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FIXED POINT THEOREM IN FUZZY METRIC SPACE USING (CLRg) PROPERTY

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Abstract: The present paper deals with the common fixed point theorem for a pair of occasionally weakly compatible mappings by using the (CLRg) property in fuzzy metric space. We also cited an example in support of our result. Our result improves the result of Sedghi et. al. [18].

Keywords: Common fixed points; fuzzy metric space; compatible maps; occasionally weakly compatible mappings and (CLRg) property.

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1. Introduction

The concept of a fuzzy set is investigated by Zadeh [28] in his seminal paper. In 1975, Kramosil and Michalek [12] introduced the concept of fuzzy metric space, which opened an avenue for further development of analysis in such spaces. Further, George and Veeramani [6] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [12] with a view to obtain a Hausdoroff topology which has very important applications in quantum particle physics, particularly in connection with both string and e^{∞} theory (see, [15–17]). Fuzzy set theory also has

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applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors viz Grabiec [7], Cho [3, 4], Subrahmanyam [26] and Vasuki [27].

In 2002, Aamri and El-Moutawakil [1] defined the notion of (E.A) property for self mappings which contained the class of non-compatible mappings in metric spaces. It was pointed out that (E.A) property allows replacing the completeness requirement of the space with a more natural condition of closedness of the range as well as relaxes the compleness of the whole space, continuity of one or more mappings and containment of the range of one mapping into the range of other which is utilized to construct the sequence of joint iterates. Many authors have proved common fixed point theorems in fuzzy metric spaces for different contractive conditions. Recently, Grabiec [7] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [21] introduced the concept of compatible mappings in Fuzzy metric space and proved the common fixed point theorem. Jungck et. al. [10] introduced the concept of compatible maps of type (A) in metric space and proved fixed point theorems. Cho [4, 5] introduced the concept of compatible maps of type (α) and compatible maps of type (β) in fuzzy metric space. In 2011, using the concept of compatible maps of type (A) and type (β), Singh et. al. [22, 23] proved fixed point theorems in a fuzzy metric space. Recently, Sintunavarat and Kumam [25] defined the notion of (CLRg) property in fuzzy metric spaces and improved the results of Mihet [13] without any requirement of the closedness of the subspace. Recently in 2012, Jain et. al. [8, 9] and Sharma et. al. [19] proved various fixed point theorems using the concepts of semi-compatible mappings, property (E.A.) and absorbing mappings.

In this paper, we prove a common fixed point theorem for a pair of occasionally weakly compatible mappings by using (CLRg) property in fuzzy metric space. Our results improve the results of Sedghi, Shobe and Aliouche [18]. We also give an example in support of our result.

For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

2. Preliminaries

Definition 2.1. [14] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for a, b, c, d $\in [0, 1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2. [14] The 3-tuple (X, M, *) is said to be a *fuzzy metric space* if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions :

for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,

(FM-6) $\lim_{t \to \infty} M(x, y, t) = 1$.

Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [14] Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$ and all $t > 0$. Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3. [14] A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is said to be a *Cauchy sequence* if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \ge n_0$.

The sequence $\{x_n\}$ is said to *converge* to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

A Fuzzy metric space (X, M, *) is said to be *complete* if every Cauchy sequence in it converges to a point in it.

Definition 2.4. [21] Self mappings A and S of a fuzzy metric space (X, M, *) are said to be *compatible* if and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition 2.6. Self maps A and S of a fuzzy metric space (X, M, *) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Definition 2.7. A pair of self mappings A and S of a fuzzy metric space (X, M, *) is said to satisfy the (CLRg) property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = Bu. \text{ for some } u \in X.$$

Proposition 2.1. [23] In a fuzzy metric space (X, M, *) limit of a sequence is unique.

Proposition 2.2. [21] Let S and T be compatible self maps of a Fuzzy metric space (X, M, *) and let $\{x_n\}$ be a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X. Then $STx_n \to Tu$ provided T is continuous.

Proposition 2.3. [21] Let S and T be compatible self maps of a Fuzzy metric space (X, M, *) and $Su = Tu$ for some u in X then

$$STu = TSu = SSu = TTu.$$

Lemma 2.1. [7] Let (X, M, *) be a fuzzy metric space. Then for all $x, y \in X$, M(x, y, .) is a non-decreasing function.

Lemma 2.2. [2] Let (X, M, *) be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$

$$M(x, y, kt) \geq M(x, y, t) \ \forall \ t > 0$$

then $x = y$.

Lemma 2.3. [23] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a number $k \in (0, 1)$ such that

$$M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t) \ \forall \ t > 0 \ \text{ and } \ n \in N.$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.4. [11] The only t-norm * satisfying $r * r \geq r$ for all $r \in [0, 1]$ is the minimum t-norm, that is $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$.

3. Main Result

Theorem 3.1. Let (X, M, *) be a fuzzy metric space, where * is a continuous t-norm. Further let f, g be mappings from X into itself and satisfying the inequality

$$M(fx, fy, t) \geq \phi\left(\min\left\{ \begin{array}{c} M(gx, gy, t) \\ \sup_{t_1+t_2=\frac{2}{k}t} \min\left\{ \begin{array}{c} M(gx, fx, t_1), \\ M(gy, fy, t_2) \end{array} \right\}, \\ \sup_{t_3+t_4=\frac{2}{k}t} \max\left\{ \begin{array}{c} M(gx, fy, t_3), \\ M(gy, fx, t_4) \end{array} \right\} \end{array} \right\} \right) \qquad (3.1)$$

for all $x, y \in X$, $t > 0$ and for some $1 \leq k < 2$.

If the pair (f, g) satisfies the (CLRg) property then f and g have a unique common fixed point provided the pair (f, g) is occasionally weakly compatible.

Proof. Since the pair (f, g) satisfies the (CLRg) property, there exists a sequence $\{x_n\}$ in x such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu, \text{ for some } u \in X.$$

Now, we assert that fu = gu.

Suppose, on the contrary, $fu \neq gu$, then there exists $t_0 > 0$ such that

$$M\left(fu, gu, \frac{2}{k}t_0\right) > M\left(fu, gu, t_0\right). \tag{3.2}$$

To support the claim, let it be untrue. Then we have

$$M\left(fu, gu, \frac{2}{k}t\right) > M\left(fu, gu, t\right), \quad \text{for all } t > 0.$$

Repeatedly, using this inequality, we obtain

$$M(fu, gu, t) = M\left(fu, gu, \frac{2}{k}t\right) = \ldots = M\left(fu, gu, \left(\frac{2}{k}\right)^n t\right) \to 1,$$

as $n \to \infty$. This shows that M(fu, gu, t) = 1 for all t > 0 which contradicts $fu \neq gu$ and hence (3.2) is proved.

On using inequality (3.1), with $x = x_n$ and $y = u$, we get

$$M(fx_n, fu, t_0) \geq \phi\left(\min\left\{\begin{array}{c} M(gx_n, gu, t_0) \\ \sup_{t_1+t_2=\frac{2}{k}t_0} \min\left\{\begin{array}{c} M(gx_n, fx_n, t_1), \\ M(gu, fu, t_2) \end{array}\right\}, \\ \sup_{t_3+t_4=\frac{2}{k}t_0} \max\left\{\begin{array}{c} M(gx_n, fu, t_3), \\ M(gu, fx_n, t_4) \end{array}\right\} \end{array}\right\}\right)$$

for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$. As $n \to \infty$, it follows that

$$M(gu, fu, t_0) \geq \phi\left(\min\left\{\begin{array}{c} M(gu, gu, t_0) \\ \min\left\{M(gu, gu, \varepsilon), M(gu, fu, \frac{2}{k}t_0 - \varepsilon)\right\}, \\ \max\left\{M(gu, fu, \frac{2}{k}t_0 - \varepsilon), M(gu, gu, \varepsilon)\right\} \end{array}\right\}\right)$$

$$= \phi\left(M\left(gu, fu, \frac{2}{k} t_0 - \varepsilon \right) \right)$$

$$> M\left(gu, fu, \frac{2}{k} t_0 - \varepsilon \right),$$

as $\varepsilon \to 0$, we have

$$M(gu, fu, t_0) \geq M\left(gu, fu, \frac{2}{k} t_0 \right),$$

which contradicts (3.2), hence, we have

$$gu = fu.$$

Next, we suppose z = fu = gu.

Since the pair (f, g) is occasionally weakly compatible, so

$$fgu = gfu$$

which implies that fz = fgu = gfu = gz.

Now, we show that z = fz.

Suppose z $\neq$ fz, then on using (3.1) with x = z and y = u, for some $t_0 > 0$, we get

$$M(fz, fu, t_0) \geq \phi\left(\min\left\{ \begin{array}{c} M(gz, gu, t_0) \\ \sup_{t_1 + t_2 = \frac{2}{k} t_0} \min\left\{ \begin{array}{l} M(gz, fz, t_1), \\ M(gu, fu, t_2) \end{array} \right\}, \\ \sup_{t_3 + t_4 = \frac{2}{k} t_0} \max\left\{ \begin{array}{l} M(gz, fu, t_3), \\ M(gu, fz, t_4) \end{array} \right\} \end{array} \right\} \right)$$

$$M(fz,z,t_0) \geq \phi\left(\min\left\{ \begin{array}{c} M(fz,z,t_0) \\ \min\left\{ M(fz,fz,\varepsilon), M(z,z,\frac{2}{k}t_0-\varepsilon)\right\}, \\ \max\left\{ M(fz,z,\varepsilon), M(z,fz,\frac{2}{k}t_0-\varepsilon)\right\} \end{array} \right\} \right),$$

for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$. As $\varepsilon \to 0$, we have

$$M(fz,z,t_0) \geq \phi\left(\min\left\{ M(fz,z,t_0), M\left(z,fz,\frac{2}{k}t_0\right)\right\}\right)$$

$$= \phi(M(fz, z, t_0)) > M(fz, z, t_0),$$

which is a contradiction. Hence fz = gz = z. Therefore, z is a common fixed point of f and g.

Uniqueness. Let w ($\neq$ z) be another common fixed point of f and g. On using inequality (3.1) with x = z and y = w, we get for some $t_0 > 0$,

$$M(fz,fw,t_0) \geq \phi\left(\min\left\{ \begin{array}{c} M(gz,gw,t_0) \\ \sup_{t_1+t_2=\frac{2}{k}t_0} \min\left\{ \begin{array}{c} M(gz,fz,t_1), \\ M(gw,fw,t_2) \end{array}\right\}, \\ \sup_{t_3+t_4=\frac{2}{k}t_0} \max\left\{ \begin{array}{c} M(gz,fw,t_3), \\ M(gw,fz,t_4) \end{array}\right\} \end{array} \right\} \right)$$

$$M(z,w,t_0) \geq \phi\left(\min\left\{ \begin{array}{c} M(z,w,t_0) \\ \min\left\{ M(z,z,\varepsilon), M(w,w,\frac{2}{k}t_0-\varepsilon)\right\}, \\ \max\left\{ M(z,w,\varepsilon), M(w,z,\frac{2}{k}t_0-\varepsilon)\right\} \end{array} \right\} \right),$$

for all $\varepsilon \in \left(0, \frac{2}{k} t_0\right)$. As $\varepsilon \to 0$, we have

$$M(z, w, t_0) \geq \phi\left(\min\left\{M(z, w, t_0), M\left(w, z, \frac{2}{k} t_0\right)\right\}\right)$$

$$= \phi(M(z, w, t_0)) > M(z, w, t_0),$$

which is a contradiction. Hence Bz = z = Tz. It implies that f and g have a unique common fixed point.

Remark 3.1. Theorem 3.1 improves the result of Sedghi et. al. [18] in the sense that our result does not require any containment of ranges amongst the involved mappings and closedness of one or more subspaces.

The following example illustrates our theorem 3.1.

Example 3.1. Let (X, M, *) be a fuzzy metric space, where X = [3, 19), with t-norm * defined by a * b = ab for all a, b $\in$ [0, 1] and

$$M(x, y, t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases}$$

for all x, y $\in$ X. Let the function ϕ : (0, 1] $\to$ (0, 1] defined by $\phi(t) = t^{\frac{1}{2}}$. Define the self mappings f and g by

$$f(x) = \begin{cases} 3, & \text{if } x \in \{3\} \cup (5, 19); \\ 12, & \text{if } x \in (3, 5] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 3, & \text{if } x = 3; \\ 11, & \text{if } x \in (3, 5]; \\ \frac{x+1}{2}, & \text{if } x \in (5, 19). \end{cases}$$

Taking or $\{x_n\} = \left\{5 + \frac{1}{n}\right\}_{n \in \mathbb{N}}$ or $\{x_n\} = \{3\}$, it is clear that the pair (f, g) satisfies the (CLRg) property.

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 3 = g(3) \in X.$$

It is noted that f(X) = {3, 12} $\not\subset$ [3, 10) $\cup${11} = g(X). Thus, all the conditions of theorem 3.1 are satisfied and 3 is a unique common fixed point of the pair (f, g). Also, all the involved mappings are even discontinuous at their unique common fixed point 3. Here, it may be pointed out that g(X) is not a closed subspace of X.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Aamri, M. and Moutawakil, D. El, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (1)(2002), 181–188.

[2] Cho, S.H., On common fixed point theorems in fuzzy metric spaces, J. Appl. Math. & Computing, 20(2006), 523-533.

[3] Cho, Y. J., Fixed points for compatible mappings of type (A), Math. Japon. 38 (3) (1993), 497–508.

[4] Cho, Y.J., Fixed point in Fuzzy metric space, J. Fuzzy Math. 5(1997), 949-962.

[5] Cho, Y.J., Pathak, H.K., Kang, S.M., Jung, J.S., Common fixed points of compatible mappings of type (β) on fuzzy metric spaces, Fuzzy sets and systems, 93 (1998), 99-111.

[6] George, A. and Veeramani, P., On some results in Fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994), 395-399.

[7] Grabiec, M., Fixed points in Fuzzy metric space, Fuzzy sets and systems, 27(1998), 385-389.

[8] Jain, A., Badshah, V.H. and Prasad, S.K., Fixed Point Theorem in Fuzzy Metric Space for Semi-Compatible Mappings, International Journal of Research and Reviews in Applied Sciences 12 (3), (2012), 523-526.

[9] Jain, A., Badshah, V.H. and Prasad, S.K., The Property (E.A.) and The Fixed Point Theorem in Fuzzy Metric, International Journal of Research and Reviews in Applied Sciences, 12 (3), (2012), 527-530.

[10] Jungck, G., Murthy, P.P. and Cho, Y.J., Compatible mappings of type (A) and common fixed points, Math. Japonica, 38 (1993), 381-390.

[11] Klement, E.P., Mesiar, R. and Pap, E., Triangular Norms, Kluwer Academic Publishers, 2000.

[12] Kramosil, I. and Michalek, J., Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975), 336-344.

[13] Mihet, D., Fixed point theorems in fuzzy metric spaces using property E.A., Nonlinear Anal. 73(7)(2010), 2184–2188.

[14] Mishra, S.N., Mishra, N. and Singh, S.L., Common fixed point of maps in fuzzy metric space, Int. J. Math. Math. Sci. 17(1994), 253-258.

[15] Naschie, M. S. El, On the uncertainty of Cantorian geometry and two-slit experiment, Chaos Solitons Fractals 9 (3) (1998), 517–529.

[16] Naschie, M. S. El, A review of E-infinity theory and the mass spectrum of high energy particle physics, Chaos Solitons Fractals 19 (2004), 209–236.

[17] Naschie, M. S. El, A review of applications and results of E-infinity theory, Int. J. Nonlinear Sci. Numer. Simul. 8 (2007), 11–20.

[18] Sedghi, S., Shobe, N. and Aliouche, A., A common fixed point theorem for weakly compatible mappings in fuzzy metric spaces, Gen. Math. 18 (3) (2010), 3–12.

[19] Sharma, A., Jain, A. and Chaudhary, S., A note on absorbing mappings and fixed point theorems in fuzzy metric space, International Journal of Theoretical and Applied Sciences, 4(1), (2012), 52-57.

[20] Sharma, S., Common fixed point theorems in fuzzy metric spaces, Fuzzy sets and System, 127 (2002), 345-352.

[21] Singh, B. and Chouhan, M.S., Common fixed points of compatible maps in Fuzzy metric spaces, Fuzzy sets and systems, 115 (2000), 471-475.

[22] Singh, B., Jain, A. and Govery, A.K., Compatibility of type (β) and fixed point theorem in Fuzzy metric space, Applied Mathematical Sciences, 5 (11)(2011), 517-528.

[23] Singh, B., Jain, A. and Govery, A.K., Compatibility of type (A) and fixed point theorem in Fuzzy metric space, Int. J. Contemp. Math. Sciences, 6 (21) (2011), 1007-1018.

[24] Singh, B., Jain, S. and Jain, S., Generalized theorems on fuzzy metric spaces, Southeast Asian Bulletin of Mathematics, 31 (2007), 963-978.

[25] Sintunavarat, W. and Kumam, P., Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math. 2011(2011), Article ID 637958.

[26] Subrahmanyam, P.V., A common fixed point theorem in fuzzy metric spaces, Inform. Sci. 83 (1995), 109–112.

[27] Vasuki, R., Common fixed points for R-weakly commuting maps in fuzzy metric spaces, Indian J. Pure Appl. Math. 30 (4) (1999), 419–423.

[28] Zadeh, L. A., Fuzzy sets, Inform and control, 89 (1965), 338-353.