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GENERALIZATIONS OF THE FUNCTIONAL FORM OF JENSEN'S INEQUALITY

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Abstract. The paper deals with the functional forms of Jensen's inequality for continuous convex functions of one variable. Some generalizations are stated by using positive linear functionals on the linear space of real functions. Obtained results are further applied to functional quasi-arithmetic means.

Keywords: convex function; Jensen's inequality; positive linear functional.

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1. Introduction

1.1. Convexity and Affinity

Let $\mathbb{X}$ be a real linear space. Combining pairs of points $a, b \in \mathbb{X}$ and coefficients $\alpha, \beta \in \mathbb{R}$, we obtain the binomial combinations

$$\alpha a + \beta b. \tag{1}$$

If $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, the combination in equation (1) is convex, and its geometric perception is the line segment between a and b. A set $\mathscr{C} \subseteq \mathbb{X}$ is convex if it contains all binomial

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convex combinations of its points. A function $f:\mathscr{C}\to\mathbb{R}$ is convex if the inequality

$$f(\alpha a+\beta b)\leq \alpha f(a)+\beta f(b) \tag{2}$$

holds for all binomial convex combinations $\alpha a+\beta b$ of the set $\mathscr{C}$.

If $\alpha+\beta=1$, the combination in equation (1) is affine, and its perception is the line passing through a and b. A set $\mathscr{A}\subseteq\mathbb{X}$ is affine if it contains all binomial affine combinations of its points. A function $f:\mathscr{A}\to\mathbb{R}$ is affine if it satisfies the equality in equation (2) for all binomial affine combinations $\alpha a+\beta b$ of the set $\mathscr{A}$.

Relying on the mathematical induction, binomial combinations can be replaced with finite combinations. In that case equation (2) becomes the famous Jensen's inequality from 1905, see [3].

1.2. Convex Sets of Functions

Let $\mathscr{X}$ be a non-empty set, and let $\mathbb{X}$ be a subspace of the linear space of all real functions on the domain $\mathscr{X}$. We assume that $\mathbb{X}$ contains the unit function defined by $\mathit{1}(x)=1$ for every $x\in\mathscr{X}$. Let $\mathscr{I}\subseteq\mathbb{R}$ be an interval, and let $\mathbb{X}_{\mathscr{I}}\subseteq\mathbb{X}$ be a subset containing all functions with the image in $\mathscr{I}$. If $\alpha g+\beta h$ is a convex combination of functions $g,h\in\mathbb{X}_{\mathscr{I}}$, then the number convex combination $\alpha g(x)+\beta h(x)$ is in $\mathscr{I}$ for every $x\in\mathscr{X}$, which indicates that the function set $\mathbb{X}_{\mathscr{I}}$ is convex.

1.3. Positive Linear Functionals

A linear functional $L:\mathbb{X}\to\mathbb{R}$ is positive (non-negative) if $L(g)\geq 0$ for every non-negative function $g\in\mathbb{X}$, and L is unital (normalized) if $L(\mathit{1})=1$. If $g\in\mathbb{X}$, then for every unital positive functional L the number $L(g)$ is in the closed interval of real numbers containing the image of the function g.

In 1931, Jessen affirmed the functional form of Jensen's inequality for convex functions of one variable, see [4]. The suitable adaptation is as follows.

Theorem A. *Let $\mathscr{I}\subseteq\mathbb{R}$ be a closed interval, and let function $g\in\mathbb{X}_{\mathscr{I}}$. Let $f:\mathscr{I}\to\mathbb{R}$ be a continuous convex function such that $f(g)\in\mathbb{X}$.*

Then every unital positive linear functional $L : \mathbb{X} \to \mathbb{R}$ *satisfies the inclusion*

$$L(g) \in \mathscr{I}, \tag{3}$$

and the inequality

$$f(L(g)) \leq L(f(g)). \tag{4}$$

If f *is concave, then the reverse inequality is valid in equation* (4)*. If* f *is affine, then the equality is valid in equation* (4)*.*

The interval $\mathscr{I}$ must be closed, otherwise it could happen that $L(g) \notin \mathscr{I}$ as noted in [5]. The function f must be continuous, otherwise it could happen that the inequality in (4) does not apply as pointed out in [9].

2. Main Results

For ease of reference, let us mention the basic formulas related to convex functions on the interval of real numbers. If a and b are different real numbers, say $a < b$, then every real number x can be uniquely presented by the affine combination

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b. \tag{5}$$

The above binomial combination is convex if, and only if, the number x belongs to the interval $[a,b]$. Let $\mathscr{I} \subseteq \mathbb{R}$ be an interval containing $[a,b]$, let $f : \mathscr{I} \to \mathbb{R}$ be a convex function, and let $f^{\text{line}}_{\{a,b\}} : \mathbb{R} \to \mathbb{R}$ be the function of the line passing through the points $(a, f(a))$ and $(b, f(b))$ of the graph of f. Using the affinity of $f^{\text{line}}_{\{a,b\}}$, we have

$$f^{\text{line}}_{\{a,b\}}(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) \ \text{ for } x \in \mathbb{R}, \tag{6}$$

and applying the convexity of f, it follows that

$$f(x) \leq f^{\text{line}}_{\{a,b\}}(x) \ \text{ if } x \in [a,b] \tag{7}$$

and

$$f(x) \geq f^{\text{line}}_{\{a,b\}}(x) \ \text{ if } x \in \mathscr{I} \setminus (a,b). \tag{8}$$

The following is an interesting result concerning a convex function and two unital positive linear functionals.

Theorem 2.1. *Let $\mathscr{I} \subseteq \mathbb{R}$ be a closed interval, let $[a,b] \subseteq \mathscr{I}$, let function $g \in \mathbb{X}_{[a,b]}$, and let function $h \in \mathbb{X}_{\mathscr{I}\setminus(a,b)}$. Let $f : \mathscr{I} \to \mathbb{R}$ be a continuous convex function such that $f(g), f(h) \in \mathbb{X}$.*

If a pair of unital positive linear functionals $L, H : \mathbb{X} \to \mathbb{R}$ satisfies

$$L(g) = H(h), \tag{9}$$

then

$$L(f(g)) \leq H(f(h)). \tag{10}$$

Proof. To prove the inequality in equation (10) we distinguish the case $a < b$, and the case $a = b$.

Suppose that $a < b$. Then it follows that $f(g) \leq f^{\text{line}}_{\{a,b\}}(g)$ by equation in (7), and $f^{\text{line}}_{\{a,b\}}(h) \leq f(h)$ by equation in (8). Using the above inequalities and applying the affinity of $f^{\text{line}}_{\{a,b\}}$, we get

$$\begin{aligned} L(f(g)) &\leq L\left(f^{\text{line}}_{\{a,b\}}(g)\right) = f^{\text{line}}_{\{a,b\}}\left(L(g)\right) \\ &= f^{\text{line}}_{\{a,b\}}\left(H(h)\right) = H\left(f^{\text{line}}_{\{a,b\}}(h)\right) \\ &\leq H(f(h)), \end{aligned} \tag{11}$$

finishing the proof of this case.

Suppose that $a = b$. If a is the interior point of $\mathscr{I}$, then the series of inequalities in equation (11) works with any supporting line function $f^{\text{line}}_{\{a\}}$ at the point a, because $f(a) = f(g) = f^{\text{line}}_{\{a\}}(g)$ and $f^{\text{line}}_{\{a\}}(h) \leq f(h)$. If a is the boundary point of $\mathscr{I}$, then using the continuity of f for every $\varepsilon > 0$ we can find the interior point c of $\mathscr{I}$ satisfying $f(a) \leq f(c) + \varepsilon$, that is, $f(g) \leq f^{\text{line}}_{\{c\}}(g) + \varepsilon$. Applying the function $f^{\text{line}}_{\{c\}}$ to equation (11), we get

$$L(f(g)) \leq H(f(h)) + \varepsilon, \tag{12}$$

and letting ε to zero, we achieve the inequality in equation (10). $\square$

Theorem 2.1 is the generalization of Theorem A, as it shows the next corollary.

Corollary 2.2. *Let $\mathscr{I} \subseteq \mathbb{R}$ be a closed interval, and let function $g \in \mathbb{X}_{\mathscr{I}}$. Let $f : \mathscr{I} \to \mathbb{R}$ be a continuous convex function such that $f(g) \in \mathbb{X}$.*

If a unital positive linear functional $L : \mathbb{X} \to \mathbb{R}$ satisfies the implication (9) $\Rightarrow$ (10) *of Theorem 2.1 for $H = L$, then*

$$f(L(g)) \leq L(f(g)). \tag{13}$$

Proof. Taking the constant function $g_0 = L(g)\mathit{1}$, and so having the condition

$$L(g_0) = L(g) \tag{14}$$

with functions g_0 and g satisfying the requirements of Theorem 2.1, we can apply the assumption, and get

$$L(f(g_0)) \leq L(f(g)). \tag{15}$$

Since $f(g_0) = f(L(g_0))\mathit{1}$ is the constant function value of $f(L(g))$, the left-hand side of the above inequality is equal to $f(L(g))$ which proves the inequality in (13). $\square$

The extension of Theorem 2.1 by using several unital functionals is as follows.

Corollary 2.3. *Let $[a_1, b_1] \subseteq \ldots \subseteq [a_{n-1}, b_{n-1}] \subseteq \mathscr{I}$. Let function $g_1 \in \mathbb{X}_{[a_1,b_1]}$, let functions $g_k \in \mathbb{X}_{[a_k,b_k] \setminus (a_{k-1},b_{k-1})}$ for $k = 2, \ldots, n-1$, and let function $g_n \in \mathbb{X}_{\mathscr{I} \setminus (a_{n-1},b_{n-1})}$. Let $f : \mathscr{I} \to \mathbb{R}$ be a continuous convex function such that $f(g_i) \in \mathbb{X}$.*

If an n-tuple of unital positive linear functionals $L_i : \mathbb{X} \to \mathbb{R}$ satisfies

$$L_1(g_1) = \ldots = L_n(g_n), \tag{16}$$

then

$$L_1(f(g_1)) \leq \ldots \leq L_n(f(g_n)). \tag{17}$$

In order to further generalize Theorem A and Theorem 2.1, we will replace the unital functional L with n functionals L_i satisfying $\sum_{i=1}^{n} L_i(\mathit{1}) = 1$.

Using the above functional collection, and combining Theorem A with the discrete form of Jensen's inequality, we get the following extension of Theorem A.

Corollary 2.4. *Let $\mathscr{I} \subseteq \mathbb{R}$ be a closed interval, and let functions $g_1, \ldots, g_n \in \mathbb{X}_{\mathscr{I}}$. Let $f : \mathscr{I} \to \mathbb{R}$ be a continuous convex function such that $f(g_i) \in \mathbb{X}$.*

Then every n-tuple of positive linear functionals $L_i : \mathbb{X} \to \mathbb{R}$ with $\sum_{i=1}^{n} L_i(1) = 1$ satisfies the inclusion

$$\sum_{i=1}^{n} L_i(g_i) \in \mathscr{I}, \tag{18}$$

and the inequality

$$f\left(\sum_{i=1}^{n} L_i(g_i)\right) \leq \sum_{i=1}^{k} L_i(f(g_i)). \tag{19}$$

Proof. Without loss of generality we can assume that all numbers $\alpha_i = L_i(1) > 0$, and take unital positive linear functionals $M_i = (1/\alpha_i)L_i$. Then all $M_i(g_i) \in \mathscr{I}$ by the inclusion in equation (3), so it is clear that the convex combination

$$\sum_{i=1}^{n} L_i(g_i) = \sum_{i=1}^{n} \alpha_i M_i(g_i) \tag{20}$$

belongs to the interval $\mathscr{I}$. Applying the discrete form of Jensen's inequality to equation (20), and using the inequalities $f(M_i(g_i)) \leq M_i(f(g_i))$ resulting from equation (4), we achieve the inequality in equation (19). □

The extension of Theorem 2.1 follows by using two collections of positive linear functionals. Each collection must satisfy the sum condition respecting the unit function.

Theorem 2.5. *Let $\mathscr{I} \subseteq \mathbb{R}$ be a closed interval, and let $[a,b] \subseteq \mathscr{I}$. Let functions $g_1, \ldots, g_n \in \mathbb{X}_{[a,b]}$, and let functions $h_1, \ldots, h_m \in \mathbb{X}_{\mathscr{I} \setminus (a,b)}$. Let $f : \mathscr{I} \to \mathbb{R}$ be a continuous convex function such that $f(g_i), f(h_j) \in \mathbb{X}$.*

If a pair of n-tuple and m-tuple of positive linear functionals $L_i, H_j : \mathbb{X} \to \mathbb{R}$ with $\sum_{i=1}^{n} L_i(1) = \sum_{j=1}^{m} H_j(1) = 1$ satisfies

$$\sum_{i=1}^{n} L_i(g_i) = \sum_{j=1}^{m} H_j(h_j), \tag{21}$$

then

$$\sum_{i=1}^{n} L_i(f(g_i)) \leq \sum_{j=1}^{m} H_j(f(h_j)). \tag{22}$$

Proof. The sum $\sum_{i=1}^{n} L_i(g_i)$ is in $[a,b]$ by equation (20). The proof goes as in Theorem 2.1. In the case $a<b$, we use equation (11) including the equality

$$\sum_{i=1}^{n} L_i\big(f_{\{a,b\}}^{\mathrm{line}}(g_i)\big) = f_{\{a,b\}}^{\mathrm{line}}\left(\sum_{i=1}^{n} L_i(g_i)\right) \tag{23}$$

for n-tuples g_i and L_i, and the same equality for m-tuples h_j and H_j. $\square$

3. Applications of Theorem 2.1

First, we explore Theorem 2.1 to get a discrete form of the Jensen type inequality.

Corollary 3.1. *Let $\mathscr{I} \subseteq \mathbb{R}$ be an interval, and let $[a,b] \subseteq \mathscr{I}$. Let $\sum_{i=1}^{n} \alpha_i a_i$ be a convex combination where $a_i \in [a,b]$, and let $\sum_{j=1}^{m} \beta_j b_j$ be a convex combination where $b_j \in \mathscr{I} \setminus (a,b)$.*

If the above convex combinations have the common center

$$\sum_{i=1}^{n} \alpha_i a_i = \sum_{j=1}^{m} \beta_j b_j, \tag{24}$$

then every continuous convex function $f : \mathscr{I} \to \mathbb{R}$ satisfies the inequality

$$\sum_{i=1}^{n} \alpha_i f(a_i) \le \sum_{j=1}^{m} \beta_j f(b_j). \tag{25}$$

Proof. We may assume that the interval $\mathscr{I}$ is closed, otherwise we use the convex hull of points b_j. Using $\mathbb{X}$ as the space of all functions on the closed interval $\mathscr{I}$, and applying Theorem 2.1 to the summarizing functional

$$L(g) = \sum_{i=1}^{n} \alpha_i g(a_i) \tag{26}$$

with the function $g(x) = x$ if $x \in [a,b]$ and $g(x) = a$ if $x \in \mathscr{I} \setminus [a,b]$, as well as the functional

$$H(h) = \sum_{j=1}^{m} \beta_j h(b_j) \tag{27}$$

with the function $h(x) = a$ if $x \in (a,b)$ and $h(x) = x$ if $x \in \mathscr{I} \setminus (a,b)$, we realize that the equality in equation (24) implies the inequality in equation (25). $\square$

The inequality in equation (25) also holds for discontinuous convex functions. Corollary 3.1 is the generalization of the discrete form of Jensen's inequality by Corollary 2.2. The related integral forms of inequalities can be obtained by using the integrating functionals.

The implication (9) $\Rightarrow$ (10) of Theorem 2.1 can be extended in a way that

$$c = L(g) = H(h) \tag{28}$$

implies

$$f(c) \leq L(f(g)) \leq H(f(h)). \tag{29}$$

Using Theorem 2.1 with the above implication, we can present the Hermite-Hadamard inequality as follows.

Corollary 3.2. *Let $a < b$, let $g : [a,b] \to \mathbb{R}$ be an integrable function with the image in $[a,b]$, and let $f : [a,b] \to \mathbb{R}$ be a continuous convex function.*

If

$$\frac{1}{b-a}\int_a^b g(x)\,dx = \alpha a + \beta b, \tag{30}$$

then

$$f(\alpha a + \beta b) \leq \frac{1}{b-a}\int_a^b f(g(x))\,dx \leq \alpha f(a) + \beta f(b). \tag{31}$$

Proof. Using $\mathbb{X}$ as the space of all integrable functions on the interval $[a,b]$, and applying the implication (28) $\Rightarrow$ (29) to the integrating functional

$$L(g) = \frac{1}{b-a}\int_a^b g(x)\,dx \tag{32}$$

with the given function g, and the functional

$$H(h) = \alpha h(a) + \beta h(b) \tag{33}$$

with the function $h(a) = a$ and $h(x) = b$ if $x \in (a,b]$, we achieve the inequality in equation (31). □

Putting the identity function g in equation (30), we obtain the classic Hermite-Hadamard's inequality in equation (31). More on this important and interesting inequality can be read in [6] and [2].

4. Functional Quasi-Arithmetic Means

Let $\mathscr{I} \subseteq \mathbb{R}$ be a closed interval, let $g_1, \ldots, g_n \in \mathbb{X}_{\mathscr{I}}$ be functions, and let $\varphi : \mathscr{I} :\to \mathbb{R}$ be a strictly monotone continuous function such that $\varphi(g_i) \in \mathbb{X}$. Let $L_1, \ldots, L_n : \mathbb{X} \to \mathbb{R}$ be positive linear functionals with $\sum_{i=1}^n L_i(\mathit{1}) = 1$. The quasi-arithmetic mean of functions g_i respecting the function φ and functionals L_i can be defined by

$$M_\varphi(L_i, g_i) = \varphi^{-1}\left(\sum_{i=1}^n L_i(\varphi(g_i))\right). \tag{34}$$

The term in parentheses belongs to the interval $\varphi(\mathscr{I})$, and therefore the quasi-arithmetic mean $M_\varphi(L_i, g_i)$ belongs to the interval $\mathscr{I}$.

If we have two strictly monotone continuous functions $\varphi, \psi : \mathscr{I} \to \mathbb{R}$, then we say that ψ is φ-convex if the function $f = \psi \circ \varphi^{-1}$ is convex on the interval $\mathscr{J} = \varphi(\mathscr{I})$. This terminology is taken from [8, Definition 1.19], and the same notation we use for concavity.

The interrelation between different functional quasi-arithmetic means can be established by using Theorem 2.5.

Corollary 4.1. *Let $\mathscr{I} \subseteq \mathbb{R}$ be a closed interval, and let $[a,b] \subseteq \mathscr{I}$. Let functions $g_1, \ldots, g_n \in \mathbb{X}_{[a,b]}$, and let functions $h_1, \ldots, h_m \in \mathbb{X}_{\mathscr{I}\setminus(a,b)}$. Let $\varphi, \psi : \mathscr{I} \to \mathbb{R}$ be strictly monotone continuous functions such that $\varphi(g_i), \varphi(h_j), \psi(g_i), \psi(h_j) \in \mathbb{X}$. Let $L_i, H_j : \mathbb{X} \to \mathbb{R}$ be positive linear functionals with $\sum_{i=1}^n L_i(1) = \sum_{j=1}^m H_j(1) = 1$.*

If ψ is either φ-convex and increasing or φ-concave and decreasing, and if

$$M_\varphi(L_i, g_i) = M_\varphi(H_j, h_j), \tag{35}$$

then

$$M_\psi(L_i, g_i) \leq M_\psi(H_j, h_j). \tag{36}$$

If ψ is either φ-convex and decreasing or φ-concave and increasing, then the reverse inequality is valid in (36).

Proof. Take $\mathscr{J} = \varphi(\mathscr{I})$, $[c,d] = \varphi([a,b])$, and $f = \psi(\varphi^{-1}) : \mathscr{J} \to \mathbb{R}$. We will apply Theorem 2.5 to the functions $u_i = \varphi(g_i) \in \mathbb{X}_{\mathscr{J}}$, and functions $v_j = \varphi(h_j) \in \mathbb{X}_{\mathscr{J}\setminus(c,d)}$. Prove the case where the function ψ is φ-convex and increasing.

Starting with the equality $\varphi(M_\varphi(L_i,g_i)) = \varphi(M_\varphi(H_j,h_j))$, that is,

$$\sum_{i=1}^{n} L_i(u_i) = \sum_{j=1}^{m} H_j(v_j),$$

and relying on Theorem 2.5, we get

$$\sum_{i=1}^{n} L_i(f(u_i)) \le \sum_{j=1}^{m} H_j(f(v_j)).$$

Acting with the increasing function ψ^{-1} to the above inequality, it follows that

$$\psi^{-1}\left(\sum_{i=1}^{n} L_i(f(u_i))\right) \le \psi^{-1}\left(\sum_{j=1}^{m} H_j(f(v_j))\right),$$

which is actually the inequality in (36) since $f(u_i) = \psi(g_i)$ and $f(v_j) = \psi(h_j)$. $\square$

A special case of the quasi-arithmetic means in equation (34) are power means depending on real exponents r. Thus, using the functions

$$\varphi_r(x) = \begin{cases} x^r, & r \neq 0 \\ \ln x, & r = 0 \end{cases} \tag{37}$$

where $x \in (0,\infty)$, we get the power means of order r in the form

$$M_r(L_i,g_i) = \begin{cases} \left(\sum_{i=1}^{n} L_i(g_i^r)\right)^{\frac{1}{r}}, & r \neq 0 \\ \exp\left(\sum_{i=1}^{n} L_i(\ln g_i)\right), & r = 0. \end{cases} \tag{38}$$

To determine the interrelation between different functional power means we will apply Corollary 4.1. In doing so, we will use a closed interval $\mathscr{I} = [\varepsilon,\infty)$ where ε is a positive number, and the equality

$$M_1(L,g_i) = \sum_{i=1}^{n} L_i(g_i). \tag{39}$$

Corollary 4.2. *Let $\mathscr{I} = [\varepsilon,\infty)$ where $\varepsilon > 0$, and let $[a,b] \subset \mathscr{I}$. Let functions $g_1,\dots,g_n \in \mathbb{X}_{[a,b]}$, and let functions $h_1,\dots,h_m \in \mathbb{X}_{\mathscr{I}\setminus(a,b)}$. Let $L_i, H_j : \mathbb{X} \to \mathbb{R}$ be positive linear functionals with $\sum_{i=1}^{n} L_i(1) = \sum_{j=1}^{m} H_j(1) = 1$.*

If

$$M_1(L_i, g_i) = M_1(H_j, h_j), \tag{40}$$

then

$$M_r(L_i, g_i) \leq M_r(H_j, h_j) \ \ if\ r \geq 1 \tag{41}$$

and

$$M_r(L_i, g_i) \geq M_r(H_j, h_j) \ \ if\ r \leq 1. \tag{42}$$

Proof. The proof follows from Corollary 4.1 using the functions $\varphi(x) = x$, and $\psi(x) = x^r$ for $r \neq 0$ or $\psi(x) = \ln x$ for $r = 0$. □

The basic facts relating to quasi-arithmetic and power means can be found in [1]. For more details on different forms of quasi-arithmetic and power means, as well as their refinements, see [5].

A similar consideration can be done for convex functions of several variables relying on McShane's functional results, see [10]. Very general forms of Hölder's and Minkowski's inequality were obtained in [7] by direct application of McShane's form. Further generalizations could be realized by applying the version of Theorem 2.5 for continuous convex functions of several variables.

Conflict of Interests

The author declares that there is no conflict of interests.

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