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COMMON FIXED POINT THEOREMS FOR CYCLIC WEAK (ϕ,ψ)-CONTRACTIONS IN MENGER SPACES

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Abstract. Cyclic weak (ϕ,ψ)-contraction mappings are extended to Menger spaces and fixed point theorem for such mappings are studied in Menger spaces.

Keywords: Menger space; weak compatible mappings; cyclic weak (ϕ,ψ)-contraction.

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1. Introduction

In 1969, Boyd and Wong introduced the notion of ϕ-contraction in metric space. Alber and Guerre-Delabriere, gave the definition of weak ϕ-contraction for Hilbert space and proved the existence of fixed points in Hilbert space in 1997.

The existence application potential of fixed point theory in various fields resulted in several generalizations of the metric spaces. One such generalization is Menger space initiated by Menger. It was observed by many authors that contraction condition in metric space could be extended to Menger space. V. M. Sehgal and A. T. Bharucha-Reid first introduced the contraction mapping principle in probabilistic metric spaces which is a milestone for the development

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of fixed point theory in Menger space. Sunny Chauhan proved common fixed point theorem for weakly compatible mappings in Menger space satisfying ϕ-contractive conditions.

Recently, Rhoades proved interesting fixed point theorems for ψ- weak contraction in complete metric space. The significance of this kind of contraction can also be derived from the fact that they are strictly relative to famous Banach's fixed point theorem and to some other significant results. Also, motivated by the results of Rhoades and on the lines of Khan et. al. employing the idea of altering distances, Vetro et. al. extended the notion of (ϕ, ψ)- weak contraction to fuzzy metric space and proved common fixed point theorem for weakly compatible maps in fuzzy metric space. Thus an altering distance function is a control function which alter the metric distances between two points enabling one to deal with relatively new classes of fixed point problems. But, the uniqueness of control function creates difficulties in proving the existence of fixed point under contractive conditions.

The main objective of this paper is to study cyclic weak (ϕ, ψ)-contraction in Menger space.We obtained unique common fixed point theorem for the sequence of self mappings in Menger space using weak contractive condition by altering distances between points.

2. Preliminary Notes

Before giving our main results, we recall some of the basic concepts and results in Menger space

Definition 2.1. A mapping $F : R \to R^+$ is called a distribution if it is non-decreasing left continuous with $inf\{F(t) : t \in R\} = 0$ and

$$sup\{F(t) : t \in R\} = 1.$$

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, \ t \leq 0 \\ 1, \ t > 0 \end{cases}$$

Definition 2.2. A probabilistic metric space (PM-space) is an ordered pair (X,F) where X is an arbitrary set of elements and $F : X \times X \longrightarrow L$ is defined by $(p,q) \longmapsto F_{p,q}$, where L is the set of all distribution functions, that is, $L = \{F_{p,q} : p,q \in X\}$,where the functions $F_{p,q}$ satisfy:

i) $F_{p,q}(x) = 1$ for all $x > 0$, if and only if $p = q$;

ii) $F_{p,q}(0) = 0$;

iii) $F_{p,q} = F_{q,p}$;

iv) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x+y) = 1$.

Definition 2.3. A mapping $t : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a $t-$norm if

i) $t(a,1) = a,\ t(0,0) = 0$;

ii) $t(a,b) = t(b,a)$;

iii) $t(c,d) \geq t(a,b)$ for $c \geq a,\ d \geq b$;

iv) $t(t(a,b),c) = t(a,t(b,c))$.

Definition 2.4. A Menger space is a triplet (X,F,t), where (X,F) is a PM-space and t is a $t-$norm such that for all $p,q,r \in X$ and for all $x,y \geq 0$

$$F_{p,r}(x+y) \geq t\left(F_{p,q}(x), F_{q,r}(y)\right).$$

Example 2.5. If (X,d) is a metric space then the metric d induces a mapping $X \times X \longrightarrow L$, defined by $F_{p,q}(x) = H(x - d(p,q)), p,q \in X$ and $x \in R$. Further, if the $t-$norm $t : [0,1] \times [0,1] \longrightarrow [0,1]$ is defined by $t(a,b) = min\{a,b\}$, then (X,F,t) is a Menger space. The space (X,F,t) so obtained is called the induced Menger space.

Example 2.6. Let (X,d) be a metric space. We define the $t-$norm $*$ by $a*b = ab$ for all $a,b \in [0,1]$ and $F_{x,y}(t) = \frac{t}{t+d(x,y)}$ for all $x,y \in X$and $t > 0$. Then $(X,F,*)$ is a Menger space.

Definition 2.7. Let $(X,F,*)$ is a Menger space. Then

i) a sequence $\{x_n\}$ in X is said a G-Cauchy sequence if and only if

$$\lim_{n\to+\infty} F_{x_{n+1},x_n}(t) = 1$$

for any $p > 0$ and $t > 0$.

ii) The Menger space $(X, F, *)$ is called G - complete if every G - Cauchy sequence is convergent.

Definition 2.8. Self maps A and S of a Menger space (X, F, t) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Ap = Sp$ for some $p \in X$ then $ASp = SAp$.

Definition 2.9. Let X be a non-empty set, m a positive integer and
$f : X \to X$ an operator. $X = \bigcup\limits_{i=1}^{m} X_i$ is a cyclic representation of X with respect to f if

i) X_i, $i = 1, 2, \cdots m$ are non empty sets.

ii) $f(X_1) \subset X_2, \cdots, f(X_{m-1}) \subset X_m, f(X_m) \subset X_1$.

Definition 2.10. Let $(X, F, *)$ be a Menger space, $A_1, A_2, \cdots, A_m$ be closed subsets of X and $Y = \bigcup\limits_{i=1}^{m} A_i$. An operator $f : X \to X$ is called a cyclic weak (ϕ, ψ) contraction if

i) $Y = \bigcup\limits_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to f.

ii) for the function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$, $\psi(0) = 0$ and an altering distance function ϕ such that for $i > 1$, the relation

$$\phi\left(\frac{1}{F_{fx,fy}(t)} - 1\right) \leq \phi\left(\frac{1}{F_{x,y}(t)} - 1\right) - \psi\left(\frac{1}{F_{x,y}(t)} - 1\right)$$

holds for every $x, y \in X$ and each $t > 0$.

If $\{A_i\}$, $i = 1, 2, \cdots$, S and T are self maps on a Menger space $(X, F, *)$, we shall denote

$F1i_{x,y}(t) = \min\left\{F_{A_1x,Sx}(t), F_{A_iy,Ty}(t), F_{Sx,Ty}(t), F_{A_1x,Ty}(t), F_{Sx,A_iy}(t)\right\}$ for all $x, y \in X$ and all $t > 0$.

3. Results and Discussion

In this section, we prove the main result related to common fixed point theorem for (ϕ, ψ) - weak contraction in Menger space.

Theorem 3.11. Let $\{A_i\}$, $i = 1, 2, \cdots$, S and T be self maps on a Menger space $(X, F, *)$ such that

i) $A_1X \subseteq TX$, $A_iX \subseteq SX$ for $i > 1$ and

ii) for the function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(r) > 0$ for $r > 0$, $\psi(0) = 0$ and an altering distance function ϕ such that for $i > 1$, the relation

$$\phi\left(\frac{1}{F_{A_1x,A_iy}(t)} - 1\right) \leq \phi\left(\frac{1}{F1i_{x,y}(t)} - 1\right) - \psi\left(\frac{1}{F1i_{x,y}(t)} - 1\right)$$

holds for every $x, y \in X$ and each $t > 0$.

If one of A_iX, SX and TX is a G- complete subspace of X and the pairs (A_1, S), (A_i, T) for $i > 1$ are weakly compatible, then all the mappings A_i, S and T have a unique common fixed point in X.

Proof. Since $A_1X \subseteq TX$, for any $x_0 \in X$, there exists $x_1 \in X$ such that $A_1x_0 = Tx_1$. Since $A_iX \subseteq SX$, for this point x_1, we can choose $x_2 \in X$ such that $A_ix_1 = Sx_2$ for some $i > 1$. Inductively we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = A_1x_{2n} = Tx_{2n+1}$; $y_{2n+1} = A_kx_{2n+1} = Sx_{2n+2}$ for some $k > 1$.

Next we prove $\{y_n\}$ is a G- Cauchy sequence.

If $y_{2n} = y_{2n+1}$ for some n. Then using condition ii), we get $y_{2n+1} = y_{2n+2}$ and so $y_m = y_{2n}$ for each $m > 2n$. Thus $\{y_n\}$ is G- Cauchy.

If $y_n \neq y_{n+1}$ for all n, then, for some $k > 1$, setting $x = x_{2n}$ and $y = x_{2n-1}$ in condition ii), we get

$$\phi\left(\frac{1}{F_{A_1x_{2n},A_kx_{2n-1}}(t)} - 1\right) \leq \phi\left(\frac{1}{F1k_{x_{2n},x_{2n-1}}(t)} - 1\right) - \psi\left(\frac{1}{F1k_{x_{2n},x_{2n-1}}(t)} - 1\right)$$

That is,

$$\phi\left(\frac{1}{F_{y_{2n},y_{2n-1}}(t)} - 1\right) \leq \phi\left(\frac{1}{F1k_{x_{2n},x_{2n-1}}(t)} - 1\right) - \psi\left(\frac{1}{F1k_{x_{2n},x_{2n-1}}(t)} - 1\right) \tag{1}$$

Since $\psi(r) > 0$ for $r > 0$,the above inequality becomes

$$\phi\left(\frac{1}{F_{y_{2n},y_{2n-1}}(t)} - 1\right) < \phi\left(\frac{1}{F1k_{x_{2n},x_{2n-1}}(t)} - 1\right)$$

Since ϕ is non decreasing, we get

$$F_{y_{2n},y_{2n-1}}(t) > F1k_{x_{2n},x_{2n-1}}(t) > F_{y_{2n-1},y_{2n-2}}(t)$$

Therefore, $F_{y_n,y_{n-1}}(t) > F_{y_{n-1},y_{n-2}}(t)$ for all n.

Hence the sequence $\{F_{y_n,y_{n-1}}(t)\}$ is an increasing sequence of positive real numbers in $(0,1]$.

Let $S(t) = \lim_{n\to\infty} F_{y_n,y_{n-1}}(t)$

Now we show that $S(t) = 1$ for all $t > 0$.

If not, there exists some $t > 0$ such that $S(t) < 1$.

Then, on making $n \to \infty$ in (1), we obtain
$\phi\left(\frac{1}{S(t)} - 1\right) \le \phi\left(\frac{1}{S(t)} - 1\right) - \psi\left(\frac{1}{S(t)} - 1\right)$, which is a contradiction.
Therefore$F_{y_n,y_{n-1}}(t) \to 1$ as $n \to \infty$

So that for each positive integer p, we have

$$F_{y_n,y_{n+p}}(t) \ge F_{y_n,y_{n+1}}\left({}^{t}\!/_{p}\right) * F_{y_{n+1},y_{n+2}}\left({}^{t}\!/_{p}\right) * \cdots * F_{y_{n+p-1},y_{n+p}}\left({}^{t}\!/_{p}\right)$$

It follows that $\lim_{n\to\infty} F_{y_n,y_{n+p}}(t) \ge 1 * 1 * \cdots * 1 = 1$. So that $\{y_n\}$ is a G- Cauchy sequence.
Now assume that SX is G- complete. Then by definition, there exists $z \in SX$ such that $y_n \to z$ as $n \to \infty$.

So that, $A_1x_{2n} = Tx_{2n+1} = y_{2n} \to z$ and $A_kx_{2n+1} = Sx_{2n+2} = y_{2n+1} \to z$ for some $k > 1$

Let $v \in X$ be such that $Sv = z$.

Then we show that $A_1v = z$.

Suppose that $A_1v \neq Sv$.

For some $k > 1$ and for all $t > 0$, setting $x = v$ and $y = x_{2n+1}$ in condition ii) we get

$$\phi\left(\frac{1}{F_{A_1v,A_kx_{2n+1}}(t)} - 1\right) \le \phi\left(\frac{1}{F1k_{v,x_{2n+1}}(t)} - 1\right) - \psi\left(\frac{1}{F1k_{v,x_{2n+1}}(t)} - 1\right)$$

That is, $\phi\left(\frac{1}{F_{A_1v,y_{2n+1}}(t)} - 1\right) \le \phi\left(\frac{1}{F1k_{v,x_{2n+1}}(t)} - 1\right) - \psi\left(\frac{1}{F1k_{v,x_{2n+1}}(t)} - 1\right)$ where

$$\begin{aligned} F1k_{v,x_{2n+1}}(t) &= \min\left\{F_{A_1v,Sv}(t), F_{A_kx_{2n+1},Tx_{2n+1}}(t), F_{Sv,Tx_{2n+1}}(t),\right. \\ &\qquad \left. F_{A_1v,Tx_{2n+1}}(t), F_{Sv,A_kx_{2n+1}}(t)\right\} \end{aligned}$$

As $n \to \infty$,

$$\begin{aligned} \phi\left(\frac{1}{F_{A_1v,z}(t)} - 1\right) &\le \phi\left(\frac{1}{F_{A_1v,z}(t)} - 1\right) - \psi\left(\frac{1}{F_{A_1v,z}(t)} - 1\right) \\ &< \phi\left(\frac{1}{F_{A_1v,z}(t)} - 1\right), a\ contradiction \end{aligned}$$

Thus $A_1v = z = Sv$.

Since the pair (A_1, S) is weakly compatible, $Sz = SA_1v = A_1S = A_1z$.

Next we prove $A_1z = z$.

If not, then for some $k > 1$ and for all $t > 0$, setting $x = z$ and $y = x_{2n+1}$,

$$\begin{aligned} F1k_{z,x_{2n+1}}(t) &= \min\left\{F_{A_1z,Sz}(t), F_{A_kx_{2n+1},Tx_{2n+1}}(t), F_{Sz,Tx_{2n+1}}(t),\right. \\ &\qquad \left. F_{A_1z,Tx_{2n+1}}(t), F_{Sz,A_kx_{2n+1}}(t)\right\} \end{aligned}$$

and $\phi\left(\frac{1}{F_{A_1z,A_kx_{2n+1}}(t)} - 1\right) \leq \phi\left(\frac{1}{F1k_{z,x_{2n+1}}(t)} - 1\right) - \psi\left(\frac{1}{F1k_{z,x_{2n+1}}(t)} - 1\right)$
which on taking $n \to \infty$, reduces to $\phi\left(\frac{1}{F_{A_1z,z}(t)} - 1\right) < \phi\left(\frac{1}{F_{A_1z,z}(t)} - 1\right)$, a contradiction. Therefore, we have $A_1z = z$.

Again, since $A_1X \subseteq TX$, there exists some $u \in X$ such that $A_1z = Tu$.

Therefore we have $z = A_1v = Sv = Tu$.

We claim that $A_ku = z$ for some $k > 1$.

If not, then for some $k > 1$ and for all $t > 0$, setting $x = z$ and $y = u$,
in condition ii),we get $\phi\left(\frac{1}{F_{A_1z,A_ku}(t)} - 1\right) \leq \phi\left(\frac{1}{F1k_{z,u}(t)} - 1\right) - \psi\left(\frac{1}{F1k_{z,u}(t)} - 1\right)$

$$\begin{aligned} F1k_{z,u}(t) &= \min\left\{F_{A_1z,Sz}(t), F_{A_ku,Tu}(t), F_{Sz,Tu}(t),\right. \\ &\qquad \left. F_{A_1z,Tu}(t), F_{Sz,A_ku}(t)\right\} \end{aligned}$$

Therefore the above inequality reduces to $\phi\left(\frac{1}{F_{z,A_kz}(t)} - 1\right) < \phi\left(\frac{1}{F_{z,A_kz}(t)} - 1\right)$, a contradiction.

Therefore, we have $A_ku = z$.

Thus we have $z = A_1v = Sv = Tu = A_ku$.

Since the pair (A_k, T) is weakly compatible, for some $k > 1$, we have

$$A_kz = A_kTu = TA_ku = Tz.$$

Similarly, using condition ii) and for all $t > 0$, we can prove that $A_kz = z$. Thus we have $z = A_1z = Sz = A_kz = Tz$ for some $k > 0$ and so z is a common fixed point of all mappings A_i, S and T in X.

In order to prove the uniqueness of fixed point, let w be another fixed point for mappings A_i, S and T. Setting $x = z$ and $y = w$ in condition ii), we get

$$\begin{aligned} F1k_{z,w}(t) &= \min\left\{F_{A_1z,Sz}(t), F_{A_kw,Tw}(t), F_{Sz,Tw}(t),\right. \\ &\qquad \left. F_{A_1z,Tw}(t), F_{Sz,A_kw}(t)\right\} \end{aligned}$$

and so,$\phi\left(\frac{1}{F_{A_1z,A_kw}(t)} - 1\right) \leq \phi\left(\frac{1}{F1k_{z,w}(t)} - 1\right) - \psi\left(\frac{1}{F1k_{z,w}(t)} - 1\right)$ which reduces to $\phi\left(\frac{1}{F_{z,w}(t)} - 1\right) < \phi\left(\frac{1}{F_{z,w}(t)} - 1\right)$, a contradiction. Hence $z = w$.
Similarly, instead of SX, if one of A_iX or TX is assumed to be G- complete subspace of X, one can prove that all mappings A_i, S and T have a unique common fixed point in X. □

Example 3.12. Let $X = [2, 20]$, $F_{x,y}(t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$, $t > 0$ and the t-norm $*$ is defined as $a * b = ab$. Then $(X, F, *)$ is a Menger space. Define A_i, S and T from X to X as follows

$A_1x = 2$ for each x;

$Sx = x$ if $x \leq 8$, $Sx = 8$ if, $8 < x < 14$, $Sx = \frac{x+10}{3}$ if, $14 \leq x \leq 17$ and $Sx = \frac{x+7}{3}$ if $x > 17$;

$Tx = 2$ if $x = 2$,or $x > 6$, $Tx = x + 12$ if, $2 < x < 4$, $Tx = \frac{x+9}{3}$ if, $4 \leq x \leq 5$ and $Tx = 8$ if $5 \leq x \leq 6$;

$A_2x = 2$ if $x < 4$ or $x > 6$, $A_2x = x + 3$ if,$4 \leq x \leq 5$, $A_2x = x + 2$ if, $5 \leq x \leq 6$;

and for each $i > 2$, $A_ix = 2$ if, $x = 2$ or $x \geq 4$, $A_ix = \frac{x+30}{4}$ if $2 < x < 4$.

Also we define $F_{A_1x,A_ky}(t) = \frac{t}{t+|x-y|}$. Define $\phi(t) = t$ for all $t > 0$ and $\psi(t) = \frac{t}{2}$. Then all the self mappings satisfies all the conditions of the above theorem with a unique common fixed point $x = 2$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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