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COMMON FIXED POINTS OF FOUR MAPPINGS IN CONE METRIC SPACES

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Abstract. In this paper, we prove a unique common fixed point theorem for four self mappings in cone metric spaces without commutativity conditions.

Keywords: coincidence point, commutativity condition; fixed point, cone metric space.

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1. Introduction

In 2007, Huang and Zhang [1] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space. They obtained some fixed point theorems for nonlinear mappings which satisfy different contractive conditions. Subsequently, many authors studied common fixed point theorems in cone metric spaces; see [2-7] and the references therein.

In 2008, Abbas and Jungck [2] established the existence of coincidence points and common fixed points for mappings which satisfy certain contractive conditions in cone metric spaces.

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In 2009, Abbas and Rhoades [4] obtained several fixed point theorems for mappings without appealing to commutativity conditions in cone metric spaces. In this paper, motivated by the above results, we prove a common fixed point theorem for four self mappings in cone metric spaces without commutativity conditions.

2. Preliminaries

The following definitions are due to Huang and Zhang [1].

Definition 1.1. Let B be a real Banach Space and P a subset of B. The set P is called a cone if and only if:

(a) P is closed, non empty and $P \neq \{0\}$;

(b) $a,b \in R$, $a,b \geq 0$,$x,y \in P$ implies $ax+by \in P$;

(c) $x \in P$ and $-x \in P$ implies $x = 0$.

Definition 1.2. Let P be a cone in a Banach Space B, define partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y-x \in P$. We shall write $x < y$ to indicate $x = y$ but $x \neq y$ while $x << y$ will stand for $y-x \in IntP$, where $IntP$ denotes the interior of the set P. This Cone P is called an order cone.

Definition 1.3. Let B be a Banach Space and $P \in B$ be an order cone. The order cone P is called normal if there exists $L > 0$ such that for all $x,y \in B$, $0 \leq x \leq y$ implies $\| x \| \leq L \| y \|$. The least positive number L satisfying the above inequality is called the normal constant of p.

Definition 1.4. Let X be a nonempty set of B . Suppose that the map $d : X \times X \to B$ satisfies:

(d1) $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(d3) $d(x,y) \leq d(x,z)+d(y,z)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Example 1.1. [1] Let $B = R^2$, $P = \{(x,y) \in B \; such \; that : x,y \geq 0\} \subset R^2$, $X = R$ and $d : X \times X \to B$ such that $d(x,y) = (|\, x-y \,|, \alpha \,|\, x-y \,|)$, where $\alpha = 0$ is a constant. Then (X,d) is a cone metric space.

Definition 1.5. Let (X,d) be a cone metric space. We say that $\{x_n\}$ is
(i) a Cauchy sequence if for every c in B with $c >> 0$, there is N such that for all $n,m > N$, $d(x_n,x_m) << c$;
(ii) a convergent sequence if for any $c >> 0$, there is an N such that for all $n > N$, $d(x_n,x) << c$, for some fixed x in X. We denote this $x_n \to x$ as $n \to \infty$.

Lemma 1.1. Let (X,d) be a cone metric space, and let P be a normal cone with normal constant L. Let $\{x_n\}$ be a sequence in X. Then
(i) $\{x_n\}$ converges to x if and only if $d(x_n,x) \to 0$ as $n \to \infty$.
(ii) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n,x_m) \to 0$ as $n,m \to \infty$.

Definition 1.6. [2] Let X be a set and let f, g be two self-mappings of X. A point x in X is called a coincidence point of f and g iff fx = gx. We shall call $w = fx = gx$ a point of coincidence point.

3. Main results

In this section, we obtain a fixed point theorem for four mappings without appealing to commutativity conditions, defined on a cone metric space.

Theorem 3.1. *Let (X,d) be a complete cone metric space with normal constant L. Suppose that the mappings S, T, f and g are four self-maps of X such that $T(X) \subseteq f(X)$, $S(X) \subseteq g(X)$ and satisfying*

$$d(Sx,Ty) \leq \alpha d(fx,gy) + \beta[d(fx,Sx) + d(gy,Ty)] + \gamma[d(fx,Ty) + d(gy,Sx)] \tag{1}$$

for all $x,y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. If (S,f) and (T,g) have a coincidence point in X, then S, T, f and g have a unique common fixed point in X.

Proof. Suppose x_0 is an arbitrary point of X and define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = gx_{2n+1},$$

$$y_{2n+1} = Tx_{2n+1} = fx_{2n+2}.$$

By (1), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \alpha d(fx_{2n}, gx_{2n+1}) + \beta[d(fx_{2n}, Sx_{2n}) + d(gx_{2n+1}, Tx_{2n+1})] \\ &\quad + \gamma[d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sx_{2n})], \\ &\leq \alpha d(y_{2n-1}, y_{2n}) + \beta[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\ &\quad + \gamma[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})], \\ &\leq \alpha d(y_{2n-1}, y_{2n}) + \beta[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\ &\quad + \gamma[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})], \\ &\leq (\alpha + \beta + \gamma) d(y_{2n-1}, y_{2n}) + (\beta + \gamma) d(y_{2n}, y_{2n+1}), \end{aligned}$$

which implies that

$$d(y_{2n}, y_{2n+1}) \leq \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} d(y_{2n-1}, y_{2n})$$

or

$$d(y_{2n}, y_{2n+1}) \leq \lambda d(y_{2n-1}, y_{2n}),$$

where $\lambda = \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} < 1$. Similarly, we find that

$$d(y_{2n+1}, y_{2n+2}) \leq \lambda d(y_{2n}, y_{2n+1}).$$

Therefore, we have

$$d(y_{n+1}, y_{n+2}) \leq \lambda d(y_n, y_{n+1}) \leq ... \leq \lambda^{n+1} d(y_0, y_1).$$

It follows that

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{m-1}, y_m), \\ &\leq [\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1}] d(y_0, y_1), \\ &\leq \frac{\lambda^m}{1 - \lambda} d(y_0, y_1). \end{aligned}$$

Hence, we have

$$\| d(y_n, y_m) \| \leq \frac{\lambda^m}{1-\lambda} L \| d(y_0, y_1) \|,$$

which implies that $d(y_n, y_m) \to 0$ as $n, m \to \infty$. It follows that $\{y_n\}$ is a Cauchy sequence. Since X is complete, we see that there exists z in X such that $\lim_{n\to\infty} y_n = z$. Since $T(X) \subset f(X)$, we find that there exists a point $p \in X$ such that $z = fp$. Let us prove that $z = Sp$. By (1), we have

$$\begin{aligned} d(Sp,z) &\leq d(Sp, Tx_{2n-1}) + d(Tx_{2n-1}, z) \\ &\leq \alpha d(fp, gx_{2n-1}) + \beta[d(fp, Sp) + d(gx_{2n-1}, Tx_{2n-1})] \\ &\quad + \gamma[d(fp, Tx_{2n-1}) + d(gx_{2n-1}, Sp)] + d(Tx_{2n-1}, z). \end{aligned}$$

It follows that

$$\begin{aligned} \| d(Sp,z) \| \leq L\{\alpha \| d(fp, gx_{2n-1}) \| &+ \beta[\| d(fp, Sp) \| + \| d(gx_{2n-1}, Tx_{2n-1}) \|] \\ &+ \gamma[\| d(fp, Tx_{2n-1}) \| + \| d(gx_{2n-1}, Sp) \|] + \| d(Tx_{2n-1}, z) \|]\}. \end{aligned}$$

Letting $n \to \infty$, we get that

$$\begin{aligned} \| d(Sp,z) \| &\leq L\{\alpha \| d(z,z) + \beta[\| d(z,Sp) \| + \| d(z,z) \|] \\ &\quad + \gamma[\| d(z,z) \| + \| d(z,Sp) \|] + L \| d(z,z) \| \\ &\leq L(\beta+\gamma) \| d(z,Sp) \| . \end{aligned}$$

In view of $\alpha + 2\beta + 2\gamma < 1$, we see the contradiction. Therefore, $Sp = z$, $Sp = fp = z$, p is a coincidence point of (S, f). Since $S(X) \subset g(X)$, we see that there exists a point $q \in X$ such that $z = gq$. Next, let us prove that $z = Tq$. By (1), we have

$$\begin{aligned} d(z, Tq) &\leq d(Sp, Tq) \\ &\leq \alpha d(fp, gq) + \beta[d(fp, Sp) + d(gq, Tq)] + \gamma[d(fp, Tq) + d(gq, Sp)]. \end{aligned}$$

From (1.3), we have

$$\begin{aligned}\| d(z,Tq) \| &\leq L\{\| \alpha d(fp,gq)+\beta[d(fp,Sp)+d(gq,Tq)]+\gamma[d(fp,Tq)+d(gq,Sp)] \|\},\\ &\leq L\{\alpha \| d(fp,gq) \| +\beta[\| d(fp,Sp) \| + \| d(gq,Tq) \|]\\ &\quad +\gamma[\| d(fp,Tq) \| + \| d(gq,Sp) \|]\},\\ &\leq L\{\alpha \| d(z,z) \| +\beta[\| d(z,z) \| + \| d(z,Tq) \|]+\gamma[\| d(z,Tq) \| + \| d(z,z) \|]\},\\ &\leq L(\beta+\gamma) \| d(z,Tq) \|,\end{aligned}$$

which is a contradiction since $\alpha+2\beta+2\gamma<1$. Therefore, $Tq=z$, $Tq=gq=z$. q is a coincidence point of (T,g). It follows that $Sp=fp=Tq=gq(=z)$. On the other hand, we have

$$\begin{aligned}d(ffp,fp) &= d(ffp,y_{2n+1})+d(y_{2n+1},fp)\\ &= d(ffp,y_{2n+1})+d(Tx_{2n+1},Sp)\\ &= d(ffp,y_{2n+1})+d(Sp,Tx_{2n+1})\\ &\leq d(ffp,y_{2n+1})+\alpha d(fp,gx_{2n+1})+\beta[d(fp,Sp)+d(gx_{2n+1},Tx_{2n+1})]\\ &\quad +\gamma[d(fp,Tx_{2n+1})+d(gx_{2n+1},Sp)],\\ &\leq d(ffp,y_{2n+1})+\alpha d(fp,gx_{2n+1})+\beta[d(fp,Sp)+d(gx_{2n+1},Tx_{2n+1})]\\ &\quad +\gamma[d(fp,Tx_{2n+1})+d(gx_{2n+1},fp)+d(fp,Sp)],\\ &\leq d(ffp,y_{2n+1})+(\alpha+\gamma)d(fp,gx_{2n+1})+\beta[d(fp,Sp)+d(gx_{2n+1},Tx_{2n+1})]\\ &\quad +\gamma[d(fp,Tx_{2n+1})+d(fp,Sp)].\end{aligned}$$

It follows that

$$\begin{aligned}\| d(ffp,fp) \| \leq L\{&\| d(ffp,y_{2n+1}) \| +(\alpha+\gamma) \| d(fp,gx_{2n+1}) \|\\ &+\beta[\| d(fp,Sp) \| + \| d(gx_{2n+1},Tx_{2n+1} \|)]\\ &+\gamma[\| d(fp,Tx_{2n+1}) \| + \| d(fp,Sp) \|]\}.\end{aligned}$$

Letting $n \to \infty$, we find that

$$\begin{aligned}
\| d(ffp, fp) \| &\leq L\{\| d(ffp,z) \| + (\alpha+\gamma) \| d(fp,z) \| + \beta[\| d(fp,fp) \| + \| d(z,z) \|] \\
&\quad + \gamma[\| d(fp,z) \| + \| d(fp,fp) \|]\}, \\
&\leq L\{\| d(ffp,fp) \| + (\alpha+\gamma) \| d(fp,fp) \| + \beta[\| d(fp,fp) \| + \| d(z,z) \|] \\
&\quad + \gamma[\| d(fp,fp) \| + \| d(fp,fp) \|]\}, \\
&\leq L \| d(ffp,fp) \| .
\end{aligned}$$

Hence, we have $ffp = fp(=z)$, $fp(=z)$ is a fixed point of f. Note that

$$\begin{aligned}
d(SSp, Sp) &= d(SSp, Tq) \\
&\leq \alpha d(fSp, gq) + \beta[d(fSp, SSp) + d(gq, Tq)] + \gamma[d(fSp, Tq) + d(gp, SSp)], \\
&\leq \alpha d(SSp, Sp) + \beta[d(SSp, SSp) + d(Tq, Tq)] + \gamma[d(SSp, Sp) + d(Sp, SSp)], \\
&\leq (\alpha+\beta+2\gamma) d(SSp, Sp),
\end{aligned}$$

which yields from the definition of partial ordering of cone P that $d(SSp,Sp) = 0$ and $SSp = Sp(=z)$. Therefore, we have $SSp = Sp(=z)$, $Sp(=z)$ is a fixed point of S. Note that

$$\begin{aligned}
d(ggq, gq) &= d(ggq, y_{2n}) + d(y_{2n}, gq) \\
&= d(ggq, y_{2n}) + d(Sx_{2n}, Tq) \\
&\leq d(ggq, y_{2n}) + \alpha d(fx_{2n}, gq) + \beta[d(fx_{2n}, Sx_{2n}) + d(gq, Tq)] \\
&\quad + \gamma[d(fx_{2n}, Tq) + d(gq, Sx_{2n})], \\
&\leq d(ggq, y_{2n}) + \alpha d(fx_{2n}, gq) + \beta[d(fx_{2n}, gq) + d(gq, Sx_{2n}) + d(gq, Tq)] \\
&\quad + \gamma[d(fx_{2n}, Tq) + d(gq, Sx_{2n})], \\
&\leq d(ggq, y_{2n}) + (\alpha+\beta) d(fx_{2n}, gq) + \beta[d(gq, Sx_{2n}) + d(gq, Tq)] \\
&\quad + \gamma[d(fx_{2n}, Tq) + d(gq, Sx_{2n})].
\end{aligned}$$

It follows that

$$\begin{aligned}\| d(ggq,gq) \| \leq L\{\| d(ggq,y_{2n}) \| &+ (\alpha+\beta) \| d(fx_{2n},gq) \| \\ &+\beta[\| d(gq,Sx_{2n}) \| + \| d(gq,Tq) \|] \\ &+\gamma[\| d(fx_{2n},Tq) \| + \| d(gq,Sx_{2n}) \|]\}.\end{aligned}$$

Letting $n \to \infty$, we find

$$\begin{aligned}\| d(ggq,gq) \| &\leq L\{\| d(ggq,z) \| + (\alpha+\beta) \| d(z,z) \| \\ &\quad + \beta[\| d(z,z) \| + \| d(z,z) \|] + \gamma[\| d(z,z) \| + \| d(gq,Sx_{2n}) \|]\}, \\ &\leq L \| d(ggq,gq) \| .\end{aligned}$$

Hence, $ggq = gq(= z)$, $gq(= z)$ is a fixed point of g. Note that

$$\begin{aligned}d(Tq,TTq) &= d(Sp,TTq) \\ &\leq \alpha d(fSp,gTq) + \beta[d(fSp,SSp) + d(gq,Tq)] \\ &\quad + \gamma[d(fTq,TTq) + d(gTq,SSp)], \\ &\leq \alpha d(TTq,TTq) + \beta[d(SSp,SSp) + d(Tq,Tq)] + \\ &\quad \gamma[d(TTq,TTq) + d(TTq,TTq)] \\ &\leq 0.\end{aligned}$$

Hence, we have $TTq = Tq(= z)$, $Tq(= z)$ is a fixed point of T. In view of $Sp = fp = Tq = gq(= z)$, we obtain that S, T,f, and g have a common fixed point. Now, we are in a position to show the uniqueness. Let z_1 be another fixed point of S,T, f and g. Then

$$\begin{aligned}d(z,z_1) &= d(Sz,Tz_1) \\ &\leq \alpha d(fz,gz_1) + \beta[d(fz,Sz) + d(gz_1,Tz_1)] + \gamma[d(fz,Tz_1) + d(gz_1,Sz)], \\ &\leq \alpha d(z,z_1) + \beta[d(z,z) + d(z_1,z_1)] + \gamma[d(z,z_1) + d(z_1,z)]. \\ &\leq 0.\end{aligned}$$

Hence $z = z_1$. Therefore S, T, f and g have a unique common fixed point. This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

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