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REFINEMENT OF JENSEN'S INEQUALITY FOR OPERATOR CONVEX FUNCTIONS

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Abstract. In this paper, we give a refinement of discrete Jensen's inequality for the operator convex functions. We launch the corresponding mixed symmetric means for positive self-adjoint operators defined on Hilbert space and also establish the refinement of inequality between power means of strictly positive operators.

Keywords: self-adjoint operators; operator convex functions; operator means; symbolic calculus.

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1. INTRODUCTION-PRELIMINARIES

H will from now on denote a complex Hilbert space. $S(I)$ means the class of all self-adjoint bounded operators on H whose spectra are contained in an interval $I \subset \mathbb{R}$. The spectrum of a bounded operator A on H is denoted by $\mathrm{Sp}(A)$.

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Let $f : D_f(\subset \mathbb{R}) \to \mathbb{R}$ be a function and let $I \subset D_f$ be an interval. f is said to be operator monotone on I if f is continuous on I and $A, B \in S(I)$, $A \leq B$ (i.e. $B-A$ is a positive operator) imply $f(A) \leq f(B)$. The function f is said to be operator convex on I if f is continuous on I and

$$f(sA+tB) \leq sf(A)+tf(B)$$

for all A, $B \in S(I)$ and for all positive numbers s and t such that $s+t=1$. The function f is called operator concave on J if $-f$ is operator convex on J.

Theorem 1.1. *Jensen's operator inequality: Let $I \subset \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be an operator convex function on I. If $C_i \in S(I)$ $(i = 1,\ldots,n)$, and $w_i > 0$ $(i = 1,...,n)$ such that $\sum_{i=1}^{n} w_i = 1$, then*

$$f\left(\sum_{i=1}^{n} w_i C_i\right) \leq \sum_{i=1}^{n} w_i f(C_i). \tag{1}$$

If f is an operator concave function on I, then the inequality in (1) is reversed.

Some interpolations of (1) are given in [3] as follows.

Theorem 1.2. *Under the conditions of the Jensen's operator inequality*

$$f(\sum_{i=1}^{n} w_i C_i) = f_{n,n} \leq ... \leq f_{k,n} \leq ... \leq f_{1,n} = \sum_{i=1}^{n} w_i f(C_i), \tag{2}$$

where for $1 \leq k \leq n$

$$f_{k,n} := \frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \left(\sum_{j=1}^{k} w_{i_j}\right) f\left(\frac{\sum\limits_{j=1}^{k} w_{i_j} C_{i_j}}{\sum\limits_{j=1}^{k} w_{i_j}}\right). \tag{3}$$

Theorem 1.3. *If the conditions of the Jensen's operator inequality are satisfied, then*

$$f(\sum_{i=1}^{n} w_i C_i) \leq ... \leq \overline{f}_{k+1,n} \leq \overline{f}_{k,n} \leq ... \leq \overline{f}_{1,n} = \sum_{i=1}^{n} w_i f(C_i), \tag{4}$$

where for $k \geq 1$

$$\overline{f}_{k,n} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{1\le i_1\le\ldots\le i_k\le n} \left(\sum_{j=1}^{k} w_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} w_{i_j} C_{i_j}}{\sum_{j=1}^{k} w_{i_j}}\right). \tag{5}$$

A self-adjoint bounded operator A on H is called strictly positive if it is positive and invertible, or equivalently, $\mathrm{Sp}(A)\subset[m,M]$ for some $0<m<M$.

The power means for strictly positive operators $\mathbf{C}:=(C_1,\ldots,C_n)$ with positive weights $\mathbf{w}:=(w_1,\ldots,w_n)$ are defined in [3] as follows:

$$M_r(\mathbf{C},\mathbf{w}) = M_r(C_1,\ldots,C_n;w_1,\ldots,w_n) := \left(\frac{1}{W_n}\sum_{i=1}^{n} w_i C_i^r\right)^{\frac{1}{r}}, \tag{6}$$

where $r\in\mathbb{R}\setminus\{0\}$ and $W_n := \sum_{i=1}^{n} w_i$. The following result about the monotonicity of power means is also given in [3]:

$$M_s(\mathbf{C},\mathbf{w}) \le M_r(\mathbf{C},\mathbf{w}) \tag{7}$$

holds if either $s\le r$, $s\notin(-1,1)$, $r\notin(-1,1)$ or $1/2\le s\le 1\le r$ or $s\le -1\le r\le -1/2$.

Some symmetric mixed means, corresponding to the expressions (3) and (5) are introduced in [3]: for $r,s\in\mathbb{R}\setminus\{0\}$ and for $W_n=1$, define

$$\begin{aligned} &M_n(s,r;k) := \\ &\left(\frac{1}{\binom{n-1}{k-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^{k} w_{i_j}\right) M_r^s(C_{i_1},\ldots,C_{i_k};w_{i_1},\ldots,w_{i_k})\right)^{\frac{1}{s}}, \end{aligned} \tag{8}$$

where $1\le k\le n$, and

$$\begin{aligned} &\overline{M}_n(s,r;k) := \\ &\left(\frac{1}{\binom{n+k-1}{k-1}} \sum_{1\le i_1\le\ldots\le i_k\le n} \left(\sum_{j=1}^{k} w_{i_j}\right) M_r^s(C_{i_1},\ldots,C_{i_k};w_{i_1},\ldots,w_{i_k})\right)^{\frac{1}{s}}, \end{aligned} \tag{9}$$

where $k\ge 1$.

The following result from [3] gives some refinements of (7).

Theorem 1.4. *Let* $\mathbf{C}$ *be an n-tuple of strictly positive operators, and let* $w_i > 0$ $(i = 1, ..., n)$ *such that* $W_n = 1$. *Then the following inequalities are valid*

$$M_s(\mathbf{C}, \mathbf{w}) = M_n(s, r; 1) \leq ... \leq M_n(s, r; k) \leq ... \leq M_n(s, r; n) = M_r(\mathbf{C}, \mathbf{w}), \tag{10}$$

and

$$M_s(\mathbf{C}, \mathbf{w}) = \overline{M}_n(s, r; 1) \leq ... \leq \overline{M}_n(s, r; k) \leq ... \leq M_r(\mathbf{C}, \mathbf{w}), \tag{11}$$

if either

(i) $1 \leq s \leq r$ *or*

(ii) $-r \leq s \leq -1$ *or*

(iii) $s \leq -1$, $r \geq s \geq 2r$;

while the reverse inequalities are valid if either

(iv) $r \leq s \leq -1$ *or*

(v) $1 \leq s \leq -r$ *or*

(vi) $s \geq 1$, $r \leq s \leq 2r$.

In this paper, we generalize the above results by using a new refinement of the Jensen's inequality from [2]. First, we give the notations introduced in [2]:

Let X be a set. The power set of X is denoted by $P(X)$. $|X|$ means the number of elements in X.

The usual symbol $\mathbb{N}$ is used for the set of natural numbers (including 0).

Let $u \geq 1$ and $v \geq 2$ be fixed integers. Define the functions

$$S_{v,w} : \{1, \ldots, u\}^v \to \{1, \ldots, u\}^{v-1}, \quad 1 \leq w \leq v,$$

$$S_v : \{1, \ldots, u\}^v \to P\left(\{1, \ldots, u\}^{v-1}\right),$$

and

$$T_v : P(\{1, \ldots, u\}^v) \to P\left(\{1, \ldots, u\}^{v-1}\right)$$

by

$$S_{v,w}(i_1, \ldots, i_v) := (i_1, i_2, \ldots, i_{w-1}, i_{w+1}, \ldots, i_v), \quad 1 \leq w \leq v,$$

$$S_v(i_1,\ldots,i_v) := \bigcup_{w=1}^{v} \{S_{v,w}(i_1,\ldots,i_v)\},$$

and

$$T_v(I) := \begin{cases} \varnothing, & \text{if} \quad I = \varnothing \\ \bigcup\limits_{(i_1,\ldots,i_v)\in I} S_v(i_1,\ldots,i_v), & \text{if} \quad I \neq \varnothing \end{cases}.$$

Next, let the function

$$\alpha_{v,i} : \{1,\ldots,u\}^v \to \mathbb{N}, \quad 1 \le i \le u,$$

be given by: $\alpha_{v,i}(i_1,\ldots,i_v)$ means the number of occurrences of i in the sequence $(i_1,\ldots,i_v)$. For each $I \in P(\{1,\ldots,u\}^v)$ let

$$\alpha_{I,i} := \sum_{(i_1,\ldots,i_v)\in I} \alpha_{v,i}(i_1,\ldots,i_v), \quad 1 \le i \le u.$$

It is easy to see that the dependence of the functions $S_{v,w}$, S_v, T_v and $\alpha_{v,i}$ on u does not play an important role, so we can use simplified notations.

The following hypotheses will give the basic context of our results.

(H$_1$) Let $n \ge 1$ and $k \ge 2$ be fixed integers, and let I_k be a subset of $\{1,\ldots,n\}^k$ such that

$$\alpha_{I_k,i} \ge 1, \quad 1 \le i \le n. \tag{12}$$

(H$_2$) Let $I \subset \mathbb{R}$ be an interval, and let $C_i \in S(I)$ $(1 \le i \le n)$.

(H$_3$) Let $w_1,\ldots,w_n$ be positive numbers such that $\sum\limits_{j=1}^{n} w_j = 1$.

(H$_4$) Let the function $f : I \to \mathbb{R}$ be operator convex.

(H$_5$) Let h, $g : I \to \mathbb{R}$ be continuous and strictly operator monotone functions.

We need some further preparations.

Starting from I_k, we introduce the sets $I_l \subset \{1,\ldots,n\}^l$ $(k-1 \ge l \ge 1)$ inductively by

$$I_{l-1} := T_l(I_l), \quad k \ge l \ge 2.$$

Obviously, $I_1 = \{1,\ldots,n\}$, by (12), and this insures that $\alpha_{I_1,i} = 1$ $(1 \le i \le n)$. From (12) again, we have that $\alpha_{I_l,i} \ge 1$ $(k-1 \ge l \ge 1, 1 \le i \le n)$.

For any $k \geq l \geq 2$ and for any $(j_1, \ldots, j_{l-1}) \in I_{l-1}$, let

$$H_{I_l}(j_1, \ldots, j_{l-1})$$
$$:= \left\{ ((i_1, \ldots, i_l), m) \in I_l \times \{1, \ldots, l\} \mid S_{l,m}(i_1, \ldots, i_l) = (j_1, \ldots, j_{l-1}) \right\}.$$

Using these sets we define the functions $t_{I_k,l} : I_l \to \mathbb{N}$ $(k \geq l \geq 1)$ inductively by

$$t_{I_k,k}(i_1, \ldots, i_k) := 1, \quad (i_1, \ldots, i_k) \in I_k; \tag{13}$$

$$t_{I_k,l-1}(j_1, \ldots, j_{l-1}) := \sum_{((i_1,\ldots,i_l),m) \in H_{I_l}(j_1,\ldots,j_{l-1})} t_{I_k,l}(i_1, \ldots, i_l). \tag{14}$$

2. MAIN RESULTS

The main results of this paper involve some special expressions, which we now describe. Suppose (H_1)-(H_4). For any $k \geq l \geq 1$ let

$$A_{l,l} = A_{l,l}(I_k, C_1, \ldots, C_n, w_1, \ldots, w_n)$$
$$:= \sum_{(i_1,\ldots,i_l) \in I_l} \left(\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_l,i_s}} \right) f \left(\frac{\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_l,i_s}} C_{i_s}}{\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_l,i_s}}} \right), \tag{15}$$

and associate to each $k-1 \geq l \geq 1$ the operator

$$A_{k,l} = A_{k,l}(I_k, C_1, \ldots, C_n, w_1, \ldots, w_n)$$
$$:= \frac{1}{(k-1)\ldots l} \sum_{(i_1,\ldots,i_l) \in I_l} t_{I_k,l}(i_1, \ldots, i_l) \left(\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}} \right) f \left(\frac{\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}} C_{i_s}}{\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}}} \right). \tag{16}$$

With these preparations out of the way we come to

Theorem 2.1. *Assume (H_1)-(H_4). Then*

(a)

$$f\left(\sum_{r=1}^{n} w_r C_r \right) \leq A_{k,k} \leq A_{k,k-1} \leq \ldots \leq A_{k,2} \leq A_{k,1} = \sum_{r=1}^{n} w_r f(C_r). \tag{17}$$

(b) Suppose $|H_{I_l}(j_1,\ldots,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. *Then*

$$A_{k,l} = A_{l,l} = \frac{n}{l\,|I_l|} \sum_{(i_1,\ldots,i_l)\in I_l} \left(\sum_{s=1}^{l} w_{i_s}\right) f\left(\frac{\sum_{s=1}^{l} w_{i_s} C_{i_s}}{\sum_{s=1}^{l} w_{i_s}}\right), \quad (k \geq l \geq 1), \tag{18}$$

and thus

$$f\left(\sum_{r=1}^{n} w_r C_r\right) \leq A_{k,k} \leq A_{k-1,k-1} \leq \ldots \leq A_{2,2} \leq A_{1,1} = \sum_{r=1}^{n} w_r f(C_r).$$

To prove these results we can use the same method as in the proof of the main result (Theorem 1) in [2], so we omit the proofs.

3. Discussion, and applications

Throughout Examples (3.1-3.6) (based on the examples in [2]) the conditions (H_2)-(H_4) will be assumed.

Theorem 2.1 contains Theorem 1.2, as the first example shows.

Example 3.1. *Let*

$$I_k := \left\{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 < \ldots < i_k\right\}, \quad 1 \leq k \leq n.$$

Then $\alpha_{I_n,i} = 1$ $(i = 1,\ldots,n)$ *ensuring* (H_1) *with* $k = n$. *It is easy to check that* $T_k(I_k) = I_{k-1}$ $(k = 2,\ldots,n)$, $|I_k| = \binom{n}{k}$ $(k = 1,\ldots,n)$, *and for every* $k = 2,\ldots,n$

$$|H_{I_k}(j_1,\ldots,j_{k-1})| = n-(k-1), \quad (j_1,\ldots,j_{k-1}) \in I_{k-1},$$

and therefore, thanks to Theorem 2.1 (b),

$$A_{k,k} = \frac{1}{\binom{n-1}{k-1}} \sum_{1\leq i_1<\ldots<i_k\leq n} \left(\sum_{s=1}^{k} w_{i_s}\right) f\left(\frac{\sum_{s=1}^{k} w_{i_s} C_{i_s}}{\sum_{s=1}^{k} w_{i_s}}\right), \quad k = 1,\ldots,n.$$

and

$$f\left(\sum_{r=1}^{n} w_r C_r\right) \leq A_{k,k} \leq A_{k-1,k-1} \leq \ldots \leq A_{2,2} \leq A_{1,1} = \sum_{r=1}^{n} w_r f(C_r). \tag{19}$$

If $w_1 = \ldots = w_n = \frac{1}{n}$*, then*

$$A_{k,k} = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \ldots < i_k \le n} f\left(\frac{C_{i_1} + \ldots + C_{i_k}}{k}\right), \quad k = 1, \ldots, n,$$

and thus (19) gives Theorem 1.2.

The next example illustrates that Theorem 1.3 is a also special case of Theorem 2.1.

Example 3.2. *Let*

$$I_k := \left\{ (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 \le \ldots \le i_k \right\}, \quad k \ge 1.$$

Obviously, $\alpha_{I_k,i} \ge 1$ $(i = 1, \ldots, n)$*, and therefore* (H_1) *is satisfied. It is not hard to see that* $T_k(I_k) = I_{k-1}$ $(k = 2, \ldots)$, $|I_k| = \binom{n+k-1}{k}$ $(k = 1, \ldots)$*, and for each* $l = 2, \ldots, k$

$$|H_{I_l}(j_1, \ldots, j_{l-1})| = n, \quad (j_1, \ldots, j_{l-1}) \in I_{l-1}.$$

Consequently, by applying Theorem 2.1 (b), we deduce that

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \le i_1 \le \ldots \le i_k \le n} \left(\sum_{s=1}^{k} w_{i_s}\right) f\left(\frac{\sum_{s=1}^{k} w_{i_s} C_{i_s}}{\sum_{s=1}^{k} w_{i_s}}\right), \quad k \ge 1,$$

and

$$f\left(\sum_{r=1}^{n} w_r C_r\right) \le \ldots \le A_{k,k} \le \ldots \le A_{k,1} = \sum_{r=1}^{n} w_r f(C_r). \tag{20}$$

By taking $w_1 = \ldots = w_n = \frac{1}{n}$*, we obtain that*

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k}} \sum_{1 \le i_1 \le \ldots \le i_k \le n} f\left(\frac{C_{i_1} + \ldots + C_{i_k}}{k}\right), \quad k \ge 1,$$

and thus (20) gives Theorem 1.3.

The following two examples are particular cases of Theorem 2.1 (b).

Example 3.3. *Let*

$$I_k := \{1, \ldots, n\}^k, \quad k \ge 1.$$

Trivially, $\alpha_{I_k,i} \geq 1$ $(i = 1,\ldots,n)$*, hence* (H_1) *holds. It is evident that* $T_k(I_k) = I_{k-1}$ $(k = 2,\ldots)$*,* $|I_k| = n^k$ $(k = 1,\ldots)$*, and for every* $l = 2,\ldots,k$

$$|H_{I_l}(j_1,\ldots,j_{l-1})| = n^l, \quad (j_1,\ldots,j_{l-1}) \in I_{l-1},$$

and therefore Theorem 2.1 (b) leads to

$$A_{k,k} = \frac{1}{kn^{k-1}} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} w_{i_s}\right) f\left(\frac{\sum_{s=1}^{k} w_{i_s} C_{i_s}}{\sum_{s=1}^{k} w_{i_s}}\right), \quad k \geq 1,$$

and

$$f\left(\sum_{r=1}^{n} w_r C_r\right) \leq \ldots \leq A_{k,k} \leq \ldots \leq A_{1,1} = \sum_{r=1}^{n} w_r f(C_r), \quad k \geq 1.$$

Especially, for $w_1 = \ldots w_n = \frac{1}{n}$ *we find that*

$$A_{k,k} = \frac{1}{n^k} \sum_{(i_1,\ldots,i_k)\in I_k} f\left(\frac{C_{i_1} + \ldots + C_{i_k}}{k}\right), \quad k = 1,\ldots,n.$$

Example 3.4. *For* $1 \leq k \leq n$ *let* I_k *consist of all sequences* $(i_1,\ldots,i_k)$ *of* k *distinct numbers from* $\{1,\ldots,n\}$*. Then* $\alpha_{I_n,i} \geq 1$ $(i = 1,\ldots,n)$*, hence* (H_1) *is valid. It is immediate that* $T_k(I_k) = I_{k-1}$ $(k = 2,\ldots,n)$*,* $|I_k| = n(n-1)\ldots(n-k+1)$ $(k = 1,\ldots,n)$*, and for each* $k = 2,\ldots,n$

$$|H_{I_k}(j_1,\ldots,j_{k-1})| = (n-(k-1))\,k, \quad (j_1,\ldots,j_{k-1}) \in I_{k-1}.$$

and from them, on account of Theorem 2.1 (b), follows

$$A_{k,k} = \frac{n}{kn(n-1)\ldots(n-k+1)} \cdot \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} w_{i_s}\right) f\left(\frac{\sum_{s=1}^{k} w_{i_s} C_{i_s}}{\sum_{s=1}^{k} w_{i_s}}\right), \quad k = 1,\ldots,n$$

and

$$f\left(\sum_{r=1}^{n} w_r C_r\right) \leq A_{n,n} \leq \ldots \leq A_{k,k} \leq \ldots \leq A_{1,1} = \sum_{r=1}^{n} w_r f(C_r).$$

If we set $w_1 = \ldots = w_n = \frac{1}{n}$*, then*

$$A_{k,k} = \frac{1}{n(n-1)\ldots(n-k+1)} \sum_{(i_1,\ldots,i_k)\in I_k} f\left(\frac{C_{i_1} + \ldots + C_{i_k}}{k}\right), \quad k = 1,\ldots,n.$$

In the sequel two interesting consequences of Theorem 2.1 (a) are given.

Example 3.5. *Let $c_i \geq 1$ be an integer $(i = 1, \ldots, n)$, let $k := \sum_{i=1}^{n} c_i$, and let $I_k = P^{c_1,\ldots,c_n}$ consist of all sequences $(i_1, \ldots, i_k)$ in which the number of occurrences of $i \in \{1, \ldots, n\}$ is c_i $(i = 1, \ldots, n)$. Evidently, (H_1) is satisfied. A simple calculation shows that*

$$I_{k-1} = \bigcup_{i=1}^{n} P^{c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_n}, \quad \alpha_{I_k,i} = \frac{k!}{c_1! \ldots c_n!} c_i, \quad i = 1, \ldots, n,$$

and

$$t_{I_k,k-1}(i_1, \ldots, i_{k-1}) = k,$$

$$\text{if} \quad (i_1, \ldots, i_{k-1}) \in P^{c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_n}, \quad i = 1, \ldots, n,$$

and

$$f\left(\sum_{r=1}^{n} w_r C_r\right) = A_{k,k}$$

$$= \frac{c_1! \ldots c_n!}{k!} \sum_{(i_1,\ldots,i_k) \in I_k} \left(\sum_{s=1}^{k} \frac{w_{i_s}}{c_{i_s}}\right) f\left(\frac{\sum_{s=1}^{k} \frac{w_{i_s}}{c_{i_s}} C_{i_s}}{\sum_{s=1}^{k} \frac{w_{i_s}}{c_{i_s}}}\right).$$

According to Theorem 2.1 (a)

$$f\left(\sum_{r=1}^{n} w_r C_r\right) \leq A_{k,k-1} \leq \sum_{r=1}^{n} w_r f(C_r),$$

where

$$A_{k,k-1} = \frac{1}{k-1} \sum_{i=1}^{n} (c_i - w_i) f\left(\frac{\sum_{r=1}^{n} w_r C_r - \frac{w_i}{c_i} C_i}{1 - \frac{w_i}{c_i}}\right).$$

Example 3.6. *Let*

$$I_2 := \left\{(i_1, i_2) \in \{1, \ldots, n\}^2 \mid i_1 | i_2\right\}.$$

The notation $i_1 | i_2$ means that i_1 divides i_2. Since $i | i$ $(i = 1, \ldots, n)$, (H_1) holds. In this case

$$\alpha_{I_2,i} = \left[\frac{n}{i}\right] + d(i), \quad i = 1, \ldots, n,$$

where $\left[\frac{n}{i}\right]$ is the largest natural number that does not exceed $\frac{n}{i}$, and $d(i)$ denotes the number of positive divisors of i. By Theorem 2.1 (a), we have

$$f\left(\sum_{r=1}^{n} w_r C_r\right) \leq \sum_{(i_1,i_2)\in I_2} \left(\frac{w_{i_1}}{\left[\frac{n}{i_1}\right]+d(i_1)} + \frac{w_{i_2}}{\left[\frac{n}{i_2}\right]+d(i_2)}\right)$$

$$\cdot f\left(\frac{\frac{w_{i_1}}{\left[\frac{n}{i_1}\right]+d(i_1)}C_{i_1} + \frac{w_{i_2}}{\left[\frac{n}{i_2}\right]+d(i_2)}C_{i_2}}{\frac{w_{i_1}}{\left[\frac{n}{i_1}\right]+d(i_1)} + \frac{w_{i_2}}{\left[\frac{n}{i_2}\right]+d(i_2)}}\right) \leq \sum_{r=1}^{n} w_r f(C_r).$$

4. Symmetric Means

Assume (H_1)-(H_3). The power means corresponding to $\mathbf{i}^l := (i_1,\ldots,i_l) \in I_l$ $(l = 1,\ldots,k)$ are given as:

(21)
$$M_r(I_k,\mathbf{i}^l) := \left(\frac{\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}} C_{i_s}^r}{\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}}}\right)^{\frac{1}{r}}, \quad r \neq 0.$$

Next, we introduce the mixed symmetric means corresponding to the expressions (15) and (16) as follows:

(22)
$$M_{s,r}^1(I_k,k) := \left(\sum_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k} \left(\sum_{j=1}^{k} \frac{w_{i_j}}{\alpha_{I_k,i_j}}\right) \left(M_r(I_k,\mathbf{i}^k)\right)^s\right)^{\frac{1}{s}}, \ s \neq 0,$$

and for $k-1 \geq l \geq 1$

(23)
$$M_{s,r}^1(I_k,l) :=$$
$$\left(\frac{1}{(k-1)\ldots l} \sum_{\mathbf{i}^l=(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(\mathbf{i}^l) \left(\sum_{j=1}^{l} \frac{w_{i_j}}{\alpha_{I_k,i_j}}\right) \left(M_r(I_k,\mathbf{i}^l)\right)^s\right)^{\frac{1}{s}}, s \neq 0.$$

The following result is a comprehensive generalization of Theorem 1.4.

Theorem 4.1. *Assume (H_1)-(H_3) for an n-tuple* $\mathbf{C}$ *of strictly positive operators. Then*

(24)
$$M_s(\mathbf{C},\mathbf{w}) = M_{s,r}^1(I_k,1) \leq \ldots. \leq M_{s,r}^1(I_k,k) \leq M_r(\mathbf{C},\mathbf{w}).$$

holds if either

(i) $1 \leq s \leq r$ *or*

(ii) $-r \leq s \leq -1$ *or*

(iii) $s \leq -1,\ r \geq s \geq 2r$*;*

while the reverse inequalities hold in (24) if either

(iv) $r \leq s \leq -1$ *or*

(v) $1 \leq s \leq -r$ *or*

(vi) $s \geq 1,\ r \leq s \leq 2r$*.*

Proof. It is well known (see [1]) that the function $f : D_f(\subset \mathbb{R}) \to \mathbb{R},\ f(x) = x^p$ is operator convex on $(0,\infty)$ if either $1 \leq p \leq 2$ or $-1 \leq p \leq 0$, and operator concave on $(0,\infty)$ if $0 \leq p \leq 1$, while f is operator monotone on $(0,\infty)$ if $0 \leq p \leq 1$. It is also true that $-f$ is operator monotone on $(0,\infty)$ if $-1 \leq p \leq 0$. By using these facts, we can apply Theorem 2.1 (a) to the function $f(x) = x^{\frac{s}{r}}$, and the operators C_i^r $(i = 1,\dots,n)$. □

Assume (H_1)-(H_3) and (H_5). Then we define the quasi-arithmetic means with respect to (15) and (16) as follows:

$$M_{h,g}^1(I_k,k) := h^{-1}\left(\sum_{(i_1,\dots,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{w_{i_s}}{\alpha_{I_k,i_s}}\right)h\circ g^{-1}\left(\frac{\sum_{s=1}^{k}\frac{w_{i_s}}{\alpha_{I_k,i_s}}g(C_{i_s})}{\sum_{s=1}^{k}\frac{w_{i_s}}{\alpha_{I_k,i_s}}}\right)\right), \tag{25}$$

and for $k-1 \geq l \geq 1$

$$M_{h,g}^1(I_k,l) := h^{-1}\left(\frac{1}{(k-1)\dots l}\sum_{\mathbf{i}^l=(i_1,\dots,i_l)\in I_l} t_{I_k,l}(\mathbf{i}^l)\left(\sum_{s=1}^{l}\frac{w_{i_s}}{\alpha_{I_k,i_s}}\right)h\circ g^{-1}\left(\frac{\sum_{s=1}^{l}\frac{w_{i_s}}{\alpha_{I_k,i_s}}g(C_{i_s})}{\sum_{s=1}^{l}\frac{w_{i_s}}{\alpha_{I_k,i_s}}}\right)\right). \tag{26}$$

The monotonicity of these generalized means is obtained in the next corollary.

Corollary 4.2. *Assume (H_1)-(H_3) and (H_5). For a continuous and strictly operator monotone function* $q : I \to \mathbb{R}$ *we define*

$$M_q := q^{-1}\left(\sum_{i=1}^{n} w_i q(C_i)\right).$$

Then

$$M_h = M^1_{h,g}(I_k,1) \geq ... \geq M^1_{h,g}(I_k,k) \geq M_g, \tag{27}$$

if either $h\circ g^{-1}$ is operator convex and h^{-1} is operator monotone or $h\circ g^{-1}$ is operator concave and $-h^{-1}$ is operator monotone;

$$M_g = M^1_{g,h}(I_k,1) \leq ... \leq M^1_{g,h}(I_k,k) \leq M_h, \tag{28}$$

if either $g\circ h^{-1}$ is operator convex and $-g^{-1}$ is operator monotone or $g\circ h^{-1}$ is operator concave and g^{-1} is operator monotone.

Proof. First, we apply Theorem 2.1 (a) to the function $h\circ g^{-1}$ and replace C_i to $g(C_i)$, then we apply h^{-1} to the inequality coming from (17). This gives (27). A similar argument gives (28): $g\circ h^{-1}$, $C_i = h(C_i)$ and g^{-1}can be used. □

Assume (H$_1$)-(H$_3$), and suppose $|H_{I_l}(j_1,...,j_{l-1})| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. In this case the power means corresponding to $\mathbf{i}^l := (i_1,\ldots,i_l) \in I_l$ $(l = 1,\ldots,k)$ has the form

$$M_r(I_l,\mathbf{i}^l) = M_r(I_k,\mathbf{i}^l) = \left(\frac{\sum\limits_{s=1}^{l} w_{i_j} C^r_{i_j}}{\sum\limits_{s=1}^{l} w_{i_j}}\right)^{\frac{1}{r}}, \quad r \neq 0.$$

Now, for $k \geq l \geq 1$ we introduce the mixed symmetric means related to (18) as follows:

$$M^2_{s,r}(I_l) := \left[\frac{n}{l\,|I_l|}\sum_{\mathbf{i}^l=(i_1,...,i_l)\in I_l}\left(\sum_{j=1}^{l} w_{i_j}\right)\left(M_r\left(I_l,\mathbf{i}^l\right)\right)^s\right]^{\frac{1}{s}}, \quad s \neq 0. \tag{29}$$

Corollary 4.3. *Assume (H$_1$)-(H$_3$), and suppose $|H_{I_l}(j_1,...,j_{l-1})| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. Then*

$$M_s(\mathbf{C},\mathbf{w}) = M^2_{s,r}(I_1) \leq \ldots \leq M^2_{s,r}(I_k) \leq M_r(\mathbf{C},\mathbf{w}). \tag{30}$$

holds if either

(i) $1 \leq s \leq r$ *or*

(ii) $-r \leq s \leq -1$ *or*

(iii) $s \le -1$, $r \ge s \ge 2r$;

while the reverse inequalities hold in (30) if either

(iv) $r \le s \le -1$ *or*

(v) $1 \le s \le -r$ *or*

(vi) $s \ge 1$, $r \le s \le 2r$.

Proof. It comes from Theorem 4.1. □

Assume (H$_1$)-(H$_3$) and (H$_5$), and suppose $|H_{I_l}(j_1,...,j_{l-1})| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \ge l \ge 2)$. We define for $k \ge l \ge 1$ the quasi-arithmetic means with respect to (18) as follows:

$$M^2_{h,g}(I_l) := h^{-1}\left(\frac{n}{l|I_l|}\sum_{(i_1,...,i_l)\in I_l}\left(\sum_{s=1}^{l} w_{i_s}\right) h\circ g^{-1}\left(\frac{\sum_{s=1}^{l} w_{i_s} g(C_{i_s})}{\sum_{s=1}^{l} w_{i_s}}\right)\right). \tag{31}$$

Corollary 4.4. *Assume (H$_1$)-(H$_3$) and (H$_5$), and suppose $|H_{I_l}(j_1,...,j_{l-1})| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \ge l \ge 2)$. Then*

$$M_h = M^2_{h,g}(I_1) \ge \ldots \ge M^2_{h,g}(I_k) \ge M_g, \tag{32}$$

where either $h\circ g^{-1}$ is operator convex and h^{-1} is operator monotone or $h\circ g^{-1}$ is operator concave and $-h^{-1}$ is operator monotone;

$$M_g = M^2_{g,h}(I_1) \le \ldots \le M^2_{g,h}(I_k) \le M_h, \tag{33}$$

where either $g\circ h^{-1}$ is operator convex and $-g^{-1}$ is operator monotone or $g\circ h^{-1}$ is operator concave and g^{-1} is operator monotone.

Proof. Similar to the proof of Corollary 4.2. □

Finally, we apply the results of this section in some special cases. Throughout Remarks 4.5-4.8 and 4.10-4.9, which are based on examples in [2], the conditions (H$_2$)-(H$_3$) (in the mixed symmetric means) and (H$_5$) (in the quasi-arithmetic means) will be assumed.

Remark 4.5. *In the case of Example 3.1, for $n \ge k \ge 1$ (29) becomes*

$$M^2_{s,r}(I_k) = \left(\frac{1}{\binom{n-1}{k-1}}\sum_{1\le i_1<\ldots<i_k\le n}\left(\sum_{j=1}^{k} w_{i_j}\right)\left(M_r(I_k,\mathbf{i}^k)\right)^s\right)^{\frac{1}{s}}, \quad s \ne 0. \tag{34}$$

and (31) has the form

$$M^2_{h,g}(I_k) = h^{-1}\left(\frac{1}{\binom{n-1}{k-1}}\sum_{1\le i_1<\ldots<i_k\le n}\left(\sum_{s=1}^{k} w_{i_s}\right) h\circ g^{-1}\left(\frac{\sum\limits_{s=1}^{k} w_{i_s} g(C_{i_s})}{\sum\limits_{s=1}^{k} w_{i_s}}\right)\right). \tag{35}$$

Remark 4.6. *Under the setting of Example 3.2, for $k \geq 1$ (29) becomes*

$$M^2_{s,r}(I_k) = \left(\frac{1}{\binom{n+k-1}{k-1}}\sum_{1\le i_1\le\ldots\le i_k\le n}\left(\sum_{j=1}^{k} w_{i_j}\right)\left(M_r(I_k,\mathbf{i}^k)\right)^s\right)^{\frac{1}{s}}, \quad s\neq 0. \tag{36}$$

and (31) has the form

$$M^2_{h,g}(I_k) = h^{-1}\left(\frac{1}{\binom{n+k-1}{k-1}}\sum_{1\le i_1\le\ldots\le i_k\le n}\left(\sum_{s=1}^{k} w_{i_s}\right) h\circ g^{-1}\left(\frac{\sum\limits_{s=1}^{k} w_{i_s} g(C_{i_s})}{\sum\limits_{s=1}^{k} w_{i_s}}\right)\right). \tag{37}$$

(34) and (36) represents mixed symmetric means as given in [3]. *Therefore Corollary 4.3 is a generalization of results given in* [3].

Remark 4.7. *Under the setting of Example 3.3, for $k \geq 1$, (29) leads to*

$$M^2_{s,r}(I_k) = \left(\frac{1}{kn^{k-1}}\sum_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k}\left(\sum_{j=1}^{k} w_{i_j}\right)\left(M_r(I_k,\mathbf{i}^k)\right)^s\right)^{\frac{1}{s}}, \quad s\neq 0. \tag{38}$$

and (31) gives

$$M^2_{h,g}(I_k) = h^{-1}\left(\frac{1}{kn^{k-1}}\sum_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k} w_{i_s}\right) h\circ g^{-1}\left(\frac{\sum\limits_{s=1}^{k} w_{i_s} g(C_{i_s})}{\sum\limits_{s=1}^{k} w_{i_s}}\right)\right), \tag{39}$$

respectively.

Remark 4.8. *Under the setting of Example 3.4, for $k = 1,\ldots,n$, (29) gives*

$$M^2_{s,r}(I_k) = \left(\frac{n}{kn(n-1)\ldots(n-k+1)}\sum_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k}\left(\sum_{j=1}^{k} w_{i_j}\right)\left(M_r(I_k,\mathbf{i}^k)\right)^s\right)^{\frac{1}{s}}, \quad s\neq 0. \tag{40}$$

and (31) has the form

$$M^2_{h,g}(I_k) = h^{-1}\left(\frac{n}{kn(n-1)\ldots(n-k+1)}\sum_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k} w_{i_s}\right) h\circ g^{-1}\left(\frac{\sum\limits_{s=1}^{k} w_{i_s} g(C_{i_s})}{\sum\limits_{s=1}^{k} w_{i_s}}\right)\right), \tag{41}$$

respectively.

Remark 4.9. *Under the construction of Example 3.5, (23) is written as*

$$M^1_{s,r}(I_k,k-1)=\left(\frac{1}{k-1}\sum_{i=1}^{n}(c_i-w_i)\left(\frac{\sum_{j=1}^{n}w_jC_j^r-\frac{w_i}{c_i}C_i^r}{1-\frac{w_i}{c_i}}\right)^{\frac{s}{r}}\right)^{\frac{1}{s}},\quad s\neq 0, r\neq 0, \tag{42}$$

while (26) becomes

$$M^1_{h,g}(I_k,k-1)=h^{-1}\left(\frac{1}{k-1}\sum_{i=1}^{n}(c_i-w_i)h\circ g^{-1}\left(\frac{\sum_{r=1}^{n}w_rg(C_r)-\frac{w_i}{c_i}g(C_i)}{1-\frac{w_i}{c_i}}\right)\right). \tag{43}$$

Remark 4.10. *Under the construction of Example 3.6, (22) gives*

$$M^1_{s,r}(I_2,2)=\left(\sum_{\mathbf{i}^2=(i_1,i_2)\in I_2}\left(\sum_{j=1}^{2}\frac{w_{i_j}}{\left[\frac{n}{i_j}\right]+d(i_j)}\right)\left(M_r(I_2,\mathbf{i}^2)\right)^s\right)^{\frac{1}{s}},\quad s\neq 0, \tag{44}$$

while (25) gives

$$\begin{aligned} &M^1_{h,g}(I_2,2)\\ &=h^{-1}\left(\sum_{(i_1,i_2)\in I_2}\left(\sum_{s=1}^{2}\frac{w_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}\right)h\circ g^{-1}\left(\frac{\sum_{s=1}^{2}\frac{w_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}g(C_{i_s})}{\sum_{s=1}^{2}\frac{w_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}}\right)\right). \end{aligned} \tag{45}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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