Available online at http://scik.org
Adv. Inequal. Appl. 2014, 2014:31
ISSN: 2050-7461

FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN SYMMETRIC SPACES

LOKESH K. JOSHI[1,*], MAHESH C. JOSHI[2], AND ANUJ KUMAR[3]

[1]Department of Applied Mathematics, Faculty of Engineering & Technology, Gurukula Kangri Vishwavidyalaya, Haridwar-249404, INDIA

[2]Department of Mathematics, Kumaun University, Nainital-263002, INDIA

[3]Department of Mathematics, University of Petroleum & Energy Studies, Dehradun-248007, INDIA

Copyright © 2014 L. Joshi, M. Joshi and A. Kumar. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract: In this paper we prove some fixed point theorems for multivalued mappings in the setting of symmetric spaces generalizing some well-known results in metric spaces.

Keywords: Symmetric Spaces, Multivalued maps.

2010 AMS Subject Classification: 47H10, 54H25

1. Introduction

Fixed Point Theory is one of the famous and traditional theory in mathematics and has a broad area of application. In this theory contraction is one of the important tool to prove the existence and uniqueness of a fixed point. Banach contraction principle is one of the most fascinating and classical result of the last century in the field of non linear analysis. It provides a powerful technique for solving a variety of problems in mathematical sciences and engineering. There are many generalizations of Banach contraction principle in the literature. Following Banach contraction mapping Nadler [6] introduced the concept of multivalued contraction mapping and established that a multivalued contraction mapping

*Corresponding author

Received May 5, 2014

possesses a fixed point in a complete metric space. Subsequently, a number of fixed point theorems in metric spaces have been proved for multivalued mappings satisfying contractive type conditions.

On the other hand, Hicks [2], and Hicks and Rhoades [3] started the study of existence of fixed points in symmetric spaces. Throughout this paper (X, d) be symmetric space and H denote the Hausdorff distance function on CL(X) induced by metric d, where CL(X) is the collection of all nonempty closed subsets of X. The aim of this paper is to establish some fixed point theorems for multivalued mapping using the multivalued analogous of the contractive condition introduced by Jaggi [1]. These results generalize the results of Hicks [2] and Moutawakil [5].

2. Preliminaries

Definition 1 [3] A sym-metric on a set X is a nonnegative real valued function d on X×X such that

(i) $d(x, y) = 0$ iff $x=y$

(ii) $d(x, y) = d(y, x)$.

Let d be a sym-metric on a set X and for r>0 and any $x \in X$, let $B(x,r) = \{y \in X : d(x,y) < r\}$. A topology t(d) on X is given by $U \in t(d)$ if and only if, for each $x \in U, B(x,r) \subset U$ for some r>0.

We need following two axioms (W.3) and (W.4) given by Wilson [7] in a sym-metric space (X, d).

(W.3) Given $\{x_n\}$, x and y in X, $d(x_n, x) \to 0$ and $d(x_n, y) \to 0$ imply that $x=y$.

(W.4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $d(x_n, x) = 0$ and $d(x_n, y_n) = 0$ imply that $\lim d(y_n, x) = 0$.

Let (X, d) be a symmetric space. CB(X) (resp. CL(X)) denotes the collection of all closed bounded (resp. closed) subsets of X.

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(A,x)\}, \text{ for all } A,B \in CL(X).$$

Then clearly $(CL(X), H)$ is a symmetric space.

We now site some definitions and lemmas from Hicks [2] and Hicks & Rhoades[3].

Definition 2[2]. A sequence $\{x_n\}$ in X is d-Cauchy sequence if it satisfies the usual metric condition.

Definition 3. [2]. Let *(X, d)* be a symmetric space.

(a) (X, d) is S-complete if for every d-Cauchy sequence {xn}, there exists x in X with $\lim d(x_n, x) = 0$.

(b) $f : X \longrightarrow X$ is d-continuous if $\lim d(x_n, x) = 0$ implies $\lim d(fx_n, fx) = 0$.

Definition 4[2].

A symmetric space (X, d) is complete if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that there exists x in X such that $d(x_n, x) \to 0$.

Lemma 1.[2].

Suppose $T:X \to CL(X)$ where d is a bounded symmetric. Then $\lim d(x_n, Tx) = 0$ iff there exist $y_n \in Tx$ such that $\lim d(x_n, y_n) = 0$.

It is well known that for *A*, *B* in *CB(X)* then for $\in > 0$ and $a \in A$, there exists $b \in B$ such that

$d(a,b) \leq H(A,B) + \in$

If *A, B* are in *C(X),* set of all compact subsets of X, then one can choose $b \in B$ such that $d(a,b) \leq H(A,B)$

Theorem 1.[3]. Let *(X, d)* be a complete symmetric space with *d* bounded and suppose (W.4) holds. Let $T:X \to CL(X)$ where T satisfies $\lim d(x_n, x) = 0$ implies $\lim H(Tx_n, Tx) = 0$. Then there exists x in X with $x \in Tx$ iff there exists a sequence $\{x_n\}$ in X with $x_{n+1} \in Tx_n$ and $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ or $\lim d(x_n, x) = 0$.

For our main result we follow the concept of multivalued contraction mappings given by Nadler [6] and extend the contractive condition introduced by Jaggi [1] (also refer to Rhoades [4]).

3. Main results

Taking a singlevalued mapping $f : X \to X$ and a multivalued mapping $T : X \to 2^X$, we establish the following results.

Theorem 3.1. *Suppose (X, d) is a symmetric space with d-bounded and (W.4) holds and* $f : X \to X$. *If* $T : X \to CL(X)$, *such that*

(i) $H(Tx,Ty) \le \dfrac{\alpha d(fx,Tx)d(fy,Ty)}{d(fx,fy)} + \beta d(fx,fy)$

for all $x, y \in X$, $fx \ne fy$, $\alpha, \beta \ge 0$, *and* $\alpha + \beta < 1$, *and*

(ii) $T(x) \subseteq f(x)$,

(iii) $f(x)$ *is S-complete,*

(iv) T *satisfies* $\lim d(x_n, x) = 0 \Rightarrow \lim H(Tx_n, Tx) = 0$.

Then there exists a point u in X such that $fu \in Tu$ *i.e. u is a coincidence point of f and T.*

Proof.

Pick $x_0 \in X$. Construct a sequence $\{x_n\}$ of points of X as follows:

Since $T(x) \subseteq f(x)$, one can choose a point x_1 in X such that $fx_1 \in Tx_0$. If $Tx_0 = Tx_1$, then $x_1 = u$ is the coincidence point. If $Tx_0 \ne Tx_1$, choose $x_2 \in X$ such that $d(fx_1, fx_2) \le \lambda H(Tx_0, Tx_1)$ where $\lambda > 1$ and $\lambda(\alpha + \beta) < 1$.

Continuing this process, we can choose $fx_{n+2} \in Tx_{n+1}$ such that

$$d(fx_{n+1}, fx_{n+2}) \le \lambda H(Tx_n, Tx_{n+1}).$$

By (i)

$$\begin{aligned} d(fx_{n+1}, fx_{n+2}) &\le \lambda H(Tx_n, Tx_{n+1}) \\ &\le \lambda \left[\frac{\alpha d(fx_n, Tx_n) d(fx_{n+1}, Tx_{n+1})}{d(fx_n, fx_{n+1})} + \beta d(fx_n, fx_{n+1}) \right] \\ &\le \lambda \left[\frac{\alpha d(fx_n, fx_{n+1}) d(fx_{n+1}, fx_{n+2})}{d(fx_n, fx_{n+1})} + \beta d(fx_n, fx_{n+1}) \right] \\ &\le \lambda\alpha\, d(fx_{n+1}, fx_{n+2}) + \lambda\beta\, d(fx_n, fx_{n+1}). \end{aligned}$$

Which implies

$$
\begin{aligned}
d(fx_{n+1}, fx_{n+2}) &\leq \left(\frac{\lambda\beta}{1-\lambda\alpha}\right) d(fx_n, fx_{n+1}) \\
&\leq \left(\frac{\lambda\beta}{1-\lambda\alpha}\right)^2 d(fx_{n-1}, fx_n) \\
&\dots \\
&\leq \left(\frac{\lambda\beta}{1-\lambda\alpha}\right)^{n+1} d(fx_0, fx_1)
\end{aligned}
$$

Since, $\lambda(\alpha+\beta)<1$ implies $\left(\frac{\lambda\beta}{1-\lambda\alpha}\right)^{n+1}<1$, which shows that $\{fx_n\}$ is a Cauchy sequence in *f(X).* Again f(*X)* is *S*-complete therefore $\{fx_n\}$ converges in *f(X)* at a point *b* i.e. there exists point $u \in X$ such that *f(u)=b,* by condition (iv) and Lemma 1 it implies that $fu \in Tu$.

Taking *T* a multivalued map from *X* to the set of compact subsets *C(X)* of *X* we get the following coincidence point theorem.

Theorem 3.2. *Suppose (X, d) is a symmetric space with d-bounded and (W.4) holds. If* $T : X \to C(X)$, *such that all conditions (i)-(iv) of Theorem 1 hold. Then f and T have a coincidence, i.e there exist a point u in X such that* $fu \in Tu$.

Proof.

Since $T(x) \subseteq f(x)$, and *T(x)* is compact. The only change occurs in the proof of this result is that the inequality $d(fx_{n+1}, fx_{n+2}) \leq \lambda H(Tx_n, Tx_{n+1})$ of proof of Theorem 3.1 will be replaced by the stronger inequality

$$d(fx_{n+1}, fx_{n+2}) \leq H(Tx_n, Tx_{n+1}).$$

Again using condition (iv) and Lemma 1 it is clear that there exists a point *u* in *X* such that $fu \in Tu$.

If both maps commute at coincidence point and $f(u) = f(f(u))$. Then *f(u)* is common fixed point of f and T.

In the Theorems 3.1 and 3.2 taking *f* as an identity mapping we get following fixed point Theorems 3.3 and 3. 4 respectively.

Theorem 3.3. *Suppose (X, d) is a S-complete symmetric space with d-bounded and (W.4)* *holds. If* $T : X \to CL(X)$, *such that*

(i) $$H(Tx,Ty) \le \frac{\alpha d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y)$$

for all $x, y \in X$, $x \ne y$, $\alpha, \beta \ge 0$, *and* $\alpha + \beta < 1$, *and*

(ii) T satisfies $\lim d(x_n, x) = 0$ *implies* $\lim H(Tx_n, Tx) = 0.$

Then there exists u in X such that $u \in Tu$.

Proof.

Let $x_0 \in X$ since Tx_0 is closed choose $x_1 \in Tx_0$ such that

$$d(x_0, x_1) \le \lambda H(Tx_0, Tx_1)$$

where $\lambda > 1$ and $\lambda(\alpha + \beta) < 1$. In general choose $x_{n+1} \in Tx_n$ such that

$$d(x_{n+1}, x_{n+2}) \le \lambda H(Tx_n, Tx_{n+1})$$

From condition (i), one can easily obtain that

$$d(x_{n+1}, x_{n+2}) \le \left(\frac{\lambda\beta}{1-\lambda\alpha}\right)^{n+1} d(x_0, x_1)$$

Since $\lambda(\alpha + \beta) < 1$ implies $\frac{\lambda\beta}{1-\lambda\alpha} < 1$. It is clear that $\{x_n\}$ is a d-Cauchy sequence. Since *X* is *S*-complete symmetric space, there exist $u \in X$ with $\lim d(x_n, u) = 0$.

By condition (ii) it follows that

$$\lim d(x_n, x) = 0 \text{ implies } \lim H(Tx_n, Tu) = 0$$

Therefore using (W.4) and Lemma 1, it is clear that there exists *u* in *X* such that $u \in Tu$.

Theorem 3.4. *Let (X, d) be a S-complete symmetric space with d-bounded and suppose that (W.4) holds. Let* $T : X \to C(X)$, *such that*

(i) $$H(Tx,Ty) \le \frac{\alpha d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y)$$

for all $x, y \in X$, $x \ne y$, $\alpha, \beta \ge 0$, *and* $\alpha + \beta < 1$, *and*

(ii) T satisfies $\lim d(x_n, x) = 0$ *implies* $\lim H(Tx_n, Tx) = 0.$

Then there exists u in X such that $u \in Tu$.

Proof.

Proof follows from the proof of Theorem 2 taking $d(x_0, x_1) \leq H(Tx_0, Tx_1)$ in place of $d(x_0, x_1) \leq \lambda H(Tx_0, Tx_1)$.

Taking $\alpha = 0$ in Theorem 3.1 we get following coincidence point theorem as corollary.

Corollary Suppose *(X, d)* is a *S*-complete symmetric space with *d*-bounded and assume (W.4) holds and $f : X \to X$. Let $0 < \beta < 1$, if $T : X \to CL(X)$ satisfies

$$H(Tx, Ty) \leq \beta d(fx, fy) \text{ for all } x, y \in X.$$

Then there exists *u* in *X* with $fu \in Tu$.

Remark 1. Taking *f* an identity mapping in above corollary we get the result of Moutawakil [5, Theorem 2.2.1] and the result of Hicks [2].

It is remarkable that very general probabilistic structures admit a compatible symmetric or semi metric (for more applications and details see [2] and [5]).

Remark 2. Again it is remarkable that in Theorem 1, if α is taken as zero and *(X, d)* a metric space, we get the result established by Nadler [6], and if in place of symmetric space we take metric space with $\alpha=0$ and *T* a singlevalued map we get Banach contraction principle.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] D. S. Jaggi, Some unique fixed point theorems, Indian J. Pure Appl. Math. 8(2)(1977), 223-230.

[2] T. L. Hicks, Fixed point theorems for multivalued mappings II, Indian J. Pure Appl. Math, 29(2) (1998), 133-137.

[3]] T. L. Hicks and B. E. Rhoades, Fixed point theory in symmetric spaces with application to probabilistic spaces, Nonlinear Analysis, 36(1999), 331-344.

[4] B. E. Rhoades, Proving fixed point theorems using general principles, Indian J. Pure appl. Math, 27(8), (1996), 741-770.

[5] D. El. Moutawakil,, A fixed point theorem for multivalued maps in symmetric spaces, Applied Mathematics E-Notes, 4(2004), 26-32.

[6] S. B. Nadler Jr., Multivalued contraction mappings, Pacific J. Math, 30(1969), 639-640.

[7] W. A. Wilson, On semi metric spaces, Amer. J. Math., 53(1931), 361-363.