Available online at http://scik.org
Adv. Inequal. Appl. 2014, 2014:40
ISSN: 2050-7461

STRONG CONVERGENCE THEOREMS FOR FINDING A COMMON SOLUTION OF A VARIATIONAL INEQUALITY PROBLEM AND A FIXED POINT PROBLEM IN A UNIFORMLY CONVEX AND 2-SMOOTH BANACH SPACES

RENU CHUGH, REKHA RANI* AND SANJAY KUMAR

Department of Mathematics, Maharshi Dayanand University, Rohtak (INDIA) – 124001

Copyright © 2014 Chugh, Rani and Kumar. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract: In this paper, we generalize α - inverse strongly accretive mapping to accretive and Lipschitz continuous mapping in uniformly convex and 2-smooth Banach space and prove a strong convergence result for finding common element of the set of fixed points of strictly pseudocontractive mappings and the set of solutions of variational inequality problem. With the help of an example, we find a common solution of a variational inequality problem and a fixed point problem.

Keywords: Fixed Points, Nonexpansive Mappings, Strictly Pseudo-Contractive Mappings, variational inequality problem

2010 AMS Subject Classification: 47H10, 47H09.

1. Introduction:

Throughout this paper, we use E and E^* for a real Banach space and its dual space. The mapping $J: E \to 2^{E^*}$ defined by $J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x\| = \|x\|^*\}$ for all $x \in E$, is called duality mapping. Now we give some definitions:

Definition 1.1 A Banach space E is said to be uniformly convex iff for any $\in$, $0 < \in \leq 2$, the inequalities $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \in$ imply there exists a $\delta > 0$ such that $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$

---

*Corresponding author

Received July 31, 2014

Definition 1.2 A Banach space E is said to be smooth if for each x ∈ $S_E = \{x \in E : \|x\| = 1\}$, there exists a unique functional $j_x \in E^*$ such that $\langle x, j_x \rangle = \|x\| \text{and} \|j_x\| = 1.$

It is clear that if E is smooth, then J is single-valued which is denoted by j. Also if E is a Hilbert space, then J = I, where I is the identity mapping.

Definition 1.3 Let E be a Banach space. Then a function $\rho_E : R^+ \to R^+$ is said to be modulus of smoothness of E if

$$\rho_E(t) = \sup\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = t\}.$$

A Banach space E is said to be uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0.$$

Also every uniformly smooth Banach space is smooth.

Let q > 1. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant c > 0 such that $\rho_E(t) = ct^q$. It is obvious that if E is q-uniformly smooth, then q ≤ 2 and E is uniformly smooth.

Definition 1.4 Let C be a nonempty subset of a Banach space E and T : C → C be any mapping. T is said to be nonexpansive if for all x, y ∈ C,

$$\|Tx - Ty\| \le \|x - y\|. \quad (1.1)$$

T is said to be η - strictly pseudo-contractive if there exists a constant η ∈ (0, 1) such that

$\langle Tx - Ty, j(x-y) \rangle \le \|x-y\|^2 - \eta \|(I-T)x - (I-T)y\|^2$ ∀ x, y ∈ C and for some j(x - y) ∈ J(x - y). (1.2)

(1.2) is equivalent to:

$\langle (I-T)x - (I-T)y, j(x-y) \rangle \ge \eta \|(I-T)x - (I-T)y\|^2$ ∀ x, y ∈ C and for some j(x - y) ∈ J(x - y). (1.3)

Let C and D be nonempty subsets of a Banach space E such that C is nonempty closed convex and D ⊂ C, then a mapping P : C → D is said to be sunny [9] if P(x + t(x – P(x))) = P(x) for all x ∈ C and t ≥ 0, whenever x + t(x – P(x)) ∈ C. A mapping P : C → D is said to be retraction if Px = x for all x ∈ D. P is said to be sunny nonexpansive retraction from C onto D if P is a retraction from C onto D which is also sunny and nonexpansive. The subset D of C is called sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D.

An operator A of C into E is said to be accretive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq 0, \quad \forall x, y \in C.$$

An operator A of C into E is said to be $\alpha-$ inverse strongly accretive if there exists $j(x - y) \in J(x - y)$ and $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Remark 1.5 Every $\alpha-$ inverse strongly accretive operator is accretive and Lipschitz continuous but converse is not true. Also if T is an η-strictly pseudo-contractive mapping, then I – T is η-inverse strongly accretive mapping.

In a Banach space, the variational inequality problem is to find a point $x^* \in C$ such that

$$\langle Ax^*, j(x-x^*)\rangle \geq 0, \forall x \in C \text{ and for some } j(x-x^*) \in J(x-x^*). \tag{1.4}$$

Firstly, this problem was introduced by Aoyama et al. [7]. The set of solutions of a variational inequality problem in a Banach space is denoted by S(C, A), that is,

$$S(C, A) = \{u \in C : \langle Au, J(v-u)\rangle \geq 0, \forall v \in C\}. \tag{1.5}$$

In 2005, in order to find a solution of the variational inequality (1.4), Aoyama et al. [7] obtained a weak convergence theorem as follows :

Theorem 1.6 [7] Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C, let $\alpha > 0$ and let A be inverse strongly accretive operator of C into E with $S(C, A) \neq \phi$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C (x_n - \lambda_n A x_n), \quad n \geq 0,$$

where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in [0, 1]. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in [a, \frac{\alpha}{K^2}]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then $\{x_n\}$ converges weakly to some element z of S(C, A), where K is the 2-uniformly smoothness constant of E.

In 2013, Kangtunyakarn [1] proved a strong convergence theorem for finding a common element of the set of solutions of a finite family of variational inequality problems and the set of fixed points of a nonexpansive mapping and an η-strictly pseudo-contractive mapping in uniformly convex and 2-uniformly smooth spaces.

Firstly, we give a definition.

Definition 1.7 [1] Let C be a nonempty closed convex sebset of a Banach space H. Let $\{T_i\}_{i=1}^N$ be finite family of nonexpansive mappings of C into itself and let $\lambda_1, \lambda_2, \ldots, \lambda_N$, be real numbers such that $0 \leq \lambda_i \leq 1$ for every i = 1, 2, ……, N. Define a mapping K : C → C as follows:

$\mathcal{U}_1 = \lambda_1 T_1 + (1 - \lambda_1)I,$

$\mathcal{U}_2 = \lambda_2 T_2 \mathcal{U}_1 + (1 - \lambda_2)\mathcal{U}_1,$

$\mathcal{U}_3 = \lambda_3 T_3 \mathcal{U}_2 + (1 - \lambda_3)\mathcal{U}_2,$

.

.

$\mathcal{U}_{N-1} = \lambda_{N-1} T_{N-1} \mathcal{U}_{N-2} + (1 - \lambda_{N-1})\mathcal{U}_{N-2},$

$K = \mathcal{U}_N = \lambda_N T_N \mathcal{U}_{N-1} + (1 - \lambda_N)\mathcal{U}_{N-1},$

Such a mapping K is called the K–mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$.

Theorem 1.8 [1] Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. For every i = 1, 2, ……, N, let $A_i : C \to E$ α-inverse strongly accretive mappings. Define a mapping $G_i : C \to C$ by $Q_C(I - \lambda_i A_i)x = G_i x$ for all $x \in C$ and i = 1, 2, ……, N, where $\lambda_i \in (0, \frac{\alpha_i}{K^2})$, K is the 2-uniformly smooth constant of E. Let B : C → C be the K-mapping generated by $G_1, G_2, \ldots, G_N$ and $\rho_1, \rho_2, \ldots, \rho_N$, where $\rho_i \in (0, 1), \forall$ i = 1, 2, ……, N – 1 and $\rho_N \in (0, 1]$. Let T : C → C be a nonexpansive mapping and S : C → C be an η-strictly pseudocontractive mapping with $F = F(S) \cap F(T) \bigcap_{i=1}^N S(C, A_i) \neq \phi$. Define a mapping $B_A : C \to C$ by $T((1 - \alpha)I + \alpha S)x = B_A x, \forall x \in C$ and $\alpha \in (0, \frac{\eta}{K^2})$. For arbitrarily given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \gamma_n B_A x_n, \forall n \geq 1,$

where f : C → C is a contractive mapping and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and satisfy the following conditions :

(i). $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii). $\{\gamma_n\}$, $\{\delta_n\} \subseteq [c, d] \subset (0, 1)$, for some c, d > 0, $\forall$ n ≥ 1,

(iii). $\sum_{n=1}^{\infty}\left|\beta_{n+1} - \beta_n\right| < \infty$, $\sum_{n=1}^{\infty}\left|\gamma_{n+1} - \gamma_n\right| < \infty$, $\sum_{n=1}^{\infty}\left|\delta_{n+1} - \delta_n\right| < \infty$

(iv). $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $q \in F$, whivh solves the following VIP:

$< q - f(q), j(q - p) > \le 0, \forall\ p \in F$.

In 2013, Atid Kangtunyakarn [2], introduced a new mapping, called S^A-mapping to modify the Halpern iterative scheme for finding a common element of two sets of solutions of variational inequality problem and the set of fixed points of a finite family of nonexpansive mappings and the set of fixed points of a finite family of strictly pseudo-contrctive mappings in a uniformly convex and 2-uniformly smooth Banach space.

Firstly, he gave a definition.

Definition 1.9 [2] Let C be a nonempty closed convex subset of a Banach space H. Let $\{S_i\}_{i=1}^{N}$ and $\{T_i\}_{i=1}^{N}$ be two finite families of mappings of C into itself. For each j = 1, 2, ……, N, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define $S^A : C \to C$ as follows:

$\mathcal{U}_0 = T_1 = I$,

$\mathcal{U}_1 = T_1(\alpha_1^1 S_1 \mathcal{U}_0 + \alpha_2^1 \mathcal{U}_0 + \alpha_3^1 I)$,

$\mathcal{U}_2 = T_2(\alpha_1^2 S_2 \mathcal{U}_1 + \alpha_2^2 \mathcal{U}_1 + \alpha_3^2 I)$,

$\mathcal{U}_3 = T_3(\alpha_1^3 S_3 \mathcal{U}_2 + \alpha_2^3 \mathcal{U}_2 + \alpha_3^3 I)$,

.

.

$\mathcal{U}_{N-1} = T_{N-1}(\alpha_1^{N-1} S_{N-1} \mathcal{U}_{N-2} + \alpha_2^{N-1} \mathcal{U}_{N-2} + \alpha_3^{N-1} I)$, (1.6)

$S^A = \mathcal{U}_N = T_N(\alpha_1^N S_N \mathcal{U}_{N-1} + \alpha_2^N \mathcal{U}_{N-1} + \alpha_3^N I)$,

This mapping is called the S^A –mapping generated by $S_1, S_2,\ldots, S_N, T_1, T_2,\ldots T_N$ and $\alpha_1, \alpha_2,\ldots, \alpha_N$.

Theorem 1.10 [2] Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let A, B be α- and β-inverse strongly accretive mappings of C into E, respectively. Let $\{S_i\}_{i=1}^N$ be a finite family of k_i-strict pseudocontractions of C into itself and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself such that F = $\bigcap_{i=1}^N F(S_i) \bigcap_{i=1}^N F(T_i) \cap S(C, A) \cap S(C, B) \neq \phi$ and k = min$\{k_i : i = 1, 2,\ldots,N\}$ with $K^2 \leq k$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1]$, $\alpha_2^j \in [0, 1]$, $\alpha_3^j \in (0, 1)$ for all j = 1, 2, ……, N. Let S^A be the S^A –mapping generated by $S_1, S_2,\ldots, S_N, T_1, T_2,\ldots T_N$ and $\alpha_1, \alpha_2,\ldots, \alpha_N$.

Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and

$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C (I - aA) x_n + \delta_n Q_C (I - bB) x_n + \eta_n S^A x_n$, $n \geq 1$,

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\},\{\eta_n\} \in [0, 1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$ and satisfy the following conditions:

(i). $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii). $\{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \subseteq [c, d] \subset (0, 1)$, for some c, d > 0, $\forall$ n $\geq$ 1,

(iii). $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

$\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$,

(iv). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(v). $a \in (0, \frac{\alpha}{K^2})$ and $b \in (0, \frac{\beta}{K^2})$.

Then $\{x_n\}$ converges strongly to $z_0 = Q_F u$, where Q_F is the sunny nonexpansive retraction of C onto F.

Motivated by the research going on in this direction, we generalize the above mentioned result to more general class of mappings known as accretive and Lipschitz-continuous. Also with the help of a numerical example, we prove the validity of the result.

2. Preliminaries.

In this section, we give some lemmas, which will be used to prove our main result.

Lemma 2.1 [4] Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:

$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x)\rangle + 2\|Ky\|^2$ for any x, y ∈ E.

Lemma 2.2 [11] Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \le r\}, r > 0$. Then there exists a continuous, strictly increasing and convex function g : [0, ∞] → [0, ∞], g(0) = 0 such that

$\|\alpha x + \beta y + \gamma z\|^2 \le \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g(\|x-y\|)$ for all x, y, z ∈ B_r and all α, β, γ ∈ [0, 1] with α + β + γ = 1.

Lemma 2.3 [7] Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then, for all $\lambda > 0$,

$S(C, A) = F(Q_C(I - \lambda A))$.

Lemma 2.4 [5] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying $s_{n+1} = (1-\alpha_n)s_n + \delta_n, \ \forall n \ge 0,$ where $\{\alpha_n\}$ is a sequence in (0, 1) and $\{\delta_n\}$ is a sequence such that

(i). $\sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii). $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$

Then $\lim_{n\to\infty} s_n = 0.$

Lemma 2.5 [2] Let C be a nonempty closed convex subset of a 2-uniformly smooth and uniformly convex Banach space. Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i - strict pseudo-contractions of C into itself and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(S_i) \bigcap \bigcap_{i=1}^N F(T_i) \ne \phi$ and $\kappa = \min\{\kappa_i : i = 1, 2, ..., N\} \text{ with } K^2 \le \kappa$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where I = [0,1], $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1,$ $\alpha_1^j \in (0,1], \alpha_2^j \in [0,1], \alpha_3^j \in (0,1)$ for all j = 1, 2, …, N. Let S^A be the S^A-mapping generated by

$S_1, S_2, ..., S_N, T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S^A) = \bigcap_{i=1}^{N} F(S_i) \bigcap \bigcap_{i=1}^{N} F(T_i)$ and S^A is a nonexpansive mapping.

Lemma 2.6 [2] Let C be a closed convex subset of a strictly convex Banach space E. Let T_1, T_2, T_3 be three nonexpansive mappings from C into itself with $F(T_1) \bigcap F(T_2) \bigcap F(T_3) \neq \phi$. Define a mapping S by $Sx = \alpha T_1 x + \beta T_2 x + \gamma T_3 x$, $\forall$ x ∈ C, where α, β, γ is a constant in (0, 1) and $\alpha + \beta + \gamma = 1$. Then S is nonexpansive and F(S) = $F(T_1) \bigcap F(T_2) \bigcap F(T_3)$.

Lemma 2.7 [4] Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:

$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2\|Ky\|^2$ for any x, y ∈ E.

Lemma 2.8 [10] Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in [0, 1] with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose

$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$ for all integers n ≥ 0 and

$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.9 [8] In a Banach space E, the following inequality holds:

$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle, \forall x, y \in E,$ where j(x + y) = J(x + y).

Lemma 2.10 [6] Let C be a nonempty closed convex subset of a real uniformly smooth Banach space E and let T : C → C be a nonexpansive mapping with a nonempty fixed point F(T). If $\{x_n\}$ ⊂ C is a bounded sequence such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Then there exists a unique sunny nonexpansive retraction $Q_{F(T)} : C \to F(T)$ such that

$\limsup_{n\to\infty} \langle u - Q_{F(T)}u, J(x_n - Q_{F(T)}u)\rangle \leq 0$ for any given u ∈ C.

3. Main Result

Now, we prove our main result.

Theorem 3.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let A and B be accretive and L- Lipschitz continuous mappings of C into E. Let $\{S_i\}_{i=1}^N$ be a finite family of*

k_i-strict pseudocontractions of C into itself and let $\{T_i\}_{i=1}^N$ *be a finite family of nonexpansive mappings of C into itself such that* $\boldsymbol{F} = \bigcap_{i=1}^N F(S_i) \bigcap_{i=1}^N F(T_i) \cap S(C, A) \cap S(C, B) \neq \phi$ *and* $k = \min\{k_i : i = 1, 2, \ldots, N\}$ *with* $K^2 \le k$, *where K is the 2-uniformly smooth constant of E. Let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, *where* $I \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1]$, $\alpha_2^j \in [0, 1]$, $\alpha_3^j \in (0, 1)$ *for all* $j = 1, 2, \ldots, N$. *Let* S^A *be the* S^A *–mapping generated by* $S_1, S_2, \ldots, S_N, T_1, T_2, \ldots T_N$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Let $\{x_n\}$ *be the sequence generated by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n x_1 + \beta_n x_n + \gamma_n Q_C (I - aA) x_n + \delta_n Q_C (I - bB) x_n + \eta_n S^A x_n, \quad n \ge 1,$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\},\{\eta_n\} \in [0, 1]$ *and* $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$ *and satisfy the following conditions:*

(i). $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii). $\{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \subseteq [c, d] \subset (0, 1)$, *for some* $c, d > 0$, $\forall\, n \ge 1$,

(iii). $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$

$\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_n + \delta_n| < \infty$,

(iv). $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(v). $a \in (0, \frac{\alpha}{K^2})$ *and* $b \in (0, \frac{\beta}{K^2})$.

Then $\{x_n\}$ *converges strongly to* $z_0 = Q_F x_1$, *where* Q_F *is the sunny nonexpansive retraction of C onto F.*

Proof. Let $y_n = Q_C(I - aA)x_n$ and $z_n = Q_C(I - bB)x_n$ for all n ≥ 1.

Let $u \in F = \bigcap_{i=1}^N F(S_i) \bigcap \bigcap_{i=1}^N F(T_i) \bigcap S(C, A) \bigcap S(C, B)$. Then

$$\|y_n - u\|^2 \le \|x_n - aAx_n - u\|^2 - \|x_n - aAx_n - y_n\|^2$$

$$= \|x_n - u\|^2 + \|aAx_n\|^2 - 2a\langle x_n - u, j(Ax_n)\rangle - \|x_n - y_n\|^2 - \|aAx_n\|^2 + 2a\langle x_n - y_n, j(Ax_n)\rangle$$

$$= \|x_n - u\|^2 - \|x_n - y_n\|^2 + 2a\langle x_n - y_n - x_n + u, j(Ax_n)\rangle$$

$= \|x_n - u\|^2 - \|x_n - y_n\|^2 + 2a\langle u - y_n, j(\mathrm{A}x_n)\rangle$

$= \|x_n - u\|^2 - \|x_n - y_n\|^2 + 2a(\langle Ax_n - Au, j(\mathrm{u} - \mathrm{x}_n)\rangle + \langle Au, j(\mathrm{u} - \mathrm{x}_n)\rangle + \langle Ax_n, j(\mathrm{x}_n - y_n)\rangle)$

$\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 + 2\langle aAx_n, j(\mathrm{x}_n - y_n)\rangle$

$= \|x_n - u\|^2 - \|x_n - y_n\|^2 + 2\langle x_n + aAx_n - \mathrm{y}_n, j(\mathrm{x}_n - y_n)\rangle + \langle y_n - x_n, j(\mathrm{x}_n - y_n)\rangle$

$\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|x_n - y_n\|^2$

$\leq \|x_n - u\|^2$

$\Longrightarrow \|y_n - u\| \leq \|x_n - u\|$ for $n \geq 1$. (3.1)

Similarly, we can prove that

$\|z_n - u\| \leq \|x_n - u\|$ for $n \geq 1$. (3.2)

Now by induction, we have,

$\|x_n - u\| \leq \|x_1 - u\| \;\; \forall \;\; n \geq 1.$ (3.3)

In fact when n = 1, it follows from (3.1) and (3.2) that

$\|x_2 - u\| = \|\alpha_1 \mathrm{x}_1 + \beta_1 \mathrm{x}_1 + \gamma_1 Q_C(I - aA)\,\mathrm{x}_1 + \delta_1 Q_C(I - bB)\,\mathrm{x}_1 + \eta_1 S^A \mathrm{x}_1 - \mathrm{u}\|$

$\leq \alpha_1\|\mathrm{x}_1 - \mathrm{u}\| + \beta_1\|\mathrm{x}_1 - \mathrm{u}\| + \gamma_1\|\mathrm{y}_1 - \mathrm{u}\| + \delta_1\|\mathrm{z}_1 - \mathrm{u}\| + \eta_1\|S^A \mathrm{x}_1 - \mathrm{u}\|$

$\leq \|x_1 - u\|$, which implies that (3.3) holds for n = 1. Assume that (3.3) holds for $n \geq 2$. Then we have, $\|x_n - u\| \leq \|x_1 - u\|$. Now,

$\|x_{n+1} - u\| = \|\alpha_n \mathrm{x}_1 + \beta_n \mathrm{x}_n + \gamma_n Q_C(I - aA)\,\mathrm{x}_n + \delta_n Q_C(I - bB)\,\mathrm{x}_n + \eta_n S^A \mathrm{x}_n - \mathrm{u}\|$

$\leq \alpha_n\|\mathrm{x}_1 - \mathrm{u}\| + \beta_n\|\mathrm{x}_n - \mathrm{u}\| + \gamma_n\|\mathrm{y}_n - \mathrm{u}\| + \delta_n\|\mathrm{z}_n - \mathrm{u}\| + \eta_n\|S^A \mathrm{x}_n - \mathrm{u}\|$

$\leq \|x_1 - u\|$.

Thus (3.3) holds for n + 1. Therefore (3.3) holds for all $n \geq 1$. Hence $\{x_n\}$ is bounded. And so $\{y_n\}$, $\{z_n\}$ $\{S^A x_n\}$ are bounded. Next, we shall show that

$\lim_{n\to\infty} \|\mathrm{x}_{n+1} - \mathrm{x}_n\| = 0$ (3.4)

Now,

$\|Q_C(I - aA)\,\mathrm{x}_{n+1} - Q_C(I - aA)\,\mathrm{x}_n\|^2 \leq \|(\mathrm{x}_{n+1} - \mathrm{x}_n) - a(A\mathrm{x}_{n+1} - A\mathrm{x}_n)\|^2$

$$\leq \|x_{n+1}-x_n\|^2 - 2a\langle Ax_{n+1}-Ax_n, j(x_{n+1}-x_n)\rangle + 2K^2a^2\|Ax_{n+1}-Ax_n\|^2$$

$$\leq \|x_{n+1}-x_n\|^2 + 2K^2a^2\|Ax_{n+1}-Ax_n\|^2$$

$$\leq \|x_{n+1}-x_n\|^2 + 2K^2a^2L_1^{\,2}\|x_{n+1}-x_n\|^2$$

$$= (1+2K^2a^2L_1^{\,2})\|x_{n+1}-x_n\|^2$$

$$\|Q_C(I-aA)x_{n+1} - Q_C(I-aA)x_n\| \leq (1+\sqrt{2}KaL)\|x_{n+1}-x_n\| \tag{3.5}$$

Similarly, $\|Q_C(I-bB)x_{n+1} - Q_C(I-bB)x_n\| \leq (1+\sqrt{2}KbL)\|x_{n+1}-x_n\|$. (3.6)

By definition of x_n , we can rewrite x_n as

$$x_{n+1} = \beta_n x_n + (1-\beta_n)z_n, \tag{3.7}$$

where

$$z_n = \frac{\alpha_n x_1 + \gamma_n Q_C(I-aA)x_n + \delta_n Q_C(I-bB)x_n + \eta_n S^A x_n}{1-\beta_n}.$$

Now, using (3.5) and (3.6), we have

$$\|z_{n+1}-z_n\| = \left\| \frac{\alpha_{n+1}x_1 + \gamma_{n+1}Q_C(I-aA)x_{n+1} + \delta_{n+1}Q_C(I-bB)x_{n+1} + \eta_{n+1}S^A x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n x_1 + \gamma_n Q_C(I-aA)x_n + \delta_n Q_C(I-bB)x_n + \eta_n S^A x_n}{1-\beta_n} \right\|$$

$$= \left\| \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \right\|$$

$$\leq \left\| \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_{n+1}} \right\| + \left\| \frac{x_{n+1}-\beta_n x_n}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \right\|$$

$$= \frac{1}{1-\beta_{n+1}}\|(x_{n+2}-\beta_{n+1}x_{n+1}) - (x_{n+1}-\beta_n x_n)\| + \left|\frac{1}{1-\beta_{n+1}} - \frac{1}{1-\beta_n}\right| \|x_{n+1}-\beta_n x_n\|$$

$$= \frac{1}{1-\beta_{n+1}} \left\| \begin{array}{l} \alpha_{n+1}x_1 + \gamma_{n+1}Q_C(I-aA)x_{n+1} + \delta_{n+1}Q_C(I-bB)x_{n+1} + \eta_{n+1}S^A x_{n+1} \\ -\alpha_n x_1 - \gamma_n Q_C(I-aA)x_n - \delta_n Q_C(I-bB)x_n - \eta_n S^A x_n \end{array} \right\| + \left|\frac{1}{1-\beta_{n+1}} - \frac{1}{1-\beta_n}\right| \|x_{n+1}-\beta_n x_n\|$$

$$\frac{1}{1-\beta_{n+1}}(|\alpha_{n+1}-\alpha_n|\|x_1\|+\gamma_{n+1}\|Q_C(I-aA)x_{n+1}-Q_C(I-aA)x_n\|+|\gamma_{n+1}-\gamma_n|\|Q_C(I-aA)x_n\|$$
$$+\delta_{n+1}\|Q_C(I-bB)x_{n+1}-Q_C(I-bB)x_n\|+|\delta_{n+1}-\delta_n|\|Q_C(I-bB)x_n\|$$
$$+\eta_{n+1}\|S^Ax_{n+1}-S^Ax_n\|+|\eta_{n+1}-\eta_n|\|S^Ax_n\|)+\left|\frac{1}{1-\beta_{n+1}}-\frac{1}{1-\beta_n}\right|\|x_{n+1}-\beta_nx_n\|$$

$$\leq\frac{1}{1-\beta_{n+1}}(|\alpha_{n+1}-\alpha_n|\|x_1\|+\gamma_{n+1}(1+\sqrt{2}KaL)\|x_{n+1}-x_n\|+|\gamma_{n+1}-\gamma_n|\|Q_C(I-aA)x_n\|$$
$$+\delta_{n+1}(1+\sqrt{2}KbL)\|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|\|Q_C(I-bB)x_n\|$$
$$+\eta_{n+1}\|x_{n+1}-x_n\|+|\eta_{n+1}-\eta_n|\|S^Ax_n\|)+\frac{|\beta_{n+1}-\beta_n|}{(1-\beta_{n+1})(1-\beta_n)}\|x_{n+1}-\beta_nx_n\|$$

$$=\frac{1}{1-\beta_{n+1}}(|\alpha_{n+1}-\alpha_n|\|x_1\|+(\gamma_{n+1}+\delta_{n+1}+\eta_{n+1})\|x_{n+1}-x_n\|+\sqrt{2}KL(a\gamma_{n+1}+b\delta_{n+1})\|x_{n+1}-x_n\|$$
$$+|\gamma_{n+1}-\gamma_n|\|Q_C(I-aA)x_n\|+|\delta_{n+1}-\delta_n|\|Q_C(I-bB)x_n\|$$
$$+|\eta_{n+1}-\eta_n|\|S^Ax_n\|)+\frac{|\beta_{n+1}-\beta_n|}{(1-\beta_{n+1})(1-\beta_n)}\|x_{n+1}-\beta_nx_n\|$$

$$\leq\frac{1}{1-\beta_{n+1}}(|\alpha_{n+1}-\alpha_n|\|x_1\|+\|x_{n+1}-x_n\|+\sqrt{2}KL(a\gamma_{n+1}+b\delta_{n+1})\|x_{n+1}-x_n\|$$
$$+|\gamma_{n+1}-\gamma_n|\|Q_C(I-aA)x_n\|+|\delta_{n+1}-\delta_n|\|Q_C(I-bB)x_n\|$$
$$+|\eta_{n+1}-\eta_n|\|S^Ax_n\|)+\frac{|\beta_{n+1}-\beta_n|}{(1-\beta_{n+1})(1-\beta_n)}\|x_{n+1}-\beta_nx_n\|$$

Now using conditions (i) - (iv), we obtain,

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\leq 0$$

Using Lemma 2.8 and (3.7), we obtain

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.8}$$

Also, by (3.7), we have

$$\|x_{n+1}-x_n\|=(1-\beta_n)\|z_n-x_n\|$$

By condition (iv) and (3.8), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$$

Next, we shall show that

$$\lim_{n\to\infty}\| Q_C(I-aA)x_n - x_n\| = \lim_{n\to\infty}\| Q_C(I-bB)x_n - x_n\| = \lim_{n\to\infty}\| S^A x_n - x_n\| = 0 \tag{3.9}$$

Using definition of x_n, we can write

$$\|x_{n+1}-u\|^2 =$$

$$\|\alpha_n(x_1-u)+\beta_n(x_n-u)+\gamma_n(Q_C(I-aA)x_n-u)+\delta_n(Q_C(I-bB)x_n-u)+\eta_n(S^A x_n-u)\|^2$$

$$= \left\| \begin{array}{l} \beta_n(x_n-u)+\gamma_n(Q_C(I-aA)x_n-u) \\ +(\alpha_n+\delta_n+\eta_n)(\frac{\alpha_n(x_1-u)}{\alpha_n+\delta_n+\eta_n}+\frac{\delta_n(Q_C(I-bB)x_n-u)}{\alpha_n+\delta_n+\eta_n}+\frac{\eta_n(S^A x_n-u)}{\alpha_n+\delta_n+\eta_n} \end{array} \right\|^2$$

$= \| \beta_n(x_n-u)+\gamma_n(Q_C(I-aA)x_n-u)+c_n z_n \|^2$, where $c_n = \alpha_n+\delta_n+\eta_n$ and

$$z_n = \frac{\alpha_n(x_1-u)}{\alpha_n+\delta_n+\eta_n}+\frac{\delta_n(Q_C(I-bB)x_n-u)}{\alpha_n+\delta_n+\eta_n}+\frac{\eta_n(S^A x_n-u)}{\alpha_n+\delta_n+\eta_n}.$$

By Lemma 2.2, we have

$$\|x_{n+1}-u\|^2 \le \beta_n\|x_n-u\|^2+\gamma_n\|Q_C(I-aA)x_n-u\|^2+c_n\|z_n-u\|^2-\beta_n\gamma_n g_1(\|x_n-Q_C(I-aA)x_n\|)$$

$$\le \begin{array}{l} (\beta_n+\gamma_n)\|x_n-u\|^2+2K^2a^2L^2\gamma_n\|x_n-u\|^2+\alpha_n\|x_1-u\|^2+\delta_n\|x_n-u\|^2 \\ +2K^2b^2L^2\delta_n\|x_n-u\|^2+\eta_n\|x_n-u\|^2-\beta_n\gamma_n g_1(\|x_n-Q_C(I-aA)x_n\|) \end{array}$$

$$= \begin{array}{l} (\beta_n+\gamma_n+\delta_n+\eta_n)\|x_n-u\|^2+\alpha_n\|x_1-u\|^2+2K^2L^2(a^2\gamma_n+b^2\delta_n)\|x_n-u\|^2 \\ -\beta_n\gamma_n g_1(\|x_n-Q_C(I-aA)x_n\|) \end{array}$$

$$\le \|x_n-u\|^2+\alpha_n\|x_1-u\|^2+2K^2L^2(a^2\gamma_n+b^2\delta_n)\|x_n-u\|^2-\beta_n\gamma_n g_1(\|x_n-Q_C(I-aA)x_n\|)$$

$$\Rightarrow \beta_n\gamma_n g_1(\|x_n-Q_C(I-aA)x_n\|)$$

$$\le \|x_n-u\|^2-\|x_{n+1}-u\|^2+\alpha_n\|x_1-u\|^2+2K^2L^2(a^2\gamma_n+b^2\delta_n)\|x_n-u\|^2$$

$$\le (\|x_n-u\|+\|x_{n+1}-u\|)\|x_{n+1}-x_n\|+\alpha_n\|x_1-u\|^2+2K^2L^2(a^2\gamma_n+b^2\delta_n)\|x_n-u\|^2$$

Using (3.4) and conditions (i) and (iii), we get

$$\lim_{n\to\infty} g_1(\|x_n-Q_C(I-aA)x_n\|)=0.$$

By using property of g_1, we obtain

$$\lim_{n\to\infty}\|x_n-Q_C(I-aA)x_n\|=0. \tag{3.10}$$

Applying the same method as in (3.10), we can obtain

$\lim_{n\to\infty}\| Q_C(I-bB)x_n - x_n\| = \lim_{n\to\infty}\| S^A x_n - x_n\| = 0$.

Set Gx = $\alpha S^A x + \beta Q_C(I-aA)x + \gamma Q_C(I-bB)x$, $\forall$ x $\in$ C and $\alpha + \beta + \gamma = 1$. By Lemma 2.6, we obtain, $F(G) = \bigcap F(Q_C(I-aA))\bigcap F(Q_C(I-bB))\bigcap F(Q_C S^A)$. Using Lemma 2.3 and 2.5, we can say that

F = $\bigcap_{i=1}^{N} F(S_i)\bigcap_{i=1}^{N} F(T_i) \cap S(C, A) \cap S(C, B) = F(G)$.

By definition of G,

$$\|Gx_n - x_n\| \le \alpha\|S^A x_n - x_n\| + \beta\|Q_C(I-aA)x_n - x_n\| + \gamma\|Q_C(I-bB)x_n - x_n\|.$$

Using (3.9), we can say that

$$\lim_{n\to\infty}\|Gx_n - x_n\| = 0. \tag{3.11}$$

By Lemma (2.10) and (3.11), we obtain

$$\limsup_{n\to\infty}\langle x_1 - z_0, j(x_n - z_0)\rangle \le 0, \tag{3.12}$$

where $z_0 = Q_F x_1$. Now we shall prove that the sequence $\{x_n\}$ converges strongly to $z_0 = Q_F x_1$. By definition of x_n,

$$\|x_{n+1} - z_0\|^2 =$$

$$\|\alpha_n(x_1 - z_0) + \beta_n(x_n - z_0) + \gamma_n(Q_C(I-aA)x_n - z_0) + \delta_n(Q_C(I-bB)x_n - z_0) + \eta_n(S^A x_n - z_0)\|^2$$

$$= \left\| \alpha_n(x_1 - z_0) + (1-\alpha_n)\Big(\frac{\beta_n(x_n - z_0)}{1-\alpha_n} + \frac{\gamma_n(Q_C(I-aA)x_n - z_0)}{1-\alpha_n} + \frac{\delta_n(Q_C(I-bB)x_n - z_0)}{1-\alpha_n} + \frac{\eta_n(S^A x_n - z_0)}{1-\alpha_n}\Big) \right\|^2$$

$$\le \left\| (1-\alpha_n)\Big(\frac{\beta_n(x_n - z_0)}{1-\alpha_n} + \frac{\gamma_n(Q_C(I-aA)x_n - z_0)}{1-\alpha_n} + \frac{\delta_n(Q_C(I-bB)x_n - z_0)}{1-\alpha_n} + \frac{\eta_n(S^A x_n - z_0)}{1-\alpha_n}\Big) \right\|^2$$

$$+ 2\alpha_n\langle x_1 - z_0, j(x_{n+1} - z_0)\rangle$$

$$\le (1-\alpha_n)\|x_n - z_0\|^2 + 2K^2L^2(a^2\gamma_n + b^2\delta_n)\|x_n - z_0\|^2 + 2\alpha_n\langle x_1 - z_0, j(x_{n+1} - z_0)\rangle$$

Using Lemma (2.4) and conditions (i) and (iii), we obtain

$$\lim_{n\to\infty}\|x_n - z_0\| = 0.$$

$\Longrightarrow x_n \to z_0$ as $n \to \infty$.

4. Applications.

Using our main result, we prove a strong convergence theorem as in [2]. First we give a lemma.

Lemma 4.1 [2] Let C be a nonempty closed convex sebset of a uniformly convex and 2-uniformly smooth Banach space E.

Let $\{S_i\}_{i=1}^{N}$ be a finite family of k_i-strict pseudocontractions of C into itself such that F = $\bigcap_{i=1}^{N} F(S_i) \neq \phi$ and $k = \min\{k_i : i = 1, 2,\ldots\ldots,N\}$ with $K^2 \leq k$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1]$, $\alpha_2^j \in [0, 1]$, $\alpha_3^j \in (0, 1)$ for all j = 1, 2, ……, N. Let S be the S–mapping generated by S_1, S_2,….., S_N and $\alpha_1, \alpha_2,\ldots.., \alpha_N$. Then $F(S) = \bigcap_{i=1}^{N} F(S_i)$ and S is a nonexpansive mapping.

Theorem 4.2 Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let A and B be accretive and L- Lipschitz continuous mappings of C into E. Let $\{S_i\}_{i=1}^{N}$ be a finite family of k_i-strict pseudocontractions of C into itself such that **F** = $\bigcap_{i=1}^{N} F(S_i) \cap S(C, A) \cap S(C, B) \neq \phi$ and $k = \min\{k_i : i = 1, 2,\ldots\ldots,N\}$ with $K^2 \leq k$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1]$, $\alpha_2^j \in [0, 1]$, $\alpha_3^j \in (0, 1)$ for all j = 1, 2, ……, N. Let S be the S–mapping generated by S_1, S_2,….., S_N and α_1, $\alpha_2,\ldots.., \alpha_N$.

Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and

$x_{n+1} = \alpha_n x_1 + \beta_n x_n + \gamma_n Q_C (I - aA) x_n + \delta_n Q_C (I - bB) x_n + \eta_n S^A x_n$, $n \geq 1$,

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\},\{\eta_n\} \in [0, 1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$ and satisfy the following conditions:

(i). $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii). $\{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \subseteq [c, d] \subset (0, 1)$, for some $c, d > 0$, $\forall\, n \geq 1$, (iii). $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty}|\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty}|\delta_{n+1} - \delta_n| < \infty$

$\sum_{n=1}^{\infty}|\eta_{n+1} - \eta_n| < \infty$, $\sum_{n=1}^{\infty}|\gamma_n + \delta_n| < \infty$,

(iv). $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(v). $a \in (0, \frac{\alpha}{K^2})$ and $b \in (0, \frac{\beta}{K^2})$.

Then $\{x_n\}$ converges strongly to $z_0 = Q_F x_1$, where Q_F is the sunny nonexpansive retraction of C onto F.

Proof. By putting $I = T_1 = T_2 = \ldots = T_N$ in Theorem 3.1 and by using Lemma 4.1, the desired can be obtained.

Theorem 4.3 Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let A_i, A and B be accretive and L- Lipschitz continuous mappings of C into E. Define a mapping $G_i : C \to C$ by $Q_C(I - \lambda_i A_i)x = G_i x$ for all $x \in C$ and i = 1, 2, ……, N, where $\lambda_i \in (0, \frac{\alpha_i}{K^2})$, K is the 2-uniformly smooth constant of E.Let $\{S_i\}_{i=1}^N$ be a finite family of k_i-strict pseudocontractions of C into itself such that $\mathbf{F} = \bigcap_{i=1}^{N} F(S_i) \bigcap_{i=1}^{N} S(C, A_i) \cap S(C, A) \cap S(C, B) \neq \phi$ and $k = \min\{k_i : i = 1, 2, \ldots, N\}$ with $K^2 \leq k$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1]$, $\alpha_2^j \in [0, 1]$, $\alpha_3^j \in (0, 1)$ for all j = 1, 2, ……, N. Let S^A be the S^A –mapping generated by $S_1, S_2, \ldots, S_N, T_1, T_2, \ldots T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and

$x_{n+1} = \alpha_n x_1 + \beta_n x_n + \gamma_n Q_C (I - aA) x_n + \delta_n Q_C (I - bB) x_n + \eta_n S^A x_n$, $n \geq 1$,

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\},\{\eta_n\} \in [0, 1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$ and satisfy the following conditions:

(i). $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii). $\{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \subseteq [c, d] \subset (0, 1)$, for some $c, d > 0$, $\forall\, n \geq 1$, (iii). $\sum_{n=1}^{\infty}\left|\alpha_{n+1} - \alpha_n\right| < \infty$, $\sum_{n=1}^{\infty}\left|\beta_{n+1} - \beta_n\right| < \infty$, $\sum_{n=1}^{\infty}\left|\gamma_{n+1} - \gamma_n\right| < \infty$, $\sum_{n=1}^{\infty}\left|\delta_{n+1} - \delta_n\right| < \infty$

$\sum_{n=1}^{\infty}\left|\eta_{n+1} - \eta_n\right| < \infty$, $\sum_{n=1}^{\infty}\left|\gamma_n + \delta_n\right| < \infty$,

(iv). $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(v). $a \in (0, \frac{\alpha}{K^2})$ and $b \in (0, \frac{\beta}{K^2})$.

Then $\{x_n\}$ converges strongly to $z_0 = Q_F x_1$, where Q_F is the sunny nonexpansive retraction of C onto F.

Proof. By Lemma 2.3, we have $F(G_i) = S(C, A_i)$ for all $I = 1, 2,\ldots, N$. Using Theorem 3.1, the desired result can be obtained.

5. Numerical Example:

In this section, we use the iterative scheme given below.

$x_{n+1} = \alpha_n x_1 + \beta_n x_n + \gamma_n Q_C (I - aA) x_n + \delta_n Q_C (I - bB) x_n + \eta_n S^A x_n$, $n \ge 1$,

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\},\{\eta_n\} \in [0, 1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$ and satisfy the following conditions:

(i). $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii). $\{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \subseteq [c, d] \subset (0, 1)$, for some $c, d > 0$, $\forall\, n \ge 1$,

(iii). $\sum_{n=1}^{\infty}\left|\alpha_{n+1} - \alpha_n\right| < \infty$, $\sum_{n=1}^{\infty}\left|\beta_{n+1} - \beta_n\right| < \infty$, $\sum_{n=1}^{\infty}\left|\gamma_{n+1} - \gamma_n\right| < \infty$, $\sum_{n=1}^{\infty}\left|\delta_{n+1} - \delta_n\right| < \infty$,

$\sum_{n=1}^{\infty}\left|\eta_{n+1} - \eta_n\right| < \infty$, $\sum_{n=1}^{\infty}\left|\gamma_n + \delta_n\right| < \infty$,

(iv). $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(v). $a \in (0, \frac{\alpha}{K^2})$ and $b \in (0, \frac{\beta}{K^2})$.

We use the following numerical values for the above mentioned iterative scheme.

Let $\{T_i\}_{i=1}^{n}$ be the family of nonexpansive mappings defined by $T_n x = \frac{x}{n+2}$, $n \ge 1$ and $\{S_i\}_{i=1}^{n}$ be the family of pseudo contractive mappings defined as $S_n x = \frac{x^2}{1+x}$ and let

$S^A = T_N(\alpha_1^N S_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I)$. Also let Q_C be a sunny nonexpansive retraction mapping from E onto C defined as $Q_C x = \{0\}, \ \forall x \in E$, where E = [0,1] and C = {0},

Let A and B be accretive and L- Lipschitz continuous mappings of C into E defined as

$$Ax = \frac{x^2}{1+x}$$

$$Bx = \frac{x^2}{1+x}, \text{ where } x \in C$$

The initial values used in C++ program to find the solution are

α_1^i=0.7, α_2^i=0.2, α_3^i=0.1 where i=1,2,3….N.

$\alpha_n = \frac{1}{n}, n \geq 1$, $\beta_n = .000001$, $\gamma_n = .000000001$, $\delta_n = .0000000000001$, $\eta_n = 0.999999$, $x_1 = 0.5$ and $x_0 = 0.5$, by using these mappings and initial value in C++ program, we get the following observation shown in tabular form

Table 5.1

N	1	2	3	19	20	38	39	53	54	55	56	57	58
x_n	0.5	5e-007	5e-013	5e-109	5e-115	5e-223	5e-229	5.00001e-313	4.99999e-319	0	0	0	0

From the above table, we find that $x_n \to 0$ as $n \to \infty$ which is the solution of our problem.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] A Kangtunyakarn, S Suantai, A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings. Nonlinear Anal. 71 (2009), 4448-4460.

[2] A Kangtunyakarn, A new mapping for finding a common element of the sets of fixed points of two finite families of nonexpansive and strictly pseudo-contractive mappings and two sets of variational inequalities in uniformly convex and 2-smooth Banach spaces. Fixed Point Theory and Applications, 2013 (2013), Article ID 157.

[3] A Kangtunyakarn, Iterative scheme for a nonexpansive mapping, an η-strictly pseudo-contractive mapping and variational inequality problems in a uniformly convex and 2-uniformly smooth Banach space. Fixed Point Theory Appl. 2013 (2013), Article ID 23.

[4] HK Xu, Inequalities in Banach spaces with applications. Nonlinear Anal. 16 (1991), 1127-1138.

[5] HK Xu, An iterative approach to quadratic optimization. J. Optim. Theory Appl. 116 (2003), 659-678.

[6] H Zhou, Convergence theorems for λ-strict pseudocontractions in 2-uniformly smooth Banach spaces. Nonlinear Anal. 69(2008), 3160-3173.

[7] K Aoyama, H Iiduka, W Takahashi, Weak convergence of an iterative sequence for accretive operators in Banach spaces. Fixed Point Theory Appl. 2006 (2006), Article ID 35390.

[8] SS Chang, On Chidumes open questions and approximate solutions of multivalued strongly accretive mapping equations in Banach spaces. J. Math. Anal. Appl. 216 (1997), 94-111.

[9] S Reich, Asymptotic behaviour of contractions in Banach spaces. J. Math. Anal. Appl. 44 (1973), 57-70.

[10] T Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals. J. Math. Anal. Appl. 305 (2005), 227-239.

[11] YJ Cho, HY Zhou, G Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings. Comput. Math. Appl. 47 (2004), 707-717.