Available online at http://scik.org
Adv. Inequal. Appl. 2014, 2014:42
ISSN: 2050-7461

# ON CERTAIN INEQUALITIES CONCERNING THE CLASSICAL EULER'S GAMMA FUNCTION

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**Abstract.** In this paper, some monotonic functions and some inequalities concerning certain ratios of generalized gamma functions are established. The procedure utilizes the series forms of the generalized digamma functions.

**Keywords:** gamma function; $q$-gamma function; $k$-gamma function; $(p,q)$-gamma function; inequality.

**2010 AMS Subject Classification:** 33B15, 26A48.

## 1. Introduction and Preliminaries

The classical Euler's Gamma function, $\Gamma(t)$ and the digamma function, $\psi(t)$ are well-known in literature as

$$\Gamma(t)=\int_0^\infty e^{-x}x^{t-1}\,dx \quad \text{and} \quad \psi(t)=\frac{d}{dt}\ln\Gamma(t)=\frac{\Gamma'(t)}{\Gamma(t)}, \quad t>0.$$

The $p$-Gamma function, $\Gamma_p(t)$ and the $p$-digamma function, $\psi_p(t)$ are defined for $p\in N$ as (see [6])

$$\Gamma_p(t)=\frac{p!p^t}{t(t+1)\dots(t+p)} \quad \text{and} \quad \psi_p(t)=\frac{d}{dt}\ln\Gamma_p(t)=\frac{\Gamma'_p(t)}{\Gamma_p(t)}, \quad t>0.$$

Received September 2, 2014

The $q$-Gamma function, $\Gamma_q(t)$ and the $q$-digamma function, $\psi_q(t)$ are also defined for $q \in (0,1)$ as (see [2])

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{t+n}} \quad \text{and} \quad \psi_q(t) = \frac{d}{dt} \ln \Gamma_q(t) = \frac{\Gamma_q'(t)}{\Gamma_q(t)}, \quad t>0.$$

Also, the $k$-Gamma function, $\Gamma_k(t)$ and the $k$-digamma function, $\psi_k(t)$ are defined for $k>0$ as (see [1])

$$\Gamma_k(t) = \int_0^{\infty} e^{-\frac{x^k}{k}} x^{t-1}\, dx \quad \text{and} \quad \psi_k(t) = \frac{d}{dt} \ln \Gamma_k(t) = \frac{\Gamma_k'(t)}{\Gamma_k(t)}, \quad t>0.$$

In 2005, Díaz and Teruel [5] defined the $(q,k)$-Gamma and the $(q,k)$-digamma functions for $q \in (0,1)$ and $k>0$ as

$$\Gamma_{(q,k)}(t) = \frac{(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}} \quad \text{and} \quad \psi_{(q,k)}(t) = \frac{d}{dt} \ln \Gamma_{(q,k)}(t) = \frac{\Gamma_{(q,k)}'(t)}{\Gamma_{(q,k)}(t)}, \quad t>0,$$

where $(t)_{n,k} = t(t+k)(t+2k)\dots(t+(n-1)k) = \prod_{j=0}^{n-1}(t+jk)$ is the $k$-generalized Pochhammer symbol.

Also in 2012, Krasniqi and Merovci [4] defined the $(p,q)$-Gamma and the $(p,q)$-digamma functions for $p \in N$ and $q \in (0,1)$ as

$$\Gamma_{(p,q)}(t) = \frac{[p]_q^t [p]_q!}{[t]_q[t+1]_q \dots [t+p]_q} \quad \text{and} \quad \psi_{(p,q)}(t) = \frac{d}{dt} \ln \Gamma_{(p,q)}(t) = \frac{\Gamma_{(p,q)}'(t)}{\Gamma_{(p,q)}(t)}, \quad t>0.$$

where $[p]_q = \frac{1-q^p}{1-q}$.

The generalized digamma functions $\psi_q(t)$, $\psi_k(t)$, $\psi_{(p,q)}(t)$ and $\psi_{(q,k)}(t)$ as defined above exhibit the following series representations (see also [8], [9], [10], [11] and [12]).

$$\psi_q(t) = -\ln(1-q) + \ln q \sum_{n=1}^{\infty} \frac{q^{nt}}{1-q^n} \tag{1}$$

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} \tag{2}$$

$$\psi_{(p,q)}(t) = \ln[p]_q + (\ln q) \sum_{n=1}^{p} \frac{q^{nt}}{1-q^n} \tag{3}$$

$$\psi_{(q,k)}(t) = \frac{-\ln(1-q)}{k} + (\ln q) \sum_{n=1}^{\infty} \frac{q^{nkt}}{1-q^{nk}} \tag{4}$$

where $\gamma$ is the Euler-Mascheroni's constant.

In 2010, Krasniqi and Shabani [7] presented the following inequalities:

$$\frac{p^{-t}e^{-\gamma t}\Gamma(\alpha)}{\Gamma_p(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_p(\alpha+t)} < \frac{p^{1-t}e^{\gamma(1-t)}\Gamma(\alpha+1)}{\Gamma_p(\alpha+1)}$$

for $t \in (0,1)$, where $\alpha$ is a positive real number such that $\alpha + t > 1$.

Also in that same year, Krasniqi, Mansour and Shabani [6] presented the following results.

$$\frac{(1-q)^t e^{-\gamma t}\Gamma(\alpha)}{\Gamma_q(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{(1-q)^{t-1}e^{\gamma(1-t)}\Gamma(\alpha+1)}{\Gamma_q(\alpha+1)}$$

for $t \in (0,1)$, where $\alpha$ is a positive real number such that $\alpha + t > 1$ and $q \in (0,1)$.

Several results of this nature as well as some generalizations have since been established. These can be found in the papers [8], [9], [10], [11] and [12].

In the present paper, our main objective is to present similar results involving the ratios $\frac{\Gamma_q(t)}{\Gamma_{(p,q)}(t)}$, $\frac{\Gamma_q(t)}{\Gamma_{(q,k)}(t)}$, $\frac{\Gamma_k(t)}{\Gamma_{(p,q)}(t)}$ and $\frac{\Gamma_k(t)}{\Gamma_{(q,k)}(t)}$.

## 2. Auxiliary Results

**Lemma 2.1.** *Let $\alpha > 0$, $t > 0$, $p \in N$ and $q \in (0,1)$. Then,*

$$\ln(1-q) + \ln[p]_q + \psi_q(\alpha+t) - \psi_{(p,q)}(\alpha+t) \leq 0.$$

**Proof.** By the series representations (1) and (3) we have,

$$\begin{aligned}\ln(1-q) + \ln[p]_q + \psi_q(t) - \psi_{(p,q)}(t) &= (\ln q)\left[\sum_{n=1}^{\infty}\frac{q^{nt}}{1-q^n} - \sum_{n=1}^{p}\frac{q^{nt}}{1-q^n}\right] \\ &= (\ln q)\sum_{n=p+1}^{\infty}\frac{q^{nt}}{1-q^n} \leq 0.\end{aligned}$$

Replacing $t$ by $\alpha + t$ completes the proof.

**Lemma 2.2.** *Let $\alpha > 0$, $t > 0$, $q \in (0,1)$ and $k \geq 1$. Then,*

$$\ln(1-q) - \frac{1}{k}\ln(1-q) + \psi_q(\alpha+t) - \psi_{(q,k)}(\alpha+t) \leq 0.$$

**Proof.** By the series representations (1) and (4) we have,

$$\ln(1-q)^{1-\frac{1}{k}} + \psi_q(t) - \psi_{(q,k)}(t) = (\ln q)\sum_{n=1}^{\infty}\left[\frac{q^{nt}}{1-q^n} - \frac{q^{nkt}}{1-q^{nk}}\right] \leq 0.$$

Replacing $t$ by $\alpha + t$ completes the proof.

**Lemma 2.3.** *Let $\alpha > 0$, $t > 0$, $k > 0$, $p \in N$ and $q \in (0,1)$. Then,*

$$\ln[p]_q - \frac{\ln k - \gamma}{k} + \frac{1}{\alpha+t} + \psi_k(\alpha+t) - \psi_{(p,q)}(\alpha+t) > 0.$$

**Proof.** By the series representations (2) and (3) we have,

$$\ln[p]_q - \frac{\ln k - \gamma}{k} + \frac{1}{t} + \psi_k(t) - \psi_{(p,q)}(t) = \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} - (\ln q)\sum_{n=1}^{p} \frac{q^{nt}}{1-q^n} > 0.$$

Replacing $t$ by $\alpha + t$ completes the proof.

**Lemma 2.4.** *Let $\alpha > 0$, $t > 0$, $q \in (0,1)$ and $k > 0$. Then,*

$$-\frac{\ln(k(1-q))}{k} + \frac{\gamma}{k} + \frac{1}{\alpha+t} + \psi_k(\alpha+t) - \psi_{(q,k)}(\alpha+t) > 0.$$

**Proof.** By the series representations (2) and (4) we have,

$$-\frac{\ln(k(1-q))}{k} + \frac{\gamma}{k} + \frac{1}{t} + \psi_k(t) - \psi_{(q,k)}(t) = \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} - (\ln q)\sum_{n=1}^{\infty} \frac{q^{nkt}}{1-q^{nk}} > 0.$$

Replacing $t$ by $\alpha + t$ completes the proof.

## 3. Main Results

**Theorem 3.1.** *Define a function $S$ by*

$$S(t) = \frac{(1-q)^t \Gamma_q(\alpha+t)}{[p]_q^{-t} \Gamma_{(p,q)}(\alpha+t)}, \quad t \in (0,\infty) \tag{5}$$

*for $\alpha > 0$, $p \in N$ and $q \in (0,1)$. Then $S$ is non-increasing on $t \in (0,\infty)$ and the inequalities*

$$\frac{(1-q)^{-t}\Gamma_q(\alpha)}{[p]_q^{t}\Gamma_{(p,q)}(\alpha)} \geq \frac{\Gamma_q(\alpha+t)}{\Gamma_{(p,q)}(\alpha+t)} \geq \frac{(1-q)^{1-t}\Gamma_q(\alpha+1)}{[p]_q^{t-1}\Gamma_{(p,q)}(\alpha+1)} \tag{6}$$

*are valid for every $t \in (0,1)$.*

**Proof.** Let $g(t) = \ln S(t)$ for every $t \in (0,\infty)$. Then,

$$\begin{aligned} g(t) &= \ln \frac{(1-q)^t \Gamma_q(\alpha+t)}{[p]_q^{-t} \Gamma_{(p,q)}(\alpha+t)} \\ &= t\ln(1-q) + t\ln[p]_q + \ln\Gamma_q(\alpha+t) - \ln\Gamma_{(p,q)}(\alpha+t). \quad \text{Then,} \end{aligned}$$

$$g'(t) = \ln(1-q) + \ln[p]_q + \psi_q(\alpha+t) - \psi_{(p,q)}(\alpha+t) \leq 0. \quad \text{(by Lemma 2.1)}$$

That implies $g$ is non-increasing on $t \in (0,\infty)$. Hence $S$ is non-increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$ we have,

$$S(0) \geq S(t) \geq S(1)$$

yielding the result.

**Theorem 3.2.** *Define a function $T$ by*

$$T(t) = \frac{(1-q)^t \Gamma_q(\alpha+t)}{(1-q)^{\frac{t}{k}} \Gamma_{(q,k)}(\alpha+t)}, \quad t \in (0,\infty) \tag{7}$$

*for $\alpha > 0$, $q \in (0,1)$ and $k \geq 1$. Then $T$ is non-increasing on $t \in (0,\infty)$ and the inequalities*

$$\frac{(1-q)^{-t}\Gamma_q(\alpha)}{(1-q)^{-\frac{t}{k}}\Gamma_{(q,k)}(\alpha)} \geq \frac{\Gamma_q(\alpha+t)}{\Gamma_{(q,k)}(\alpha+t)} \geq \frac{(1-q)^{1-t}\Gamma_q(\alpha+1)}{(1-q)^{\frac{1}{k}(1-t)}\Gamma_{(q,k)}(\alpha+1)} \tag{8}$$

*are valid for every $t \in (0,1)$.*

**Proof.** Let $h(t) = \ln T(t)$ for every $t \in (0,\infty)$. Then,

$$\begin{aligned} h(t) &= \ln \frac{(1-q)^t \Gamma_q(\alpha+t)}{(1-q)^{\frac{t}{k}} \Gamma_{(q,k)}(\alpha+t)} \\ &= t\ln(1-q) - \frac{t}{k}\ln(1-q) + \ln\Gamma_q(\alpha+t) - \ln\Gamma_{(q,k)}(\alpha+t). \quad \text{Then,} \\ h'(t) &= \ln(1-q) - \frac{1}{k}\ln(1-q) + \psi_q(\alpha+t) - \psi_{(q,k)}(\alpha+t) \leq 0. \quad \text{(by Lemma 2.2)} \end{aligned}$$

That implies $h$ is non-increasing on $t \in (0,\infty)$. Hence $T$ is non-increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$ we have,

$$T(0) \geq T(t) \geq T(1)$$

establishing the result.

**Theorem 3.3.** *Define a function $U$ by*

$$U(t) = \frac{(\alpha+t)k^{-\frac{t}{k}} e^{\frac{\gamma t}{k}} \Gamma_k(\alpha+t)}{[p]_q^{-t} \Gamma_{(p,q)}(\alpha+t)}, \quad t \in (0,\infty) \tag{9}$$

*for $\alpha > 0$, $p \in N$, $q \in (0,1)$ and $k > 0$. Then $U$ is increasing on $t \in (0,\infty)$ and the inequalities*

$$\frac{\alpha k^{\frac{t}{k}} e^{-\frac{\gamma t}{k}} \Gamma_k(\alpha)}{(\alpha+t)[p]_q^t \Gamma_{(p,q)}(\alpha)} < \frac{\Gamma_k(\alpha+t)}{\Gamma_{(p,q)}(\alpha+t)} < \frac{(\alpha+1)k^{\frac{t-1}{k}} e^{\frac{\gamma(1-t)}{k}} \Gamma_k(\alpha+1)}{(\alpha+t)[p]_q^{t-1} \Gamma_{(p,q)}(\alpha+1)} \tag{10}$$

*are valid for every $t \in (0,1)$.*

**Proof.** Let $\mu(t) = \ln U(t)$ for every $t \in (0,\infty)$. Then,

$$\mu(t) = \ln \frac{(\alpha+t)k^{-\frac{t}{k}}e^{\frac{\gamma t}{k}}\Gamma_k(\alpha+t)}{[p]_q^{-t}\Gamma_{(p,q)}(\alpha+t)}$$
$$= \ln(\alpha+t) - \frac{t}{k}\ln k + \frac{\gamma t}{k} + t\ln[p]_q + \ln\Gamma_k(\alpha+t) - \ln\Gamma_{(p,q)}(\alpha+t). \quad \text{Then,}$$
$$\mu'(t) = \ln[p]_q - \frac{\ln k - \gamma}{k} + \frac{1}{\alpha+t} + \psi_k(\alpha+t) - \psi_{(p,q)}(\alpha+t) > 0. \text{(by Lemma 2.3)}$$

That implies $\mu$ is increasing on $t \in (0,\infty)$. As a result, $U$ is also increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$ we have,

$$U(0) < U(t) < U(1)$$

yielding the result.

**Theorem 3.4.** *Define a function $V$ by*

$$V(t) = \frac{(\alpha+t)e^{\frac{\gamma t}{k}}\Gamma_k(\alpha+t)}{k^{\frac{t}{k}}(1-q)^{\frac{t}{k}}\Gamma_{(q,k)}(\alpha+t)}, \quad t \in (0,\infty) \tag{11}$$

*for $\alpha > 0$, $q \in (0,1)$ and $k > 0$. Then $V$ is increasing on $t \in (0,\infty)$ and the inequalities*

$$\frac{\alpha k^{\frac{t}{k}}e^{-\frac{\gamma t}{k}}\Gamma_k(\alpha)}{(\alpha+t)(1-q)^{-\frac{t}{k}}\Gamma_{(q,k)}(\alpha)} < \frac{\Gamma_k(\alpha+t)}{\Gamma_{(q,k)}(\alpha+t)} < \frac{(\alpha+1)k^{\frac{t-1}{k}}e^{\gamma\frac{1-t}{k}}\Gamma_k(\alpha+1)}{(\alpha+t)(1-q)^{\frac{1-t}{k}}\Gamma_{(q,k)}(\alpha+1)} \tag{12}$$

*are valid for every $t \in (0,1)$.*

**Proof.** Let $\delta(t) = \ln V(t)$ for every $t \in (0,\infty)$. Then,

$$\delta(t) = \ln \frac{(\alpha+t)e^{\frac{\gamma t}{k}}\Gamma_k(\alpha+t)}{k^{\frac{t}{k}}(1-q)^{\frac{t}{k}}\Gamma_{(q,k)}(\alpha+t)}$$
$$= \ln(\alpha+t) + \frac{\gamma t}{k} - \frac{t}{k}\ln(k(1-q)) + \ln\Gamma_k(\alpha+t) - \ln\Gamma_{(q,k)}(\alpha+t). \quad \text{Then,}$$
$$\delta'(t) = -\frac{\ln(k(1-q))}{k} + \frac{\gamma}{k} + \frac{1}{\alpha+t} + \psi_k(\alpha+t) - \psi_{(q,k)}(\alpha+t) > 0. \text{(by Lemma 2.4)}$$

That implies $\delta$ is increasing on $t \in (0,\infty)$. As a result, $V$ is also increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$ we have,

$$V(0) < V(t) < V(1)$$

establishing the result.

**Conflict of Interests.**

The author declares that there is no conflict of interests.

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