Available online at http://scik.org
Adv. Inequal. Appl. 2015, 2015:14
ISSN: 2050-7461

POWER TYPE OSCILLATORY MEAN AND ITS DUAL

V. LOKESHA[1,*], S. KUMAR[2], U.K. MISRA[3], K.M. NAGARAJA[4]

[1]Department of Mathematics, V.S.K. University, Bellary-583105, India

[2]Department Of Mathematics, Acharya Institute of Technology, Bangalore-560107, India

[3]Department Of Mathematics, Berhampur University, Berhampur India

[4]Department of Mathematics, JSSAT, Bangalore, India

Abstract. In this paper, we define power oscillatory mean and its dual form in n variables and prove their monotonicity.

Keywords: Monotonicity; Inequality; Oscillatory mean; Dual.

2010 AMS Subject Classification: 26D15, 26D10.

1. Introduction

In the literature, arithmetic mean and geometric means are thoroughly studied by various researchers.

For a, $b > 0$

$$A(a,b) = F_1(a,b) = \frac{a+b}{2},$$

$$G(a,b) = F_2(a,b) = \sqrt{ab},$$

*Corresponding author

Received September 2, 2014

$$L(a,b)=\begin{cases} \frac{a-b}{\ln a-\ln b} & a\neq b,\\ a & a=b,\end{cases}$$

$$I(a,b)=\begin{cases} e^{\left(\frac{a\ln a-b\ln b}{a-b}-1\right)} & a\neq b,\\ a & a=b,\end{cases}$$

$$M_r(a,b)=\begin{cases} \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}} & r\neq 0,\\ \sqrt{ab} & r=0,\end{cases}$$

$$H(a,b)=\frac{a+\sqrt{ab}+b}{3},$$

$$H_k(a,b)=\left(\frac{a^k+(ab)^{\frac{k}{2}}+b^k}{3}\right)^{\frac{1}{k}}$$

are respectively called arithmetic mean, geometric mean. logarithmic mean, identric mean, power mean, Heron mean and power type Heron mean.

All these means defined above have been studied and many remarkable inequalities with some identities have been established. For more details the interested reader is referred to [1, 2, 4, 5, 6, 7, 8, 9].

In [3, 10], the authors defined oscillatory mean, r^{th} oscillatory mean and its dual's and established some new inequalities and the best possible values of these means with logarithmic mean, identric mean and power mean are obtained.

Definition 1.1. [10] For $a,b>0$ and $\alpha\in(0,1)$, Oscillatory mean and its dual form are as follows;

$$O(a,b;\alpha)=\alpha G(a,b)+(1-\alpha)A(a,b),$$

$$O^{(d)}(a,b;\alpha)=G(a,b)^{\alpha}A(a,b)^{1-\alpha}.$$

Definition 1.2. [3] For $a,b>0$ and $\alpha\in(0,1)$, r^{th} oscillatory mean and its dual form are as follows;

$$O(a,b;\alpha,r)=\alpha M_r(a,b)+(1-\alpha)A(a,b),$$

$$O^{(d)}(a,b;\alpha,r) = M_r(a,b)^{\alpha} A(a,b)^{1-\alpha}.$$

Enlightened from oscillatory mean and r^{th} oscillatory mean, in the forthcoming sections, we introduce power oscillatory mean and its dual. Also, establish monotonicity and some inter-related inequalities. In concluding section, consequent examples are appended.

2. Preliminaries

In this section, the power type oscillatory mean and its dual are defined as follows.

Definition 2.1. For $a,b>0$, a real number $k \in [0,\infty)$, and $\alpha \in (0,1)$ then power type oscillatory mean is denoted by $\mathrm{O}(a,b;\alpha,k)$ and defined as;

$$O(a,b;\alpha,k) = \begin{cases} \left[\alpha G(a^k,b^k) + (1-\alpha)A(a^k,b^k)\right]^{\frac{1}{k}} & k \neq 0; \\ H_k(a,b) & \alpha = \frac{1}{3}; \\ G(a,b) & k = 0; \end{cases}$$

equivalently,

$$O(a,b;\alpha,k) = \begin{cases} \left[\alpha (ab)^{\frac{k}{2}} + (1-\alpha)\left(\frac{a^k+b^k}{2}\right)\right]^{\frac{1}{k}} & k \neq 0; \\ \left(\frac{a^k+(ab)^{\frac{k}{2}}+b^k}{3}\right)^{\frac{1}{k}} & \alpha = \frac{1}{3}; \\ \sqrt{ab} & k = 0. \end{cases}$$

Definition 2.2. For a real number $k \in [0,\infty)$, and $\alpha \in (0,1)$ then power type dual oscillatory mean is denoted by $\mathrm{O}^{(d)}(a,b;\alpha,k)$ and is defined by

$$O^{(d)}(a,b;\alpha,k) = \begin{cases} \left[G^{\alpha}(a^k,b^k)A^{1-\alpha}(a^k,b^k)\right]^{\frac{1}{k}} & k \neq 0; \\ \left(G^{\frac{1}{3}}(a^k,b^k)A^{\frac{2}{3}}(a^k,b^k)\right)^{\frac{1}{k}} & \alpha = \frac{1}{3}; \\ G(a,b) & k = 0; \end{cases}$$

equivalently,

$$O^{(d)}(a,b;\alpha,k)=\begin{cases}(ab)^{\frac{\alpha}{2}}\left(\frac{a^k+b^k}{2}\right)^{\frac{1-\alpha}{k}} & k\neq 0;\\ (\sqrt{a}b)^{\frac{1}{3}}\left(\frac{a^k+b^k}{2}\right)^{\frac{2}{3k}} & \alpha=\frac{1}{3};\\ \sqrt{ab} & k=0.\end{cases}$$

Now, we state properties of power type oscillatory mean and its dual.

For a real number $k\in[0,\infty)$ and $\alpha\in(0,1)$ the power type oscillatory mean and its dual are satisfies the following properties.

Property 2.3. The means $O(a,b;\alpha,k)$ and $O^d(a,b;\alpha,k)$ are

(1) **Symmetric** :
$O(a,b;\alpha,k)=O(b,a;\alpha,k)$ and $O^{(d)}(a,b;\alpha,k)=O^{(d)}(b,a;\alpha,k)$.

(2) **Homogeneous** :
$O(at,bt;\alpha,k)=tO(a,b;\alpha,k)$ and $O^{(d)}(at,bt;\alpha,k)=tO^{(d)}(a,b;\alpha,k)$.

Property 2.4. According to Definition 2.1 and Definition 2.2, the following characteristic properties for $O(a,b;\alpha,k)$ and $O^{(d)}(a,b;\alpha,k)$ are straightforward.

For a real number $k\in[0,\infty)$ and $\alpha\in(0,1)$ then

(1) $O(a,b;\frac{1}{3},k)=\left(\frac{a^k+(ab)^{\frac{1}{k}}+b^k}{3}\right)=H_k(a,b)$.
(2) $O(a,b;\alpha,0)=G(a,b)=O(a,b;1,k)$.
(3) $O(a,b;\alpha,1)=O(a,b;\alpha)$.
(4) $O(a,b;0,k)=M_k(a,b)=O^{(d)}(a,b;0,k)$.
(5) $O(a,b;\frac{1}{2},k)=M_{\frac{k}{2}}(a,b)$.
(6) $O(a,b;\frac{1}{2},2)=A(a,b)$.
(7) $O^{(d)}(a,b;\frac{1}{3},k)=[G(a,b)M_{\frac{k}{2}}(a,b)]^{\frac{1}{3}}$.
(8) $O^{(d)}(a,b;\frac{1}{3},1)=[G(a,b)M_{\frac{1}{2}}(a,b)]^{\frac{1}{3}}$.
(9) $O^{(d)}(a,b;\alpha,1)=O^{(d)}(a,b;\alpha)$.
(10) $O^{(d)}(a,b;\alpha,0)=G(a,b)=O^{(d)}(a,b;1,k)$.
(11) $O^{(d)}(a,b;\frac{1}{2},k)=[G(a,b)M_k(a,b)]^{\frac{1}{2}}$.
(12) $O^{(d)}(a,b;\frac{1}{2},1)=[G(a,b)A(a,b)]^{\frac{1}{2}}$.

(13) $min(a,b) \leq O^{(d)}(a,b;\alpha,k) \leq O(a,b;\alpha,k) \leq max(a,b).$

3. Monotonic Results

In this section, the monotonicity and behavior of the power type oscillatory means in different situations are studied.

Theorem 3.1. *For $\alpha \in (0,1)$ and $k \in [0,\infty)$ a real number and for a,b > 0 , then*

$$O^{(d)}(a,b;\alpha,k) \leq O(a,b;\alpha,k).$$

Proof. The proof of Theorem 3.1 follows from well known power mean inequality.

$$M_r(a,b) = \begin{cases} \left(\dfrac{a^k+b^k}{2}\right)^{\frac{1}{k}}, & r \neq 0; \\ \sqrt{ab}, & r = 0. \end{cases}$$

Theorem 3.2. *The power type oscillatory mean $O(a,b;\alpha,k)$ and the power type dual oscillatory mean $O^{(d)}(a,b;\alpha,k)$ are decreasing functions with $\alpha \in (0,1)$, for a,b > 0 and $k \in [0,\infty)$,*

$$O(a,b;\alpha,k) \geqslant O(a,b;\alpha+1,k)$$

and

$$O^{(d)}(a,b;\alpha,k) \geqslant O^{(d)}(a,b;\alpha+1,k).$$

Proof. From Definition 2.1, we find that

$$\begin{aligned} O(a,b;\alpha,k) &= \Big[\alpha G(a^k,b^k) + (1-\alpha)A(a^k,b^k)\Big]^{\frac{1}{k}} \\ &= \left[\alpha(a^k b^k)^{\frac{1}{2}} + (1-\alpha)\left(\frac{a^k+b^k}{2}\right)\right]^{\frac{1}{k}} \\ &\geqslant \left[\alpha\left[(a^k b^k)^{\frac{1}{2}} - \left(\frac{a^k+b^k}{2}\right)\right] + (a^k b^k)^{\frac{1}{2}}\right]^{\frac{1}{k}} \\ &= \left[(1+\alpha)(a^k b^k)^{\frac{1}{2}} - \alpha\left(\frac{a^k+b^k}{2}\right)\right]^{\frac{1}{k}} \\ &= O(a,b;\alpha+1,k). \end{aligned}$$

From Definition 2.2, we have

$$\begin{aligned}
O^{(d)}(a,b;\alpha,k) &= \left[(a^k b^k)^{\alpha/2}\left(\frac{a^k+b^k}{2}\right)^{(1-\alpha)}\right]^{\frac{1}{k}} \\
&= \left[\left[\frac{(a^k b^k)^{\frac{1}{2}}}{\left(\frac{a^k+b^k}{2}\right)}\right]^{\alpha}\left(\frac{a^k+b^k}{2}\right)\right]^{\frac{1}{k}} \\
&\geqslant \left[\left[\frac{(a^k b^k)^{\frac{1}{2}}}{\left(\frac{a^k+b^k}{2}\right)}\right]^{\alpha}(a^k b^k)^{\frac{1}{2}}\right]^{\frac{1}{k}} \\
&= \left[\left(\frac{a^k+b^k}{2}\right)^{-\alpha}(a^k b^k)^{\frac{\alpha+1}{2}}\right]^{\frac{1}{k}} \\
&= O^{(d)}(a,b;\alpha+1,k).
\end{aligned}$$

This completes the proof of Theorem 3.2.

Theorem 3.3. *For a,b > 0, $\alpha \in (0,1)$, $k \in [-\infty,\infty)$ be a real number then the power type oscillatory mean $O(a,b;\alpha,k)$ and the power type dual oscillatory mean $O^{(d)}(a,b;\alpha,k)$ are monotonically increasing with respect to k, for fixed α.*

Proof. From Definition 2.1, we have

$$\begin{aligned}
O(a,b;\alpha,k) &= \left[\alpha G(a^k,b^k)+(1-\alpha)A(a^k,b^k)\right]^{\frac{1}{k}} \\
&= \left[\alpha(ab)^{\frac{k}{2}}+(1-\alpha)\left(\frac{a^k+b^k}{2}\right)\right]^{\frac{1}{k}}
\end{aligned}$$

on differentiating with respect to k on both sides gives,

$$\begin{aligned}
&\frac{\partial}{\partial k}[O(a,b;\alpha,k)] \\
&= \frac{1}{k}\left[\alpha(ab)^{\frac{k}{2}}+(1-\alpha)\left(\frac{a^k+b^k}{2}\right)\right]^{\frac{1}{k}-1}\left[\alpha(ab)^{\frac{k}{2}}log\sqrt{ab}+(1-\alpha)\frac{a^k loga+b^k logb}{2}\right] \\
&= \frac{1}{k}\left[\alpha(ab)^{\frac{k}{2}}+(1-\alpha)\left(\frac{a^k+b^k}{2}\right)\right]^{\frac{1}{k}-1} log\left[\frac{(a^{a^k}b^{b^k})^{\frac{1}{2}}}{(a^{a^k}b^{b^k})^{\frac{\alpha}{2}}}(\sqrt{ab})^{\alpha(\sqrt{ab})^k}\right]
\end{aligned}$$

hence for all $k \geqslant 0$ and fixed α,

$$\begin{aligned} O^{(d)}(a,b;\alpha,k) =&[G^{\alpha}(a^k,b^k)A^{1-\alpha}(a^k,b^k)]^{\frac{1}{k}} \\ =&\left[(ab)^{\frac{k\alpha}{2}}\left(\frac{a^k+b^k}{2}\right)^{(1-\alpha)}\right]^{\frac{1}{k}} \\ =&(ab)^{\frac{\alpha}{2}}\left[\left(\frac{a^k+b^k}{2}\right)^{\frac{1}{k}}\right]^{(1-\alpha)} \\ =&G^{\alpha}(a,b)M_k^{1-\alpha}(a,b). \end{aligned}$$

Since $M_k(a,b)$ is monotonically increasing for k and fixed α, hence $O^{(d)}(a,b;\alpha,k)$ is also monotonically increasing with respect to k. This completes the proof of the Theorem.

4. Some inequalities

By replacing $a=t+1, b=1$, in definition 2.1 and 2.2, then the Taylor's series expansion of $O(a,b;\alpha,k)$ and $O^{(d)}(a,b;\alpha,k)$ are as follows

$$O(t+1,1;\alpha,k) = 1+\frac{1}{2}t+\frac{(1-\alpha)(k-1)}{8}t^2+\ldots$$

$$O^{(d)}(t+1,1;\alpha,k) = 1+\frac{1}{2}t+\frac{(1-\alpha)(k-1)}{8}t^2+\ldots,$$

$$L(a,b) = L(t+1,1) = 1+\frac{t}{2}-\frac{1}{12}t^2+\ldots,$$

$$I(a,b) = I(t+1,1) = 1+\frac{t}{2}-\frac{1}{24}t^2+\ldots,$$

$$H_p(a,b) = H_p(t+1,1) = 1+\frac{t}{2}+\frac{2p-3}{24}t^2+\ldots,$$

$$M_r(a,b) = M_r(t+1,1) = 1+\frac{t}{2}+\frac{r-1}{8}t^2+\ldots,$$

From Theorem 3.2 and Taylor's series expansion of various means as above, we compute the following inequalities.

For $a, b > 0$, $\alpha \in (0,1)$, $k_1, k_2 \in [0, \infty)$ then,

Property 4.1. *For $k_1 \leq \frac{2p}{3(1-\alpha)} \leq k_2$, following double inequality holds*

$$O^{(d)}(a,b;\alpha,k_1) \leq H_p(a,b) \leq O(a,b;\alpha,k_2) \tag{1}$$

Further more $k_1 = \frac{2p}{3(1-\alpha)} = k_2$ is the best possible for (1).

Property 4.2. *For $k_1 \leq \frac{r}{(1-\alpha)} \leq k_2$, following double inequality holds*

$$O^{(d)}(a,b;\alpha,k_1) \leq M_r(a,b) \leq O(a,b;\alpha,k_2) \tag{2}$$

Further more $k_1 = \frac{r}{(1-\alpha)} = k_2$ is the best possible for (2).

Property 4.2. *For $k_1 \leq \frac{2}{3(1-\alpha)} \leq k_2$, following double inequality holds*

$$O^{(d)}(a,b;\alpha,k_1) \leq I(a,b) \leq O(a,b;\alpha,k_2) \tag{3}$$

Further more $k_1 = \frac{2}{3(1-\alpha)} = k_2$ is the best possible for (3).

Property 4.3. *For $k_1 \leq \frac{1}{3(1-\alpha)} \leq k_2$, following double inequality holds*

$$O^{(d)}(a,b;\alpha,k_1) \leq L(a,b) \leq O(a,b;\alpha,k_2) \tag{4}$$

Further more $k_1 = \frac{1}{3(1-\alpha)} = k_2$ is the best possible for (4).

Property 4.4. *For $\alpha_1 \geqslant 1 - \frac{2p}{3k} \geqslant \alpha_2$, following double inequality holds*

$$O^{(d)}(a,b;\alpha_1,k_1) \geqslant H_p(a,b) \geqslant O(a,b;\alpha_2,k_2) \tag{5}$$

Property 4.5. *For $\alpha_1 \geqslant 1 - \frac{r}{k} \geqslant \alpha_2$, following double inequality holds*

$$O^{(d)}(a,b;\alpha_1,k_1) \geqslant M_r(a,b) \geqslant O(a,b;\alpha_2,k_2). \tag{6}$$

Property 4.6. *For $\alpha_1 \geqslant 1 - \frac{2}{3k} \geqslant \alpha_2$, following double inequality holds*

$$O^{(d)}(a,b;\alpha_1,k_1) \geqslant I(a,b) \geqslant O(a,b;\alpha_2,k_2). \tag{7}$$

Property 4.7. *For $\alpha_1 \geqslant 1 - \frac{1}{3k} \geqslant \alpha_2$, following double inequality holds*

$$O^{(d)}(a,b;\alpha_1,k_1) \geqslant L(a,b) \geqslant O(a,b;\alpha_2,k_2). \tag{8}$$

5. Some examples

By theorem we establish the following two inequality chains.

Example 5.1. *From Theorem* $O(a,b;\alpha,k)$ *is decreasing function with* α, *then*

$$O(a,b;0,k) \geq O(a,b;\frac{1}{4},k) \geq O(a,b;\frac{1}{3},k) \geq O(a,b;\frac{2}{3},k)$$

$$\geq O(a,b;\frac{3}{4},k) \geq O(a,b;\frac{1}{2},k) \geq O(a,b;1,k)$$

$$\left(\frac{a^k+b^k}{2}\right)^{\frac{1}{k}} \geq \left(\frac{(ab)^{\frac{k}{2}}+3(\frac{a^k+b^k}{2})}{4}\right)^{\frac{1}{k}} \geq H_k(a,b) \geq \left(\frac{(ab)^{\frac{k}{2}}+(\frac{a^k+b^k}{2})}{2}\right)^{\frac{1}{k}}$$

$$\geq (ab)^{\frac{k}{2}} \geq \left(\frac{2(ab)^{\frac{k}{2}}+(\frac{a^k+b^k}{2})}{3}\right)^{\frac{1}{4}} \geq \left(\frac{3(ab)^{\frac{k}{2}}+(\frac{a^k+b^k}{2})}{4}\right)^{\frac{1}{2}}$$

for k=1: The following inequality is obtained

$$A \geq \frac{G+3A}{4} \geq H \geq \frac{G+A}{2} \geq \frac{2G+A}{3} \geq \frac{3G+A}{4} \geq G.$$

Example 5.2. *From Theorem* $O^{(d)}(a,b;\alpha,k)$ *is decreasing function with* α, *then*

$$O^{(d)}(a,b;0,k) \geq O^{(d)}(a,b;\frac{1}{4},k) \geq O^{(d)}(a,b;\frac{1}{3},k) \geq O^{(d)}(a,b;\frac{2}{3},k)$$
$$\geq O^{(d)}(a,b;\frac{3}{4},k) \geq O^{(d)}(a,b;\frac{1}{2},k) \geq O^{(d)}(a,b;1,k)$$

$$\left(\frac{a^k+b^k}{2}\right) \geq (ab)^{\frac{k}{8}}\left(\frac{a^k+b^k}{2}\right)^{\frac{3}{4}} \geq (ab)^{\frac{k}{6}}\left(\frac{a^k+b^k}{2}\right)^{\frac{2}{3}} \geq (ab)^{\frac{k}{4}}\left(\frac{a^k+b^k}{2}\right)^{\frac{1}{2}}$$
$$\geq (ab)^{\frac{2k}{6}}\left(\frac{a^k+b^k}{2}\right)^{\frac{1}{3}} \geq (ab)^{\frac{3k}{8}}\left(\frac{a^k+b^k}{2}\right)^{\frac{1}{4}} \geq (ab)^{\frac{1}{2}}$$

for k=1, the following inequality is obtained

$$A \geq G^{\frac{1}{4}}A^{\frac{3}{4}} \geq G^{\frac{1}{3}}A^{\frac{2}{3}} \geq G^{\frac{1}{2}}A^{\frac{1}{2}} \geq G^{\frac{2}{3}}A^{\frac{1}{3}} \geq G^{\frac{3}{4}}A^{\frac{1}{4}} \geq G. \tag{9}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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