additive mapping $T : E_1 \to E_2$ such that
> $$\|f(x) - T(x)\| \leq \varepsilon$$
> for every $x \in E_1$"

This result is called the Hyers-Ulam Stability of the additive Cauchy equation $g(x+y) = g(x) + g(y)$.

In 1978, Rassias [14, 15] introduced a new functional inequality that we call Cauchy-Rassias inequality and succeeded in extending the result of Hyers, by weakening the condition for the Cauchy differences to unbounded map as follows:

> "If there exist $\varepsilon \geq 0$ and $0 \leq p < 1$ such that
> $$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$
> for all $x, y \in E_1$, then there exist a unique additive mapping $T : E_1 \to E_2$ such that
> $$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|x\|^p$$
> for every $x \in E_1$."

The stability phenomenon of this kind is called the Generalized Hyers-Ulam Stability.

To this end, several works has been done in the direction of differential equations as first credited to Alisina and Gar [1], who were the first to considered the Hyers-Ulam stability of differential equation. They proved the Hyers-Ulam stability of the differential equation $y' = y$: That if given $\varepsilon > 0$, f be a differentiable function on an open interval I into $\mathbb{R}$, where $\mathbb{R}$ is the real number field, with $|f'(t) - f(t)| \leq \varepsilon$ for all $t \in I$, then there exist a differentiable function $g : I \to \mathbb{R}$ such that $g'(t) = g(t)$ and $|f(t) - g(t)| \leq 3\varepsilon$ for all $t \in I$.
The above result by Alsina and Gar was generalized by Miuri, Takahasi and Choda [12], by Miura [13], also by Takahasi, Miura and Miyajima [16, 17]. Indeed they dealt with the Hyers-Ulam stability of the differential equation $y'(t) = \lambda y(y)$.

Miura *et al.* [13] proved the Hyers-Ulam stability of the first-order linear differential equation $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function, while Jung [4] proved the Hyers-Ulam stability of the differential equation of the form $\varphi(t)y'(t) = y(t)$. Furthermore, results of Hyers-Ulam stability of first-order linear differential equation has been generalized by Miura et al [11], by Takahasi et al [17] and also by Jung [5], they dealt with the nonhomogeneous linear differential equation of the form $y' + p(t)y + q(t) = 0$. Jung [11] proved the generalized Hyers-Ulam stability of the differential equation of the form $ty'(t) + \alpha y(t) + \beta t^r x_0 = 0$ and also applied the result to the investigation of the Hyers-Ulam stability of the differential equation of the form $ty''(t) + \alpha ty'(t) + \beta y(t) = 0$.

In further works, Jung and Lee [6] worked on the; Hyers-Ulam-Rassias stability of linear differential equations of Second Order, Li and Yan [8, 9] also investigated the Hyers-Ulam Stability of nonhomogeneous Linear Differential Equations of Second Order. Mohammed, Madjid, Badrkhan, Choonkil [10] worked on the Hyers-Ulam Stability of Exact Second-Order Linear Differential Equations: Pasc, Jung, Li [2] also proved the Hyers-Ulam stability for second order linear differential equation with boundary conditions. Rus [19] investigated the Ulam stability of Differential equations $x''(t) + f(t, x(t))$.

Here in this work, we prove the Hyers-Ulam-Rassias stability of a second order linear differential equation of the form $y''(t) - \beta(t)y(t) = f(t)$ similar to the equation in [2, 19], but here with initial conditions.

2. Preliminaries

Let $(X, \|.\|)$ be a real (or complex) Banach space with $a, b \in \mathbb{R}$ where $-\infty < a < b < \infty$, ε a positive real number. Let $y : (a,b) \to X$ be a continuous operator and $\varphi : I \to [0,\infty)$ be a continuous function. We consider the following differential equation:

$$y^{(n)}(t) = \sum_{k=0}^{n-1} P_k y^{(k)}(t), \quad t \in I \tag{2.1}$$

and the following differential inequality

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} P_k y^{(k)}(t) \right| \leq \varepsilon, \quad t \in I \tag{2.2}$$

and

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} P_k y^{(k)}(t) \right| \leq \varphi(t), \quad t \in I. \tag{2.3}$$

Definition 2.1. *The equation 2.1 has the Hyers-Ulam stability for any $\varepsilon > 0$, there exist a real number $K > 0$ such that for each approximate solution $y \in C^n(I, X)$ of (2.2) there exist a solution $y_0 \in C^n(I, X)$ of (2.1) with*

$$|y - y_0| \leq K\varepsilon \quad \forall\, t \in I. \tag{2.4}$$

Definition 2.2. *The equation 2.1 has the Generalized Hyers-Ulam stability (or Hyers-Ulam-Rassias stability) if there exist $\theta_\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$, such that for each approximate solution $y \in C^n(I, X)$ of (2.3) there exist a solution $y_0 \in C^n(I, X)$ of (2.1) with*

$$|y - y_0| \leq \theta_\varphi(t) \quad \forall\, t \in I. \tag{2.5}$$

In the course of this work we shall need the following lemma.

Lemma 2.1. *(Replacement Lemma) Suppose that $f:[a,b]\to\mathbb{R}$ is a continuous. Then*

$$\int_a^x\int_a^y f(t)dtdy = \int_a^x (x-t)f(t)dt, \quad x\in[a,b]$$

For details of the proof see [20]

Several linear ordinary differential equation have been investigated to be Hyers-Ulam stable. For Hyers-Ulam stability of the differential equation $y''(x)-\lambda^2 y=0$, with $\lambda\in\mathbb{R}$, see [7]. In previous work [2], the IVP

$$y''+\beta(x)y=0 \tag{2.6}$$

with initial conditions

$$y(a)=y'(a)=0, \tag{2.7}$$

where $y\in C^2[a,b]$, $\beta\in C[a,b]$. has been investigated, proven to be Hyers-Ulam stable hence the Theorem.

Theorem 2.1. *If* $\max|\beta(x)|<\dfrac{2}{(b-a)^2}$, *then (2.6) has the Hyers-Ulam stability with initial conditions (2.7)*

Proof. see [2] □

In further works, the stability of the differential equation $y'(x)+p(x)y+q(x)=r(x)$ in Hyers-Ulam sense has been established, this leads to the Hyers-Ulam stability of the equation $y''(x)+p(x)y'+q(x)y=r(x)$. For details, see [8].

3. Main Result

In this section we shall prove the Generalized Hyers-Ulam Stability of the IVP

$$y''+\beta(x)y=f(x) \tag{3.8}$$

with initial conditions

$$y(a)=y'(a)=0, \tag{3.9}$$

where $y\in C^2[a,b]$, $\beta\in C[a,b]$ and $f:[a,b]\to\mathbb{R}$ continuous.

Theorem 3.2. *Suppose $|\beta(x)|<M$ where $M=\frac{2}{(b-a)^2}$, $\varphi:[a,b]\to[0,\infty)$ in an increasing function. The equation (3.8) has the Generalized Hyers-Ulam stability if for $\theta_\varphi\in C(\mathbb{R}_+,\mathbb{R}_+)$ and for each approximate solution $y\in C^2[a,b]$ of (3.8) satisfying*

$$|y''-\beta(x)y-f(x)|\le\varphi(x) \tag{3.10}$$

there exist a solution $z_0 \in C^2[a,b]$ of (3.8) with condition (3.9) such that

$$|y(x) - z_0(x)| \leq \theta_\varphi(x). \tag{3.11}$$

Proof. From (3.10) we have that

$$\varphi(x) \leq y'' - \beta(x)y - f(x) \leq \varphi(x).$$

Integrating from a to x, and applying condition (3.9) we have

$$\int_a^x \varphi(t)dt \leq y'(x) - \int_a^x \beta(t)y(t)dt - \int_a^x f(t)dt \leq \int_a^x \varphi(t)dt.$$

On further integration and also applying condition (3.9) we have

$$\int_a^r \int_a^x \varphi(t)dtdr \leq y(x) - \int_a^r \int_a^x \beta(t)y(t)dtdr - \int_a^r \int_a^x f(t)dtdr \leq \int_a^r \int_a^x \varphi(t)dtdr.$$

Now applying Lemma 2.1, we obtain

$$\int_a^x (x-t)\varphi(t)dt \leq y(x) - \int_a^x (x-t)\beta(t)y(t)dt - \int_a^x (x-t)f(t)dt \leq \int_a^x (x-t)\varphi(t)dt.$$

Hence we have

$$\left| y(x) - \int_a^x (x-t)\beta(t)y(t)dt - \int_a^x (x-t)f(t)dt \right| \leq \int_a^x (x-t)\varphi(t)dt$$

and

$$\left| y(x) - \int_a^x (x-t)\left(\beta(t)y(t)dt + f(t)dt\right) \right| \leq \int_a^x (x-t)\varphi(t)dt.$$

If we choose $z_0(x)$ such that it solves equation (3.8) with condition (3.9) such that

$$z_0(x) = \int_a^x (x-t)\left(\beta(t)z_0(t)dt + f(t)dt\right),$$

thus we estimate

$$|y(x) - z_0(x)| \leq \left| y(x) - \int_a^x (x-t)(\beta(t)y(t)dt + f(t))dt \right|$$

$$+ \int_a^x \left| (x-t)(\beta(t)y(t)dt + f(t)dt) - \int_a^x (x-t)(\beta(t)z_0(t)dt + f(t)) \right| dt$$

$$|y(x) - z_0(x)| \leq \int_a^x (x-t)\varphi(t)dt + \int_a^x |(x-t)\beta(t)[y(t) - z_0(t)]|\, dt$$

$$|y(x) - z_0(x)| \leq \int_a^x (x-t)\varphi(t)dt + |\beta(t)| \int_a^x (x-t)|y(t) - z_0(t)|dt$$

$$|y(x) - z_0(x)| \leq \int_a^x (x-t)\varphi(t)dt + M\int_a^x (x-t)|y(t) - z_0(t)|dt.$$

Applying Gronwall inequality, we have

$$|y(x) - z_0(x)| \leq \int_a^x (x-t)\varphi(t)dt \exp\{M\int_a^x (x-t)dt\}$$

$$|y(x) - z_0(x)| \leq \int_a^x (x-t)\varphi(t)dt \exp\left\{\frac{M(x-a)^2}{2}\right\}$$

$$|y(x) - z_0(x)| \leq c\int_a^x (x-t)\varphi(t)dt$$

with

$$c = \exp\left\{\left[\frac{x-a}{b-a}\right]^2\right\}$$

and the proof is completed. □

Remark 3.1. *It is very important to note that as $x \to b$, then the above system considered is Hyers-Ulam stable.*

4. Conclusion

The definition as studied in this work has applicable significance since it means that if one is studying the Hyers-Ulam stability of a system, then one does not have to reach the exact solution (which usually is quite difficult or time consuming), all what is required is to get a function, that is, a close exact solution. Therefore Hyers-Ulam stability guarantees that there is a closed exact solution of the system under study. This is quite useful in many applications, e.g. numerical analysis, optimization, biology and economics, e.t.c. where finding the exact solution is quite difficult. It also helps if the stochastic effects are small, to use deterministic model to approximate a stochastic one.

It is very important to note that there are many other applications for Hyers-Ulam stability in other areas like, nonlinear analysis problems including partial differential equation and integral equations. Researches are still on going on the Hyers-Ulam stability of more of first, second-order and higher-order homogeneous and nonhomogeneous differential equations, in partial differential equation as well as integral equation.

Conflict of Interests

The authors declare that there are no conflict of interests.

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