Available online at http://scik.org
Adv. Inequal. Appl. 2014, 2014:41
ISSN: 2050-7461

IDENTICALNESS OF N-NORMS

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Abstract. In this paper, we discuss the concept of n-norm and n-normed spaces. Further we prove the equality of seven formulae of n-norms on a Hilbert space and eight formulae of n-norms on a separable Hilbert space. An alternative formula of n-norm on the dual of an n-normed space is introduced. Also, we show its equality with two alternative formulae.

Keywords: Hilbert space; dual space; inner product; n-norm; separable space.

2010 AMS Subject Classification: 46B20, 46C05, 46C15.

1. Introduction

Let X be a real vector space with dim $X \geq n$, where n is a positive integer. A real valued function $\|.,\dots,.\| : X^n \to \mathbb{R}$ is called an n-norm on X if the following conditions hold:

(1) $\|x_1,\dots,x_n\|$=0 iff $x_1,\dots,x_n$ are linearly dependent.

(2) $\|x_1,\dots,x_n\|$ is invariant under permutations of $x_1,\dots,x_n$.

(3) $\|\alpha x_1,x_2,\dots,x_n\| = |\alpha|\,\|x_1,\dots,x_n\|$ for any $\alpha \in \mathbb{R}$.

(4) $\|x_0+x_1,x_2,\dots,x_n\| \leq \|x_0,\dots,x_n\| + \|x_1,\dots,x_n\|$ for all $x_0,x_1,\dots,x_n \in X$.

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Received August 27, 2014

The pair $(X,\|.,...,.\|)$ is called an n-normed space. An n-norm is always non-negative. The combination of conditions (3) and (4) above gives the non-negativity of an n-norm. If X is an n-normed space with dual X',the following formula (as formulated by Gähler[2])

$$\|x_1,...,x_n\|^G = \underset{f_j\in X',\|f_j\|\le 1}{Sup}\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}$$

defines an n-norm on X.

If X is equipped with an inner product $\langle .,.\rangle$, we can define the standard n-norm on X by $\|x_1,...,x_n\|^S = \sqrt{\det\left[\langle x_i,x_j\rangle\right]}$. Note that the value of $\|x_1,...,x_n\|^S$ represents the volume of n-dimensional parallelepiped spanned by $x_1,...,x_n$.Let X be a Hilbert space with dual $X^{'}$ .Then Gählers formula on X becomes

$$\|x_1,...,x_n\|^G = \underset{y_j\in X,\|y_j\|\le 1}{Sup} \det\left[\langle x_i,y_j\rangle\right].$$

Also the function

$$\|x_1,...,x_n\|^D = \underset{y_j\in X,\|y_1,...,y_n\|^S\le 1}{Sup}\begin{vmatrix} \langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle\end{vmatrix}$$

defines an n-norm on a Hilbert space X .Then $\|.,...,.\|^G$ and $\|.,...,.\|^D$are identical on a Hilbert space X [6]. If X is a separable Hilbert space and $\{e_1,e_2,...\}$ is a complete orthonormal set in X, we can define an n-norm on X by $\|x_1,...,x_n\|_2 = \left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}\left|\det\left[\alpha_{i j_k}\right]\right|^2\right]^{\frac{1}{2}}$ Where $\alpha_{ij} = \langle x_i,e_j\rangle$ [5],[6].

Further, the function

$$\|x_1,...,x_n\|^E = \underset{y_j\in X,\|y_1,...,y_n\|^S=1}{Sup}\begin{vmatrix} \langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle\end{vmatrix}$$

defines an n-norm on a Hilbert space and the function

$$\|x_1,...,x_n\|^r = \underset{f_j\in X', \|f_j\|=1}{Sup} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

defines an n-norm on a normed space X with dual X' , Singh and Meitei [10]. If X is a Hilbert space , $\|.,...,\|^r$ becomes

$$\|x_1,...,x_n\|^r = \underset{y_j\in X, \|y_j\|=1}{Sup} \det\left[\langle x_i, y_j\rangle\right].$$

The function

$$\|x_1,...,x_n\|^F = \underset{y_j\in X, \|y_1,...,y_n\|^S\neq 0}{Sup} \frac{\begin{vmatrix} \langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle \end{vmatrix}}{\|y_1,...,y_n\|^S}$$

defines an n-norm on a Hilbert space X [11].

Then $\|.,...,.\|^D, \|.,...,.\|^E, \|.,...,.\|^F, \|.,...,.\|^G, \|.,...,.\|^r$ and $\|.,...,.\|^S$ are identical on a Hilbert space and they are identical with $\|.,...,.\|_2$ on a separable Hilbert space [10],[11]. Also,

$$\|f_1,...,f_n\|' = \underset{x_i\in X, \|x_1,...,x_n\|\leq 1}{Sup} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

and

$$\|f_1,...,f_n\|'_1 = \underset{x_i\in X, \|x_1,...,x_n\|=1}{Sup} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

are identical n-norms on X',the dual of an n-normed space X [10].

The theory of 2-normed spaces and n-normed spaces were initially developed by Gähler [1]-[4] in the 1960s.Recent works and related works can be found in [5]-[9].The most recent work can be seen in Singh and Meitei [10],[11]. Our interest here is to study alternative formulae of n-norms especially in a Hilbert space. The alternative formulae are identical with the n-norms mentioned above. In the last part we study the equality of three n-norms defined on the dual space of an n-normed space.

2. n-Norms And Their Identicalness

Proposition 2.1. Let X be a normed space with dual X'.Then the function

$$\|x_1,...,x_n\|^H = \sup_{f_j\in X',\|f_j\|\neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1\|\,\|f_2\|\ldots\|f_n\|}$$

defines an n-norm on X.

Proof. (i) $x_1,...,x_n$ are linearly dependent.
$\Leftrightarrow$ rows of the matrix $\left[f_j(x_i)\right]$ are linearly dependent.
$\Leftrightarrow \frac{\det\left[f_j(x_i)\right]}{\|f_1\|\|f_2\|\ldots\|f_n\|} = 0 \forall f_j \in X'\, with\, \|f_j\| \neq 0.$
$\Leftrightarrow \sup_{f_j\in X',\|f_j\|\neq 0} \frac{\det\left[f_j(x_i)\right]}{\|f_1\|\|f_2\|\ldots\|f_n\|} = 0\,.$
$\Leftrightarrow \|.,...,.\|^H = 0.$

(ii) By the properties of determinants and definition of supremum, $\|.,...,.\|^H$remains invariant under permutations of $x_1,...,x_n$.

(iii) $\forall\, \alpha \in \mathbb{R}$,

$$\|\alpha x_1,...,x_n\|^H = \sup_{f_j\in X',\|f_j\|\neq 0} \frac{\begin{vmatrix} f_1(\alpha x_1) & \cdots & f_n(\alpha x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1\|\,\|f_2\|\ldots\|f_n\|}$$

$$= \sup_{f_j \in X', \|f_j\| \neq 0} \frac{\alpha \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1\| \|f_2\| \ldots \|f_n\|}$$

$$= |\alpha| \sup_{f_j \in X', \|f_j\| \neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1\| \|f_2\| \ldots \|f_n\|}$$

$$= |\alpha| \|x_1, \ldots, x_n\|^H.$$

(iv) By linearity of f_j's and properties of determinants ,we have

$$\frac{\begin{vmatrix} f_1(x_0+x_1) & \cdots & f_n(x_0+x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1\| \|f_2\| \ldots \|f_n\|} = \frac{\begin{vmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix} + \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1\| \|f_2\| \ldots \|f_n\|}$$

$$= \frac{\begin{vmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1\| \|f_2\| \ldots \|f_n\|} + \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1\| \|f_2\| \ldots \|f_n\|}.$$

Taking supremums of both sides over $f_j \in X'$with$\|f_j\| \neq 0$, we have
$\|x_0 + x_1, x_2, \ldots, x_n\|^H \leq \|x_0, x_2, \ldots, x_n\|^H + \|x_1, x_2, \ldots, x_n\|^H$.
This completes the proof.

Proposition 2.2: Let X be a normed space with dual X'.Then, $\|., \ldots, .\|^r$and $\|., \ldots, .\|^H$are identical.

Proof.

$$\|x_1,...,x_n\|^r = \underset{f_j\in X',\|f_j\|=1}{Sup} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

and

$$\|x_1,...,x_n\|^H = \sup_{f_j\in X',\|f_j\|\neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1\|\|f_2\|...\|f_n\|}$$

Clearly, $\|x_1,...,x_n\|^r \leq \|.x_1...,x_n\|^H$.

Conversely, we choose $g_j = \frac{f_j}{\|f_j\|}\ for\ j = 1,2,...,n$. Now,

$$\frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1\|\|f_2\|...\|f_n\|} = \frac{\begin{vmatrix} \|f_1\|g_1(x_1) & \cdots & \|f_n\|g_n(x_1) \\ \vdots & \ddots & \vdots \\ \|f_1\|g_1(x_n) & \cdots & \|f_n\|g_n(x_n) \end{vmatrix}}{\|f_1\|\|f_2\|...\|f_n\|}$$

$$= \frac{\|f_1\|\|f_2\|...\|f_n\|\begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n) \end{vmatrix}}{\|f_1\|\|f_2\|...\|f_n\|}$$

$$= \begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n) \end{vmatrix}$$

$$\leq \underset{g_j\in X',\|g_j\|=1}{Sup} \begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n) \end{vmatrix}$$

$$= \|x_1,...,x_n\|^r \forall f_j \in X'\ with\ \|f_j\| \neq 0$$

$\Rightarrow \; \|x_1,...,x_n\|^H \leq \|x_1...,x_n\|^r$.

This completes the proof.

Note On a Hilbert space X with dual X',$\|.,...,.\|^H$ becomes

$$\|x_1,...,x_n\|^H = \sup_{y_j \in X, \|y_j\| \neq 0} \frac{\begin{vmatrix} \langle x_1, y_1 \rangle & \cdots & \langle x_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle & \cdots & \langle x_n, y_n \rangle \end{vmatrix}}{\|y_1\| \, \|y_2\| \ldots \|y_n\|}.$$

It can be proved by using Riesz-representation theorem.

Proposition 2.3. On a Hilbert space X,

$$\|x_1,...,x_n\|^H = \sup_{y_j \in X, \|y_j\| \neq 0} \frac{\begin{vmatrix} \langle x_1, y_1 \rangle & \cdots & \langle x_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle & \cdots & \langle x_n, y_n \rangle \end{vmatrix}}{\|y_1\| \, \|y_2\| \ldots \|y_n\|}$$

and

$$\|x_1,...,x_n\|^r = \sup_{y_j \in X, \|y_j\| = 1} \begin{vmatrix} \langle x_1, y_1 \rangle & \cdots & \langle x_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle & \cdots & \langle x_n, y_n \rangle \end{vmatrix}$$

are identical.

Clearly, $\|x_1,...,x_n\|^r \leq \|x_1,...,x_n\|^H$.

Conversely, we choose $z_j = \frac{y_j}{\|y_j\|}, \|y_j\| \neq 0$.

Now, $\frac{\begin{vmatrix} \langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle \end{vmatrix}}{\|y_1\|\|y_2\|...\|y_n\|} = \frac{\begin{vmatrix} \langle x_1, \|y_1\| z_1\rangle & \cdots & \langle x_1, \|y_n\| z_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, \|y_1\| z_1\rangle & \cdots & \langle x_n, \|y_n\| z_n\rangle \end{vmatrix}}{\|y_1\|\|y_2\|...\|y_n\|}$

$$= \frac{\|y_1\|\,\|y_2\|\ldots\|y_n\| \begin{vmatrix} \langle x_1, z_1\rangle & \cdots & \langle x_1, z_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1\rangle & \cdots & \langle x_n, z_n\rangle \end{vmatrix}}{\|y_1\|\,\|y_2\|\ldots\|y_n\|}$$

$$= \begin{vmatrix} \langle x_1, z_1\rangle & \cdots & \langle x_1, z_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1\rangle & \cdots & \langle x_n, z_n\rangle \end{vmatrix}$$

$$\leq \underset{z_j\in X, \|z_j\|=1}{Sup} \begin{vmatrix} \langle x_1, z_1\rangle & \cdots & \langle x_1, z_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1\rangle & \cdots & \langle x_n, z_n\rangle \end{vmatrix}$$

$$= \|x_1,...,x_n\|^r \forall\, y_j \in X\, with\, \|y_j\| \neq 0$$

$\Rightarrow \|x_1,...,x_n\|^H \leq \|x_1,...,x_n\|^r$. $\therefore \|.,...,.\|^r$ & $\|,...,.\|^H$ are identical on a Hilbert space X. This completes the proof.

Proposition 2.4. On a Hilbert space X,$\|.,...,.\|^H$ and $\|.,...,.\|^S$ are identical.

Proof. $\|.,...,.\|^r$ and $\|.,...,.\|^S$ are identical [10].But, $\|.,...,.\|^r$ & $\|.,...,.\|^H$ are identical [proposition 2.2].So, $\|.,...,.\|^H$ and $\|.,...,.\|^S$ are identical.

Corollary 2.1. $\|.,...,.\|^D, \|.,...,.\|^E, \|.,...,.\|^F, \|.,...,.\|^G, \|.,...,.\|^H, \|.,...,.\|^r$ and $\|.,...,.\|^S$ are identical on a Hilbert space X.

Corollary 2.2. On a separable Hilbert space X,$\|.,...,.\|^D$,$\|.,...,.\|^E$,$\|.,...,.\|^F$, $\|.,...,.\|^G$, $\|.,...,.\|^H$,$\|.,...,.\|^r$, $\|.,...,.\|^S$ and $\|.,...,.\|_2$ are identical.

3. EQUALITY OF THREE n-NORMS ON A DUAL SPACE

Proposition 3.1. Let $(X, \|.,...,.\|)$ be an n-normed space. Then, the function $\|.,...,.\|_2{}' :$ $(X')^n \to \mathbb{R}$ given by $\|f_1,...,f_n\|_2{}' = \sup\limits_{\|x_1,...,x_n\| \neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|x_1,...,x_n\|}$ defines an n-norm on X'.

Proof. (i) It is easy to show that $f_1, f_2, ..., f_n$ are linearly dependent iff $\|f_1,...,f_n\|_2{}' = 0$.

(ii) By properties of determinant and definition of supremum, $\|f_1,...,f_n\|_2{}'$ is invariant under permutations of $f_1, f_2, ..., f_n$.

(iii) For any $\alpha \in \mathbb{R}$, $\|\alpha f_1,...,f_n\|_2{}' = \sup\limits_{\|x_1,...,x_n\| \neq 0} \frac{\begin{vmatrix} \alpha f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ \alpha f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|x_1,...,x_n\|}$

$$= \sup_{\|x_1,...,x_n\| \neq 0} \frac{\alpha \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|x_1,...,x_n\|}$$

$$= |\alpha| \sup_{\|x_1,...,x_n\| \neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|x_1,...,x_n\|}$$

$$= |\alpha| \, \|f_1,...,f_n\|_2{}'.$$

(iv) For any $f_0, f_1, \ldots, f_n \in X'$, $\dfrac{\begin{vmatrix} (f_0+f_1)(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ (f_0+f_1)(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|x_1,\ldots,x_n\|}$

$$= \frac{\begin{vmatrix} f_0(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_0(x_n) & \cdots & f_n(x_n) \end{vmatrix} + \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|x_1,\ldots,x_n\|}$$

$$= \frac{\begin{vmatrix} f_0(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_0(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|x_1,\ldots,x_n\|} + \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|x_1,\ldots,x_n\|}$$

$$\leq \sup_{\|x_1,\ldots,x_n\|\neq 0} \frac{\begin{vmatrix} f_0(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_0(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|x_1,\ldots,x_n\|} + \sup_{\|x_1,\ldots,x_n\|\neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|x_1,\ldots,x_n\|}$$

$$= \|f_0,\ldots,f_n\|_2{}' + \|f_1,\ldots,f_n\|_2{}' \; \forall x_i \in X \,\&\, \|x_1,\ldots,x_n\| \neq 0.$$

$\therefore \|f_0+f_1,\ldots,f_n\|_2{}' \leq \|f_0,\ldots,f_n\|_2{}' + \|f_1,\ldots,f_n\|_2{}'$. This completes the proof.

Proposition 3.2. Let $(X, \|.,\ldots,.\|)$ be an n-normed space. Then, the n-norms $\|.,\ldots,.\|'_1$ and $\|.,\ldots,.\|_2{}'$ defined on the dual X' are identical.

Proof. Clearly, $\|f_1,\ldots,f_n\|'_1 \leq \|f_1,\ldots,f_n\|_2{}'$.

Conversely, we choose $y_i = \frac{x_i}{\sqrt[n]{\|x_1,...,x_n\|}} = \frac{x_i}{a}, a = \sqrt[n]{\|x_1,...,x_n\|} \neq 0.$

$$\text{Now, } \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|x_1,...,x_n\|} = \frac{\begin{vmatrix} f_1(ay_1) & \cdots & f_n(ay_1) \\ \vdots & \ddots & \vdots \\ f_1(ay_n) & \cdots & f_n(ay_n) \end{vmatrix}}{\|ay_1,...,ay_n\|}$$

$$= \frac{a^n \begin{vmatrix} f_1(y_1) & \cdots & f_n(y_1) \\ \vdots & \ddots & \vdots \\ f_1(y_n) & \cdots & f_n(y_n) \end{vmatrix}}{a^n \|y_1,...,y_n\|}$$

$$= \begin{vmatrix} f_1(y_1) & \cdots & f_n(y_1) \\ \vdots & \ddots & \vdots \\ f_1(y_n) & \cdots & f_n(y_n) \end{vmatrix}$$

$$\leq \sup_{y_i \in X, \|y_1,...,y_n\|=1} \begin{vmatrix} f_1(y_1) & \cdots & f_n(y_1) \\ \vdots & \ddots & \vdots \\ f_1(y_n) & \cdots & f_n(y_n) \end{vmatrix}$$

$$= \|f_1,...,f_n\|'_1 \forall x_i \in X \, with \; \|x_1,...,x_n\| \neq 0.$$

$\Rightarrow \;\; \|f_1,...,f_n\|_2{}' \leq \|f_1,...,f_n\|'_1.$

$\therefore \|.,...,.\|'_1and\|.,...,.\|_2{}'$ are identical. This completes the proof.

Corollary 3.1. Let $(X, \|.,...,.\|)$ be an n-normed space. Then,$\|.,...,.\|'$, $\|.,...,.\|'_1$ and $\|.,...,.\|_2{}'$ are identical on X'.

Conflict of Interests

The authors declare that there is no conflict of interests.

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