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UNIQUE FIXED POINT THEOREMS ON GENERALIZED CONE METRIC SPACES

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Abstract: In this paper we proved some fixed point theorems in G- Cone metric spaces by inspiring from the theorems proved by Mustafa-Sim and Beg-Abbas.

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1. Introduction

Improving the Dhage'(3,4,5) theory of generalized metric spaces, Mustafa and Si**m** (6,7) introduced the concept of G- metric spaces. Afterward Mustafa proved several fixed point theorems satisfying different contractive conditions. Recently Guang and Xian (8) define the concept of cone metric spaces replacing the set of real numbers by an ordered Banach spaces and proved several fixed point theorems satisfying different contractive conditions. Then Rezapour and Hamlbarani (10) generalized some results in cone metric spaces omitting the normality of cone. Ismat Beg and Mujahid Abbas (9) introduced the concept of G- cone

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metric spaces replacing real numbers by an ordered Banach spaces and used the convergence of sequence. Then proved some fixed point theorems.

2. Preliminaries

Before proving our results, we collect the relevant definitions, results, propositions and Theorems

1. **Cone:** Let E be a real Banach spaces and P $\subset E$. P is called a cone iff

(i) P is closed and non empty and P$\neq \{0\}$

(ii) $ax + by \in P$ $for\ every\ x, y \in P$ and a, b are non negative real ; more generally if a, b, c are non negative real $x, y, z \in P$ and implies $ax + by + cz \in P$

(iii) P$\cap$ -P = $\{0\}$ i.e. x$\in P$ and -x$\in P \Longrightarrow$ x= 0

2. **Cone Metric Space:** Let X be a nonempty set. Suppose the mapping d: X x X → E satisfies:

(d_1) 0 < d (x, y) for all x, y $\in X$ $and\ d(x, y) = 0\ if\ and\ only\ if\ x = y$;

(d_2) d (x, y) = d (y, x) for all x, y$\in X$;

(d_3) d(x, y)$\leq d(x, z) + d(y, z) for\ all\ x, y, z \in X$.

3. **Partial ordering:** Let cone $P \subset E$, we define partial ordering ≤ with respect to P as

(i) $x \leq y\ iff\ y - x \in P$.

(ii) $x < y \Longrightarrow x \leq y\ but\ x \neq y$

(iii) $x \ll y \Longrightarrow y - x \in intP$

4. **Normal cone:** A cone P is called normal if ∃ k > 0 for all x, y ∈ E such that

$0 \leq x \leq y \Longrightarrow \|x\| \leq K\ \|y\|$

The least +ve number satisfying the inequality is known as normal constant of P and x << y means y$-x \in int\ P$.

5. **G- Cone Metric Space:** Let X be a nonempty set. Suppose G: X^3 → E satisfies the following axiom

[G1] $G(x, y, z) = 0\ if\ x = y = z\ and\ x, y, z \in X$

[G2] $G(x, x, y) > 0\ if\ x \neq y\ and\ x, y, \in X$

[G3] $G(x, x, y) \leq G(x, y, z) if\ y \neq z\ and\ x, y, z \in X$

[G4] $G(x,y,z) = G(x,z,y) = G(y,x,z) = \cdots$ *symmetry in all*, $x,y,z \in X$

[G1] $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ *for all* $x,y,z,a \in X$

Then G is called G- cone metric on X and X is called G- cone metric space.

6. **Cauchy Sequence, Convergent Sequence and complete G- cone metric space:** Let X be a G- cone metric space and $\{x_n\}$ be a sequence in X. Then we say,

(a) If for every c ∊ E with c>> 0, there is N such that ∃ n, m, l ∊ N we have c >> G (x_n, x_m, x_l) then the sequence $\{x_n\}$ is called Cauchy sequence.

(b) If for every c ∊ E with c>> 0, there is N such that ∃ n, m ∊ N, we have c >> G (x_m, x_m, x) for some x in X then the sequence $\{x_n\}$ is called convergent sequence. x is called limit of the sequence and is denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ when $n \to \infty$.

(c) A G- cone metric space is said to be complete if every Cauchy sequence in X is convergent in X.

7. **Proposition:**

(a) $G(x,y,y) = G(y,x,x) \;\forall x,y \in X$

(b) Let X be a G- cone metric space, Now we define $d_G: X \times X \to E$ as

$d_G(x,y) = G(x,y,y) + G(y,x,x)$ Then (X, d_G) is a cone metric space.

If X is non symmetric then $G(x,y,y) = \frac{2}{3} d_G(x,y)$

If X is symmetric then $G(x,y,y) = \frac{1}{2} d_G(x,y)$

(c) If X is a G- cone metric space, then following sound the same

(i) Sequence $\{x_n\}$ converges to x

(ii) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$

(iii) $G(x_n, x, x) \to 0$ *as* $n \to \infty$

(iv) $G(x_n, x_m, x_l) \to 0$ *as* $n, m, l \to \infty$

8. **Lemmas:**

(1) If X is a G- cone metric space, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ sequences in X such that $x_n \to x, y_n \to y, z_n \to z$ then $G(x_n, x_m, x_l) \to G(x,y,z)$ *as* $n \to \infty$.

(2) If X is a G- cone metric space, $\{x_n\}$ be a sequence and x ∈ X. If $\{x_n\} \to x, \{x_n\} \to y \Rightarrow x = y$.

(3) If X is a G- cone metric space, $\{x_n\}$ be a sequence and $x \in X. If \{x_n\} \to x$ in X, then $G(x_n, x_m, x) \to 0$ as $m, n \to \infty$.

(4) If X is a G- cone metric space, $\{x_n\}$ be a sequence and x ∈ X. if $\{x_n\}$ converges to x, then, $\{x_n\}$ is a Cauchy sequence.

(5) If X is a G- cone metric space, $\{x_n\}$ be a sequence and x ∈ X. If $\{x_n\}$ is a Cauchy sequence, then $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

3. Main results

Now we prove the following theorems

Theorem3.1: Let X be a complete symmetric G- cone metric space and $T: X \times X$ be a mapping satisfying the following conditions

(3.1.1) $G(Tx, Ty, Ty) \le a_1 G(x, y, y) + a_2 G(x, Tx, Tx) + a_3 G(y, Tx, Tx) + a_4 G(x, Ty, Ty) + a_5 G(y, Ty, Ty)$

(3.1.2) $G(Ty, Tx, Tx) \le a_1 G(y, x, x) + a_2 G(y, Ty, Ty) + a_3 G(y, Tx, Tx) + a_4 G(x, Ty, Ty) + a_5 G(x, Tx, Tx)$

For all $x, y \in X$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. Then T has unique fixed point.

Proof: Since X is a symmetric G- cone metric space, therefore from (3.1.1) we have

$$d_G(Tx, Ty) \le \frac{1}{2} a_1 d_G(x, y) + \frac{1}{2} a_2 d_G(x, Tx) + \frac{1}{2} a_3 d_G(x, Ty) + \frac{1}{2} a_4 d_G(y, Tx) + \frac{1}{2} a_5 d_G(y, Ty) \quad (3.1.3)$$

From (3.1.2), we have

$$d_G(Tx, Ty) \le \frac{1}{2} a_1 d_G(x, y) + \frac{1}{2} a_2 d_G(y, Ty) + \frac{1}{2} a_3 d_G(y, Tx) + \frac{1}{2} a_4 d_G(x, Ty) + \frac{1}{2} a_5 d_G(x, Tx) \quad (3.1.4)$$

Adding (3.1.3) and (3.1.4), we have

$$d_G(Tx, Ty) \le a_1 d_G(x, y) + \frac{1}{2}(a_2 + a_5) d_G(x, Tx) + \frac{1}{2}(a_3 + a_4) d_G(x, Ty) + \frac{1}{2}(a_3 + a_4) d_G(y, Tx) + \frac{1}{2}(a_2 + a_5) d_G(y, Ty)$$

$$d_G(Tx, Ty) \le a_1 d_G(x, y) + \frac{1}{2}\{(a_2 + a_5) d_G(x, Tx) + (a_2 + a_5) d_G(y, Ty)\} + \frac{1}{2}\{(a_3 + a_4) d_G(x, Ty) + (a_3 + a_4) d_G(y, Tx)\}$$

For all $x, y \in X$ and $a_1+a_2+a_3+a_4+a_5<1$ .Consider a point x∈ X and a sequence $\{T^n x\}$. Replacing x, y

by $T^n x, T^{n-1}x$ respectively in (3.1.5), we have

$$d_G(T^{n+1}x, T^n x) \le a_1 d_G(T^n x, T^{n-1}x) + \frac{1}{2}(a_2 + a_5)\{d_G(T^n x, T^{n+1}x) + d_G(T^{n-1}x, T^n x)\}$$
$$+ \frac{1}{2}(a_3 + a_4)\{d_G(T^n x, T^n x) + d_G(T^{n-1}x, T^{n+1}x)\}$$

$$d_G(T^{n+1}x, T^n x) \le a_1 d_G(T^n x, T^{n-1}x) + \frac{1}{2}(a_2 + a_5)\{d_G(T^n x, T^{n+1}x) + d_G(T^{n-1}x, T^n x)\}$$
$$+ \frac{1}{2}(a_3 + a_4)\{d_G(T^{n-1}x, T^n x) + d_G(T^n x, T^{n+1}x)\}$$

$$\left\{1 - \frac{1}{2}[(a_2 + a_5) + (a_3 + a_4)]\right\} d_G(T^{n+1}x, T^n x) \le \{a_1 + \frac{1}{2}[(a_2 + a_5) + (a_3 + a_4)]\, d_G(T^n x, T^{n-1}x)\}$$

$$d_G(T^{n+1}x, T^n x) \le \frac{a_1 + \frac{1}{2}[(a_2 + a_5) + (a_3 + a_4)]}{1 - \frac{1}{2}[(a_2 + a_5) + (a_3 + a_4)]}\, d_G(T^n x, T^{n-1}x)$$

$$d_G(T^{n+1}x, T^n x) \le \beta\, d_G(T^n x, T^{n-1}x),$$

where $\beta = \frac{a_1 + \frac{1}{2}[(a_2 + a_5) + (a_3 + a_4)]}{1 - \frac{1}{2}[(a_2 + a_5) + (a_3 + a_4)]} < 1 \implies a_1 + a_2 + a_3 + a_4 + a_5 < 1$.

Using the above iteration n times, we have

$$d_G(T^{n+1}x, T^n x) \le \beta^n\, d_G(T^n x, T^{n-1}x).$$

Proceeding in same manner, for +ve m, n and m > n, we have

$$d_G(T^{n+1}x, T^n x) \le \frac{\beta^n}{1-\beta}\, d_G(T^n x, T^{n-1}x)$$

$\implies \frac{\beta^n}{1-\beta} d_G(Tx, x) \ll \beta \text{ since } \beta \gg 0 \text{ (given)} \implies (T^m x, T^n x) \ll 0, \text{for } m > n,$ This implies that $\{T^n x\}$

is a Cauchy sequence. Therefore there exists z ∈ X such that $T^n z \in z$.

Now our goal is to show $Tz = z$. For that first of all we are to prove that $T^{n+1}z \in Tz$. Replacing x, y by $T^n z, z$ respectively, we have

$d_G(T^{n+1}, Tz)$

$$\le a_1 d_G(T^n, z) + \frac{1}{2}(a_2 + a_5)\{d_G(T^n x, T^{n+1}x) + d_G(z, Tz)\} + \frac{1}{2}(a_3 + a_4)\{d_G(T^n x, Tz) + (z, T^{n+1}x)\}$$

$$\le a_1 d_G(T^n, z) + \frac{1}{2}(a_2 + a_5)\{d_G(T^n x, z) + d_G(z, T^{n+1}x) + d_G(z, T^{n+1}z) + d_G(T^{n+1}z, Tz)\}$$
$$+ \frac{1}{2}(a_3 + a_4)\{d_G(T^n x, z) + d_G(z, T^{n+1}z) + d_G(T^{n+1}z, Tz) + d_G(z, T^{n+1}x)\}$$

$$\left(1-\frac{a_2+a_3+a_4+a_5}{2}\right) d_G(T^{n+1},Tz) \le \left(a_1-\frac{a_2+a_3+a_4+a_5}{2}\right) d_G(T^n,z) + (a_2+a_3+a_4+a_5)\, d_G(z,T^{n+1}z)$$

$$d_G(T^{n+1},Tz) \le \frac{1}{1-\frac{a_2+a_3+a_4+a_5}{2}}\left\{\left(a_1-\frac{a_2+a_3+a_4+a_5}{2}\right) d_G(T^n,z) + \left(a_2+a_3+a_4+a_5\right) d_G(z,T^{n+1}z)\right\} \ll c$$

for any $c \in E$. which implies that $T^{n+1}x \to Tz$ as $n \to \infty$.Now

$$d_G(z,Tz) \le d_G(z,T^{n+1}z) + d_G(T^{n+1}z,Tz) \ll \frac{c}{2}+\frac{c}{2} = c \gg 0 \text{ whenever } n \in N.$$

$$\text{Thus } d_G(z,Tz) \le \frac{c}{m} \text{ for all } m \ge 1 \Rightarrow \frac{c}{m} - d_G(z,Tz) \in P. \text{ But } \frac{c}{m} \to 0 \text{ as } m \to \infty$$

$$\Rightarrow d_G(z,Tz) \in -P, \text{ which is a contradiction. Therefore } d_G(z,Tz) = 0$$

Hence $Tz = z$ i.e. z is a unique fixed point.

This completes the proof of the theorem.

Theorem3.2: Let X be a complete symmetric G- cone metric space and $T: X \times X$ be a mapping satisfying the following condition

(3.2.1) $G(Tx,Ty,Ty)$

$$\le a_1 G(x,y,y) + a_2 G(x,Tx.Tx) + a_3 G(y,Tx.Tx) + a_4 G(x,Ty.Ty) + a_5 G(y,Ty.Ty)$$

For all $x,y \in X$ and $a_1 + a_2 + 2a_3 + a_5 < 1$. Then T has unique fixed point.

Proof: Let $x_0 \in X$ and $\{x_n\}$ be a sequence such that $x_n = T^n x$.Using (3.2.1), we have

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$

$$\le a_1 G(x_{n-1}, x_n, x_n) + a_2 G(x_{n-1}, x_n, x_n) + a_3 G(x_{n-1}, x_{n+1}, x_{n+1}) + a_4 G(x_n, x_n, x_n) + a_5 G(x_n, x_{n+1}, x_{n+1})$$

$$\le a_1 G(x_{n-1}, x_n, x_n) + a_2 G(x_{n-1}, x_n, x_n) + a_3\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\} + a_5 G(x_n, x_{n+1}, x_{n+1})$$

$$(1-a_3-a_5) G(x_n, x_{n+1}, x_{n+1}) \le (a_1+a_2+a_3) G(x_{n-1}, x_n, x_n)$$

$$G(x_n, x_{n+1}, x_{n+1}) \le \frac{a_1+a_2+a_3}{1-a_3-a_5} G(x_{n-1}, x_n, x_n)$$

$$G(x_n, x_{n+1}, x_{n+1}) \le \gamma\, G(x_{n-1}, x_n, x_n)$$

Where $\gamma = \frac{a_1+a_2+a_3}{1-a_3-a_5} < 1 \Rightarrow a_1 + a_2 + 2a_3 + a_5 < 1$ *and* γ *is* $+ve$ *but less than* 1

Continuing this process n time, we have

$$G(x_n, x_{n+1}, x_{n+1}) \le \gamma^n\, G(x_{n-1}, x_n, x_n)$$

Now for all positive m, n with m > n, we have

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m)$$

$$G(x_n, x_m, x_m) \leq (\gamma^n + \gamma^{n+1} + \gamma^{n+2} + \cdots + \gamma^{m-1})\, G(x_0, x_1, x_1)$$

$$G(x_n, x_m, x_m) \leq \frac{\gamma^n}{1-\gamma} G(x_0, x_1, x_1)$$

Let c>> 0 is given. Choose $\delta > 0$ such that c $+N_\delta(0) \subseteq$ P, where $N_\delta(0) = \{y \in E: \|y\| < \delta\}$. Also choose a natural number N_Δ such that $\frac{\gamma^n}{1-\gamma} G(x_{n-1}, x_n, x_n) \in N_\delta(0)$, for all m, n $\geq N_\Delta$. Then $\frac{\gamma^n}{1-\gamma} G(x_{n-1}, x_n, x_n)$ << c for all $m, n \geq N_\delta(0)$.Therefore $G(x_n\ x_n\ x_n) \ll 0 for\ all\ m > n.$Hence $\{x_n\}$ is a Cauchy sequence. Therefore there exists v ∊ X such that sequence $\{x_n\}$ converges to v such that Tv = v.

Now using (3.2.1) again, we have

$$G(x_n, Tv, Tv)$$

$$\leq a_1 G(x_{n-1}, v, v) + a_2 G(x_{n-1}, v, v) + a_3 G(x_{n-1}, Tv.Tv) + a_4 G(v, v, v) + a_5 G(v, Tv.Tv)$$

$$\leq a_1 G(x_{n-1}, v, v) + a_2 G(x_{n-1}, v, v) + a_3 G(x_{n-1}, Tv.Tv) + a_5 G(v, Tv.Tv)$$

Taking the limit n→∞, we have

$$G(x_n, Tv, Tv) \leq (a_1 + a_2 + a_3 + a_5) G(x_{n-1}, Tv, Tv)$$,

$$G(v, Tv, Tv) \leq (a_1 + a_2 + a_3 + a_5) G(v, Tv, Tv) \Rightarrow Tv = v.$$

Uniqueness: consider $u = Tu \neq v = Tv$ be another fixed point, therefore from (5.2.1), we have

$$G(u, v, v) = G(Tu, Tv, Tv)$$

$$\leq a_1 G(u, v, v) + a_2 G(u, Tu, Tu) + a_3 G(v, Tv, Tv) + a_4 G(v, Tu, Tu) + a_5 G(v, Tv, Tv)$$

$$\leq (a_1 + a_1 + a_1)\, G(u, Tv, Tv)$$

Implies $u. = v.$ Hence v is a unique fixed point.

This completes the proof of the theorem.

Conflict of Interests

The authors declare that there is no conflict of interests.

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