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ON THE MAXIMUM MODULUS OF A POLYNOMIAL AND ITS DERIVATIVE

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Abstract. If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then it was shown by Dewan et al [K. K. Dewan and Sunil Hans, Generalization of certain well known polynomial inequalities , J. Math. Anal. Appl. 363 (2010) 38–41] that for every real or complex number β with $|\beta| \leq 1$ and $|z| = 1$,

$$|zp'(z)+\frac{n\beta}{2}p(z)| \leq \frac{n}{2}\{(|\frac{\beta}{2}|+|1+\frac{\beta}{2}|)\max_{|z|=1}|p(z)|-(|1+\frac{\beta}{2}|-|\frac{\beta}{2}|)\min_{|z|=1}|p(z)|\}.$$

In this paper, we generalize the above inequality and some related inequalities by extending them to the class of polynomials having no zeros in $|z| < 1$ except s-fold zeros at the origin where $0 \leq s \leq n$. We also establish a compact generalization of some known polynomial inequalities.

Keywords: polynomial; inequality; maximum modulus; derivative.

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1. Introduction and statement of results

According to a well known Bernstein's inequality on the derivative of a polynomial $p(z)$ of degree n, we have

$$\max_{|z|=1}|p'(z)| \leq n\max_{|z|=1}|p(z)|. \tag{1}$$

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The result is best possible and equality holds for the polynomials having all its zeros at the origin (see [14]).

The inequality (1) can be sharpened, if we restrict ourselves to the class of polynomials having no zeros in $|z|<1$.

In fact, P. Erdös conjectured and later Lax [12] proved that if $p(z)\neq 0$ in $|z|<1$, then (1) can be replaced by

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{2}$$

If the polynomial $p(z)$ has all its zeros in $|z|\leq 1$, then it was proved by Turan [15] that

$$\max_{|z|=1}|p'(z)|\geq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{3}$$

The inequalities (2) and (3) are sharp and equalities hold for polynomials having all its zeros on $|z|=1$.

Recently Aziz and Zargar [5] improved inequality (3) and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z|\leq 1$, with s-fold zeros at the origin, then

$$\max_{|z|=1}|p'(z)|\geq \frac{n+s}{2}\max_{|z|=1}|p(z)|+\frac{n-s}{2}\min_{|z|=1}|p(z)|. \tag{4}$$

As an improvement of inequality (2) Jain [11] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then

$$|zp'(z)+\frac{n\beta}{2}p(z)|\leq \frac{n}{2}(|\frac{\beta}{2}|+|1+\frac{\beta}{2}|)\max_{|z|=1}|p(z)|, \tag{5}$$

for every real or complex number β with $|\beta|\leq 1$ and $|z|=1$. The equality holds for $P(z)=az^n+b,\ \ |a|=|b|=1/2$.

Dewan et al [7] proved that if $P(z)$ is a polynomial of degree n and has all its zeros in $|z|<1$, then for every real or complex number β with $|\beta|\leq 1$,

$$\min_{|z|=1}|zp'(z)+\frac{n\beta}{2}p(z)|\geq n|1+\frac{\beta}{2}|\min_{|z|=1}|p(z)|. \tag{6}$$

In the case $p(z)$ having no zeros in $|z|<1$, as a refinment of (5),

$$|zp'(z)+\frac{n\beta}{2}p(z)|\leq \frac{n}{2}\{(|\frac{\beta}{2}|+|1+\frac{\beta}{2}|)\max_{|z|=1}|p(z)|-(|1+\frac{\beta}{2}|-|\frac{\beta}{2}|)\min_{|z|=1}|p(z)|\}, \tag{7}$$

for every real or complex number β with $|\beta|\leq 1$ and $|z|=1$.

In this paper, we first obtain the following generalization of polynomial inequality (6), as follows:

Theorem 1.1. *Let $p(z)$ be a polynomial of degree n, having all its zeros in $|z| \leq 1$, with s-fold zeros at the origin, $0 \leq s \leq n$, then*

$$\min_{|z|=1} |zp'(z) + \beta \frac{n+s}{2} p(z)| \geq |n + \beta \frac{n+s}{2}| \min_{|z|=1} |p(z)|, \tag{8}$$

for every real or complex number β with $|\beta| \leq 1$. The result is best possible and equality holds for the polynomials $p(z) = az^n$.

If we take $s = 0$ in Theorem 1.1, the inequality (8) reduce to inequality (6). According to Lemma 2.1,

$$|zp'(z)| \geq \frac{n+s}{2} |p(z)|,$$

then for suitable argument β, we have

$$|zp'(z) + \beta \frac{n+s}{2} p(z)| = |zp'(z)| - |\beta| \frac{n+s}{2} |p(z)|. \tag{9}$$

Combining (8) and (9), we have

$$\begin{aligned} |zp'(z)| - |\beta| \frac{n+s}{2} |p(z)| &= |zp'(z) + \beta \frac{n+s}{2} p(z)| \\ &\geq \min_{|z|=1} |zp'(z) + \beta \frac{n+s}{2} p(z)| \geq |n + \beta \frac{n+s}{2}| \min_{|z|=1} |p(z)| \\ &\geq \{n - |\beta| \frac{n+s}{2}\} \min_{|z|=1} |p(z)|, \end{aligned}$$

or

$$|zp'(z)| - |\beta| \frac{n+s}{2} |p(z)| \geq \{n - |\beta| \frac{n+s}{2}\} \min_{|z|=1} |p(z)|,$$

equivalently

$$|zp'(z)| \geq |\beta| \frac{n+s}{2} |p(z)| + \{n - |\beta| \frac{n+s}{2}\} \min_{|z|=1} |p(z)|.$$

Making $|\beta| \to 1$, then we have the following interesting result which improve the inequality (4).

Corollary 1.2. *Let $p(z)$ be a polynomial of degree n, having all its zeros in $|z| \leq 1$ with s-fold zeros at the origin, $0 \leq s \leq n$, then for $|z| = 1$, we have*

$$|p'(z)| \geq \frac{n+s}{2} |p(z)| + \frac{n-s}{2} \min_{|z|=1} |p(z)|. \tag{10}$$

If we take $\beta = 0$ in Theorem 1.1, then inequality (8) reduces to the following result, which proved by Aziz and Dawood [1].

Corollary 1.3. *Let $p(z)$ be a polynomial of degree n, having all its zeros in $|z| \leq 1$, then*

$$\min_{|z|=1} |p'(z)| \geq n \min_{|z|=1} |p(z)|. \tag{11}$$

If we take $\beta = -1$ in (8), then we have:

Corollary 1.4. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$, with s-fold zeros at the origin, $0 \leq s \leq n$, then*

$$\min_{|z|=1} |zp'(z) - \frac{n+s}{2} p(z)| \geq \frac{n-s}{2} \min_{|z|=1} |p(z)|. \tag{12}$$

Next by using Theorem 1.1, we generalize the inequality (7), more precisely:

Theorem 1.2. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, except s-fold zeros at the origin, $0 \leq s \leq n$, then for every real or complex number β with $|\beta| \leq 1$ and $|z| = 1$,*

$$\begin{aligned} \max_{|z|=1} |zp'(z) + \beta \frac{n+s}{2} p(z)| \leq & \frac{1}{2}\{(|n + \beta \frac{n+s}{2}| + |s + \beta \frac{n+s}{2}|) \max_{|z|=1} |p(z)| - \\ & (|n + \beta \frac{n+s}{2}| - |s + \beta \frac{n+s}{2}|) \min_{|z|=1} |p(z)|\}. \end{aligned} \tag{13}$$

The result is best possible and equality holds in (13) for $p(z) = z^n + z^s$ and $\beta \geq 0$.

If we take $s = 0$ in Theorem 1.2, then inequality (13) reduces to inequality (7).

If we take $\beta = 0$ in Theorem 1.2, we have the following result which recently proved by Aziz and Zargar [5].

Corollary 1.5. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, except s-fold zeros at the origin, $0 \leq s \leq n$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n+s}{2} \max_{|z|=1} |p(z)| - \frac{n-s}{2} \min_{|z|=1} |p(z)|. \tag{14}$$

The result is best possible and equality holds in (14) for $p(z) = z^n + z^s$.

If we take $\beta = -1$ in Theorem 1.2, we have the follwing generalization of result due to K. K. Dewan [7].

Corollary 1.6. *Let $p(z)$ be a polynomial of degree n, not vanishing in $|z| < 1$, except s-fold zeros at the origin, $0 \leq s \leq n$, then*

$$\max_{|z|=1} |zp'(z) - \frac{n+s}{2} p(z)| \leq \frac{n-s}{2} \max_{|z|=1} |p(z)|. \tag{15}$$

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 2.1. *If $p(z)$ is a polynomial of degree n, having all its zeros in the closed disk $|z| \leq 1$, with s-fold zeros at the origin, $0 \leq s \leq n$, then*

$$|zp'(z)| \geq \frac{n+s}{2} |p(z)|, \quad |z| = 1. \tag{16}$$

This lemma is due to Aziz and Zargar [5].

Lemma 2.2. *Let $F(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$, with s-fold zeros at the origin, $0 \leq s \leq n$ and $p(z)$ be a polynomial of degree not exceeding that of $F(z)$, with s-fold zeros at the origin, $0 \leq s \leq n$. If $|p(z)| \leq |F(z)|$ for $|z| = 1$, then for any $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|z| = 1$,*

$$|zp'(z) + \beta \frac{n+s}{2} p(z)| \leq |zF'(z) + \beta \frac{n+s}{2} F(z)|. \tag{17}$$

Proof. By using the inequality $|p(z)| \leq |F(z)|$ for $|z| = 1$, any zero of $F(z)$ that lies on $|z| = 1$, is the zero of $p(z)$. On the other hand, from Rouche's Theorem, it is obvious that for α with $|\alpha| < 1$, $F(z) + \alpha p(z)$ has as many zeros in $|z| < 1$ as $F(z)$, and so has all of its zeros in $|z| < 1$. Therefore $F(z) + \alpha p(z)$ has all its zeros in $|z| \leq 1$, with s-fold zeros at the origin, $0 \leq s \leq n$. On applying Lemma 2.1, we get

$$|zF'(z) + \alpha zp'(z)| \geq \frac{n+s}{2} |F(z) + \alpha p(z)| \text{ for } |z| = 1.$$

Therefore, for any β with $|\beta| < 1$, we have for $|z| = 1$,

$$(zF'(z) + \alpha zp'(z)) + \beta \frac{n+s}{2} (F(z) + \alpha p(z)) \neq 0,$$

i.e.

$$T(z) = (zF'(z) + \beta\frac{n+s}{2}F(z)) + \alpha(zp'(z) + \beta\frac{n+s}{2}p(z)), \tag{18}$$

will have no zeros on $|z| = 1$. Then for an appropriate choice of the argument of α, one get for $|z| = 1$,

$$|\alpha||zp'(z) + \beta\frac{n+s}{2}p(z)| \neq |zF'(z) + \beta\frac{n+s}{2}F(z)|.$$

Therefore on $|z| = 1$, we have

$$|zp'(z) + \beta\frac{n+s}{2}p(z)| \leq |zF'(z) + \beta\frac{n+s}{2}F(z)|. \tag{19}$$

If inequality (19) is not true, then there is a point $z = z_0$ with $|z_0| = 1$ such that

$$|z_0p'(z_0) + \beta\frac{n+s}{2}p(z_0)| > |z_0F'(z_0) + \beta\frac{n+s}{2}F(z_0)|.$$

Now take

$$\alpha = -\frac{z_0F'(z_0) + \beta\frac{n+s}{2}F(z_0)}{z_0p'(z_0) + \beta\frac{n+s}{2}p(z_0)},$$

then $|\alpha| < 1$ and with this choice of α, we have from (18), $T(z_0) = 0$ for $|z_0| = 1$. But this contradicts the fact that $T(z) \neq 0$ for $|z| = 1$. For β with $|\beta| = 1$, inequality (19) follows by continuity. This is equivalent to the desired result.

If we take $F(z) = z^n \max_{|z|=1}|p(z)|$ in the Lemma 2.2, we have the following result:

Lemma 2.3. *If $p(z)$ is a polynomial of degree n with s-fold zeros at the origin, $0 \leq s \leq n$, then for any β with $|\beta| \leq 1$ and $|z| = 1$,*

$$|zp'(z) + \beta\frac{n+s}{2}p(z)| \leq |n + \beta\frac{n+s}{2}| \max_{|z|=1}|p(z)|.$$

Lemma 2.4. *If $p(z)$ is a polynomial of degree n with s-fold zeros at the origin, $0 \leq s \leq n$, then for any β with $|\beta| \leq 1$ and $|z| = 1$,*

$$|zp'(z) + \beta\frac{n+s}{2}p(z)| + |zq'(z) + \beta\frac{n+s}{2}q(z)| \leq \{|n + \beta\frac{n+s}{2}| + |s + \beta\frac{n+s}{2}|\} \max_{|z|=1}|p(z)|,$$

where

$$q(z) = z^{n+s}\overline{p(\frac{1}{\overline{z}})}.$$

Note that $q(z)$ is a polynomial of degree n with s-fold zeros at the origin,$0 \leq s \leq n$, because $p(z) = z^s h(z)$ which $h(z)$ is a polynomial of degree $n-s$, therefore

$$q(z) = z^{n+s}\overline{p(\frac{1}{\overline{z}})} = z^{n+s}\overline{(\frac{1}{\overline{z}^s}h(\frac{1}{\overline{z}}))} = z^n\overline{h(\frac{1}{\overline{z}})} = z^s(z^{n-s}\overline{h(\frac{1}{\overline{z}})}).$$

Since $h(z)$ is a polynomial of degree $n-s$, where $h(0) \neq 0$, hence the polynomial $z^{n-s}\overline{h(\frac{1}{\overline{z}})}$ is a polynomial of degree $n-s$. Therefore $q(z)$ is a polynomial of degree n with s-fold zeros at the origin, $0 \leq s \leq n$.

Proof. Let $M = \max_{|z|=1}|p(z)|$. For α with $|\alpha| > 1$, it follows by Rouche's Theorem that the polynomial $G(z) = p(z) - \alpha M z^s$ has no zeros in $|z| < 1$, except s-fold zeros at the origin. Correspondingly the polynomial

$$H(z) = z^{n+s}\overline{G(\frac{1}{\overline{z}})},$$

has all its zeros in $|z| \leq 1$ with s-fold zeros at origin and $|G(z)| = |H(z)|$ for $|z| = 1$. Therefore, by Lemma 2.2, for $|\beta| \leq 1$ and $|z| = 1$, we have

$$|zG'(z) + \beta\frac{n+s}{2}G(z)| \leq |zH'(z) + \beta\frac{n+s}{2}H(z)|. \tag{20}$$

On the other hand

$$H(z) = z^{n+s}\overline{G(\frac{1}{\overline{z}})} = z^{n+s}(\overline{p(\frac{1}{\overline{z}})} - \overline{\alpha}Mz^{-s}) = q(z) - \overline{\alpha}Mz^n,$$

or

$$H(z) = q(z) - \overline{\alpha}Mz^n,$$

then by replacement in (20), we have

$$|zp'(z) - \alpha s z^s M + \beta\frac{n+s}{2}(p(z) - \alpha M z^s)| \leq |zq'(z) - n\overline{\alpha}Mz^n + \beta\frac{n+s}{2}(q(z) - \overline{\alpha}Mz^n)|.$$

This implies for $|z| = 1$,

$$\begin{aligned} &|zp'(z) + \beta\frac{n+s}{2}p(z)| - |\alpha||s + \beta\frac{n+s}{2}|M \leq \\ &|(zq'(z) + \beta\frac{n+s}{2}q(z)) - \overline{\alpha}Mz^n(n + \beta\frac{n+s}{2})|. \end{aligned} \tag{21}$$

As $|p(z)| = |q(z)|$ for $|z| = 1$, then $M = \max_{|z|=1} |p(z)| = \max_{|z|=1} |q(z)|$ and $q(z)$ has s-fold zeros at origin. On applying Lemma 2.3 to the polynomial $q(z)$, we have for $|z| = 1$,

$$|zq'(z) + \beta \frac{n+s}{2} q(z)| \leq |n + \beta \frac{n+s}{2}| \max_{|z|=1} |q(z)| < |\alpha||n + \beta \frac{n+s}{2}|M.$$

Therefore from inequality (21), by suitable choice of argument of α, we have $|z| = 1$,

$$|zp'(z) + \beta \frac{n+s}{2} p(z)| - |\alpha|M|s + \beta \frac{n+s}{2}| \leq |\alpha|M|n + \beta \frac{n+s}{2}| - |zq'(z) + \beta \frac{n+s}{2} q(z)|,$$

i.e.

$$|zp'(z) + \beta \frac{n+s}{2} p(z)| + |zq'(z) + \beta \frac{n+s}{2} q(z)| \leq |\alpha|(|n + \beta \frac{n+s}{2}| + |s + \beta \frac{n+s}{2}|)M.$$

Making $|\alpha| \to 1$, Lemma 2.4 follows.

The following lemma is due to Gardner, Govil and Musukula [8].

Lemma 2.5. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n*,* $p(z) \neq 0$ *in* $|z| < k$*,* $(k > 0)$*, then* $m < |p(z)|$ *for* $|z| < k$ *and in particular* $m < |a_0|$*, where* $m = \min_{|z|=k} |p(z)|$.

2. Proofs of the theorems

Proof of the Theorem 1.1. If $p(z)$ has a zero on $|z| = 1$, then inequality (8) is trivial. Therefore we assume that $p(z)$ has all its zeros in $|z| < 1$. If $m = \min_{|z|=1} |p(z)|$, then $m > 0$ and $|p(z)| \geq m$ for $|z| = 1$. Therefore, if $|\lambda| < 1$ then it follows by Rouche's Theorem that the polynomial $G(z) = p(z) - \lambda m z^n$, has all its zeros in $|z| < 1$ with s-fold zeros at the origin, $0 \leq s \leq n$. Also by using Lemma 2.5 for $k = 1$, the polynomial $G(z) = p(z) - \lambda m z^n$ is of degree n, for $|\lambda| < 1$. On applying Lemma 2.1 to the polynomial $G(z)$ of degree n, we get

$$|zG'(z)| \geq \frac{n+s}{2} |G(z)|,$$

i.e.

$$|zp'(z) - \lambda m n z^n| \geq \frac{n+s}{2} |p(z) - \lambda m z^n|,$$

where $|z|=1$.

Therefore for β with $|\beta|<1$, it can be easily verified that the polynomial

$$(zp'(z)-\lambda mnz^n)+\beta\frac{n+s}{2}\{p(z)-\lambda mz^n\},$$

i.e.

$$(zp'(z)+\beta\frac{n+s}{2}p(z))-\lambda mz^n(n+\beta\frac{n+s}{2}),$$

will have no zeros on $|z|=1$. As $|\lambda|<1$, we have for β with $|\beta|<1$ and $|z|=1$,

$$|zp'(z)+\beta\frac{n+s}{2}p(z)|>m|\lambda z^n||n+\beta\frac{n+s}{2}|,$$

i.e.

$$|zp'(z)+\beta\frac{n+s}{2}p(z)|\geq m|n+\beta\frac{n+s}{2}|. \tag{22}$$

For β with $|\beta|=1$, (22) follows by continuity. This completes the proof of Theorem 1.1.

Proof of Theorem 2.2. Let $m=\min\limits_{|z|=1}|p(z)|$, then $m\leq|p(z)|$ for $|z|\leq 1$. Now for λ with $|\lambda|<1$, we have

$$|\lambda m|<m\leq|p(z)|,$$

where $|z|=1$. Hence by Rouche's Theorem the polynomial $G(z)=p(z)-\lambda mz^s$, has no zero in $|z|<1$ except s-fold zeros at the origin. Therefore the polynomial

$$H(z)=z^{n+s}\overline{G(1/\bar{z})}=q(z)-\overline{\lambda}mz^n,$$

will have all its zeros in $|z|\leq 1$ with s-fold zeros at the origin. Also $|G(z)|=|H(z)|$ for $|z|=1$. On the other hand by using Lemma 2.5, the polynomial $q(z)-\overline{\lambda}mz^n$ is of degree n, for $|\lambda|<1$. On applying Lemma 2.2 to the polynomial $H(z)$ of degree n, we have for $|z|=1$,

$$|zG'(z)+\beta\frac{n+s}{2}G(z)|\leq|zH'(z)+\beta\frac{n+s}{2}H(z),$$

i.e.

$$|zp'(z)-\lambda smz^s+\beta\frac{n+s}{2}(p(z)-\lambda mz^s)|\leq|zq'(z)-\overline{\lambda}nmz^n+\beta\frac{n+s}{2}(q(z)-\overline{\lambda}mz^n)|.$$

This implies

$$\begin{aligned}|zp'(z)+\beta\frac{n+s}{2}p(z)-(s+\beta\frac{n+s}{2})\lambda mz^s| \leq \\ |zq'(z)+\beta\frac{n+s}{2}q(z)-(n+\beta\frac{n+s}{2})\overline{\lambda}mz^n|.\end{aligned} \tag{23}$$

Since all the zeros of $q(z)$ lie in $|z|\leq 1$ with s-fold zeros at the origin, $0\leq s\leq n$ and $|p(z)| = |q(z)|$ for $|z|=1$, hence on applying Theorem 1 to the polynomial $q(z)$, we have for $|z|=1$,

$$|zq'(z)+\beta\frac{n+s}{2}q(z)| \geq |n+\beta\frac{n+s}{2}|\min_{|z|=1}|q(z)| = |n+\beta\frac{n+s}{2}|m,$$

where $|z|=1$ and $|\beta|\leq 1$.

Then for an appropriate choice of the argument of λ, we have

$$|zq'(z)+\beta\frac{n+s}{2}q(z)-(n+\beta\frac{n+s}{2})\overline{\lambda}mz^n| = |zq'(z)+\beta\frac{n+s}{2}q(z)|-|n+\beta\frac{n+s}{2}||\lambda|m. \tag{24}$$

By combining (23) and (24), we get for $|z|=1$ and $|\beta|\leq 1$,

$$|zp'(z)+\beta\frac{n+s}{2}p(z)|-|s+\beta\frac{n+s}{2}||\lambda|m \leq |zq'(z)+\beta\frac{n+s}{2}q(z)|-|n+\beta\frac{n+s}{2}||\lambda|m.$$

Equivalently

$$|zp'(z)+\beta\frac{n+s}{2}p(z)| \leq |zq'(z)+\beta\frac{n+s}{2}q(z)|-(|n+\beta\frac{n+s}{2}|-|s+\beta\frac{n+s}{2}|)|\lambda|m.$$

As $|\lambda|\to 1$, we have

$$|zp'(z)+\beta\frac{n+s}{2}p(z)| \leq |zq'(z)+\beta\frac{n+s}{2}q(z)|-(|n+\beta\frac{n+s}{2}|-|s+\beta\frac{n+s}{2}|)m.$$

Which implies for every real or complex number β with $|\beta|\leq 1$ and $|z|=1$,

$$\begin{aligned}2|zp'(z)+\beta\frac{n+s}{2}p(z)| \leq |zp'(z)+\beta\frac{n+s}{2}p(z)|+ \\ |zq'(z)+\beta\frac{n+s}{2}q(z)|-(|n+\beta\frac{n+s}{2}|-|s+\beta\frac{n+s}{2}|)m.\end{aligned}$$

This in conjunction with Lemma 2.4 gives for $|\beta|\leq 1$ and $|z|=1$,

$$\begin{aligned}2|zp'(z)+\beta\frac{n+s}{2}p(z)| \leq (|n+\beta\frac{n+s}{2}|+|s+\beta\frac{n+s}{2}|)\max_{|z|=1}|p(z)| \\ -(|n+\beta\frac{n+s}{2}|-|s+\beta\frac{n+s}{2}|)\min_{|z|=1}|p(z)|.\end{aligned}$$

This completes the proof of Theorem 2.2.

Conflict of Interests

The authors declare that there is no conflict of interests.

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